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SINGULAR QUASILINEAR CONVECTIVE SYSTEMS INVOLVING VARIABLE EXPONENTS

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Abstract. The paper deals with the existence of solutions for quasilinear elliptic systems involving singular and convection terms with variable exponents. The approach combines the sub-supersolutions method and Schauder's fixed point theorem.

Keywords: $p(x)$-Laplacian, variable exponents, fixed point, singular system, gradient estimate, regularity.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. Given $p_i \in C^1(\overline{\Omega})$, $1 < p_i^- \leq p_i^+ < N$ with

$$p_i^- = \inf_{x\in\Omega} p_i(x) \quad \text{and} \quad p_i^+ = \sup_{x\in\Omega} p_i(x),$$

we deal with the following quasilinear elliptic system

$$\begin{cases} -\Delta_{p_i(x)} u_i = f_i(x, u_1, u_2, \nabla u_1, \nabla u_2) \;\text{ in } \Omega, \\ u_i > 0 \;\text{ in } \Omega, \;\; u_i = 0 \;\text{ on } \partial\Omega, \; i = 1, 2, \end{cases} \tag{1.1}$$

where $-\Delta_{p_i(x)}$ stands for the $p_i(x)$-Laplacian differential operator defined by

$$-\Delta_{p_i(x)} u_i = -\mathrm{div}(|\nabla u_i|^{p_i(x)-2}\nabla u_i), \text{ for } u_i \in W_0^{1,p_i(x)}(\Omega).$$

The nonlinear terms $f_1(x, u_1, u_2, \nabla u_1, \nabla u_2)$ and $f_2(x, u_1, u_2, \nabla u_1, \nabla u_2)$ which are often expressed as dealing with convection terms, are of Carathéodory type. Namely, for every $(s_1, s_2, \xi_1, \xi_2) \in (\mathbb{R}_+^\star)^2 \times \mathbb{R}^{2N}$, we assume that $f_i(\cdot, s_1, s_2, \xi_1, \xi_2)$ is Lebesgue measurable in Ω, and, for a.e. $x \in \Omega$, $f_i(x, \cdot, \cdot, \cdot, \cdot)$ is continuous in $(\mathbb{R}_+^\star)^2 \times \mathbb{R}^{2N}$. A solution of (1.1) is understood in the weak sense, that is, a pair $(u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ satisfying

$$\int_\Omega |\nabla u_i|^{p_i(x)-2}\nabla u_i \nabla\varphi_i \, dx = \int_\Omega f_i(x, u_1, u_2, \nabla u_1, \nabla u_2)\varphi_i \, dx, \tag{1.2}$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$.

Our main purpose is to establish the existence and regularity of solutions for quasilinear singular convective system (1.1) satisfying the assumption:

(H_f) There exist constants $M_i, m_i > 0$, and functions $\alpha_i, \beta_i, \gamma_i, \bar{\gamma}_i \in C(\overline{\Omega})$, such that

$$m_i s_1^{\alpha_i(x)} s_2^{\beta_i(x)} \leq f_i(x, s_1, s_2, \xi_1, \xi_2) \leq M_i(s_1^{\alpha_i(x)} s_2^{\beta_i(x)} + |\xi_1|^{\gamma_i(x)} + |\xi_2|^{\bar{\gamma}_i(x)}),$$

for a.e. $x \in \Omega$, for all $s_1, s_2 > 0$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$, $i = 1, 2$.

The dependence of the right hand side terms on the solution and its gradient deprives system (1.1) of a variational structure. Thereby variational methods are not applicable. Moreover, due to the presence of convection terms, even the so called topological methods as sub-supersolutions and fixed points technique cannot be directly implemented. Another important feature in studying problem (1.1) is that the nonlinearities can exhibit singularities when the variables u_1 and u_2 approach zero. This occur through the following condition

($\mathrm{H}_{\alpha,\beta,\gamma}$)

$$|\alpha_i^{\mp}| + |\beta_i^{\mp}| < p_i^- - 1,$$

and

$$0 \leq \min\{\gamma_i^-, \bar{\gamma}_i^-\} \leq \max\{\gamma_i^+, \bar{\gamma}_i^+\} < p_i^- - 1,$$

where

$$\alpha_i^{\mp} := \begin{cases} \alpha_i^- & \text{if } \alpha_i(\cdot) > 0, \\ \alpha_i^+ & \text{if } \alpha_i(\cdot) < 0, \end{cases} \quad \beta_i^{\mp} := \begin{cases} \beta_i^- & \text{if } \beta_i(\cdot) > 0, \\ \beta_i^+ & \text{if } \beta_i(\cdot) < 0, \end{cases} \quad i = 1, 2.$$

Precisely, singularities appear in system whenever one of the exponents at least is negative, that is, $\min\{\alpha_1, \alpha_2, \beta_1, \beta_2\} < 0$. This represents a major hurdle to overcome. This difficulty is heightened by the very emphasized singularity character of (1.1) that stems from ($\mathrm{H}_{\alpha,\beta,\gamma}$) when $\alpha_i^{\mp} + \beta_i^{\mp} < 0$. In this case, hypothesis ($\mathrm{H}_{\alpha,\beta,\gamma}$) is strengthened by assuming

$\mathrm{H}_{\alpha,\beta,\gamma}$ If $\alpha_i^{\mp} + \beta_i^{\mp} < 0$, we have

$$|\alpha_i^{\mp}| + |\beta_i^{\mp}| \leq \frac{1}{N p_i'^{+}},$$

and

$$0 \leq \gamma_i(x) \leq \frac{p_1(x)}{N p_i'(x)} \quad \text{and} \quad 0 \leq \overline{\gamma}_i(x) \leq \frac{p_2(x)}{N p_i'(x)}, \quad \text{for } x \in \Omega.$$

Quasilinear convective system (1.1) has been rarely investigated in the literature. Actually, according to our knowledge, [21] is the only paper that has addressed this issue in the regular case, that is when all exponents are positive. Existence result is obtained applying the recent topological degree of Berkovits. The virtually non-existent works devoted to the singular case of convective systems is partly due to the involvement of the $p_i(x)$-Laplacian operator. This fact results in the lack of properties such as homogeneity making it highly challenging task to establish a control on solutions and especially on their gradient. When $p_i(x)$ is reduced to be a constant, $-\Delta_{p_i(x)}$ becomes the well-known p_i-Laplacian operator. In this respect, by relying on the a priori gradient estimate in [6, 8], the existence of solutions for singular convective systems have been investigated in [7, 9]. Still in the context of constant exponents, for singular system (1.1) defined in whole space $\mathbb{R}^N$ we quote [16] while for the case of Neumann boundary condition we refer to [15]. We also mention [2] focusing on a singular system of type (1.1) corresponding to the semilinear case, that is when $p_i(x) = 2$ $(i = 1, 2)$.

When convection terms are canceled, singular system (1.1) was recently examined in [1, 3]. In this context, depending on the sign of $\alpha_i(\cdot)$ and $\beta_i(\cdot)$, two complementary structures for the system (1.1) appear: cooperative and competitive structure (see [1]). Here, the important structural disparity of the latter makes nonlinearities f_1 and f_2 (without gradient terms) behaving in a drastically different way. This fact has led in [3] to consider only the cooperative system involving logarithmic growth while in [1], a separate study corresponding to each structure is required. We emphasize that in the present work, neither cooperative nor competitive structure on the system (1.1) is imposed. In fact, these both complementary structures for the system (1.1) are handled simultaneously without referring to them.

Our main result is stated as follows.

Theorem 1.1. *Assume* (H_f) *holds. Then:*

(i) *under assumption* $(\mathrm{H}_{\alpha,\beta,\gamma})$ *with* $\alpha_i^{\mp} + \beta_i^{\mp} > 0$, *system* (1.1) *has a bounded (positive) solution* (u_1, u_2) *in* $C_0^{1,\tau}(\overline{\Omega}) \times C_0^{1,\tau}(\overline{\Omega})$, *for certain* $\tau \in (0, 1)$, *satisfying*

$$u_i(x) \geq c_0 d(x), \text{ for a constant } c_0 > 0, \; i = 1, 2, \tag{1.3}$$

(ii) *under assumption* $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$, *system* (1.1) *admits a (positive) solution* (u_1, u_2) *in* $(W_0^{1,p_1(x)}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,p_2(x)}(\Omega) \cap L^\infty(\Omega))$ *satisfying* (1.3).

Our approach is chiefly based on Schauder's fixed point theorem. In this respect, comparison arguments as well as a priori estimates are crucial to get the appropriate localization of the desired fixed point which is actually solution of (1.1). This is achieved through a control on solutions and their gradient which in itself represents a significant feature of our result. At this point, the choice of suitable functions with an adjustment of adequate constants is crucial. However, this would not be enough without making use of the new Mean Value Theorem (cf. Theorem 6.1 in Appendix) that is decisive to offset the lack of homogeneity property and to deal with the variable exponents attendance. It should be noted that the Mean Value Theorem is a key ingredient in getting the gradient estimate thus generalizing that corresponding to the case of constant exponents problems stated in [6, 8].

The rest of the paper is organized as follows. Sections 2 and 3 establish gradient estimates and a priori bounds. Section 4 deals with comparison properties. Section 5 presents the proof of the main result while Section 6 contains the new Mean Value Theorem.

2. A PRIORI ESTIMATES

Let $L^{p(x)}(\Omega)$ be the generalized Lebesgue space that consists of all measurable real-valued functions u satisfying

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < +\infty,$$

endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \tau > 0 : \rho_{p(x)} \left(\frac{u}{\tau} \right) \leq 1 \right\}.$$

Recall for any $u \in L^{p(x)}(\Omega)$ it holds

$$\begin{cases} \|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+} & \text{if } \|u\|_{p(x)} > 1, \\ \|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-} & \text{if } \|u\|_{p(x)} \leq 1, \end{cases} \tag{2.1}$$

and

$$\|u\|_{p(x)} = a \text{ if and only if } \rho_{p(x)} \left(\frac{u}{a} \right) = 1. \tag{2.2}$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$, defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

is endowed with the norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$ which makes it a Banach space. The interested reader may consult [13, 18] for more details on Orlicz–Sobolev spaces. In the sequel, $d(x) := d(x, \partial\Omega)$ denotes the euclidean distance of x with respect to the boundary $\partial\Omega$.

The next lemma is a slight modification of [21, Lemma 5.1] which will be useful later on.

Lemma 2.1. *Let $k, m \in L^{\infty}(\Omega)$ be two real and positive functions with $m^- > 0$. If $u \in L^{k(x)}(\Omega)$ then $u^{m(x)} \in L^{\frac{k(x)}{m(x)}}(\Omega)$, and there exists $x_0 \in \Omega$ such that*

$$\left\| |u|^{m(x)} \right\|_{\frac{k(x)}{m(x)}} = \|u\|_{k(x)}^{m(x_0)}.$$

Proof. On account of (2.2), the Mean Value Theorem [4, Theorem 5] ensures the existence of $x_0 \in \Omega$ such that

$$\begin{aligned} 1 = \rho_{\frac{k(x)}{m(x)}} \left(\frac{|u|^{m(x)}}{\left\| |u|^{m(x)} \right\|_{\frac{k(x)}{m(x)}}} \right) &= \int_{\Omega} \left| \frac{u}{\|u\|_{k(x)}} \right|^{k(x)} \frac{\|u\|_{k(x)}^{k(x)}}{\left\| |u_1|^{m(x)} \right\|_{\frac{k(x)}{m(x)}}^{\frac{k(x)}{m(x)}}} dx \\ = \frac{\|u\|_{k(x)}^{k(x_0)}}{\left\| |u_1|^{m(x)} \right\|_{\frac{k(x)}{m(x)}}^{\frac{k(x_0)}{m(x_0)}}} \rho_{k(x)} \left(\frac{u}{\|u\|_{k(x)}} \right) &= \frac{\|u\|_{k(x)}^{k(x_0)}}{\left\| |u|^{m(x)} \right\|_{\frac{k(x)}{m(x)}}^{\frac{k(x_0)}{m(x_0)}}}, \end{aligned}$$

showing the desired identity. ☐

A priori gradient estimate is provided in the next lemma. It is a partial extension of [6, Lemma 1] to problems involving variable exponents.

Lemma 2.2. *Let $h \in L^\infty(\Omega)$ be a nontrivial sign-constant function and let $u \in W_0^{1,p(x)}(\Omega)$ be the weak solution of the Dirichlet problem*

$$-\Delta_{p(x)} u = h(x) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Then, there exists a constant $\bar{k}_p > 0$, depending only on p, N, and Ω, such that

$$\|\nabla u\|_\infty \leq \bar{k}_p \|h\|_\infty^{\frac{1}{p^\pm - 1}} \tag{2.4}$$

with

$$p^\pm := \begin{cases} p^- & \text{if } \|h\|_\infty > 1, \\ p^+ & \text{if } \|h\|_\infty \leq 1. \end{cases}$$

Proof. First, assume that $\|h\|_\infty \leq 1$. Multiplying (2.3) by $\varphi \in W_0^{1,p(\cdot)}(\Omega)$, with $\varphi \geq 0$, and integrating over Ω we obtain

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = \int_\Omega h(x)\varphi \, dx \leq \int_\Omega \varphi \, dx = \int_\Omega |\nabla \xi|^{p(x)-2} \nabla \xi \nabla \varphi \, dx,$$

where $\xi(x)$ is the $p(x)$-torsion function defined by

$$-\Delta_{p(x)} \xi = 1 \text{ in } \Omega, \quad \xi = 0 \text{ on } \partial\Omega.$$

The weak comparison principle implies $\|u\|_\infty \leq \|\xi\|_\infty$ while the regularity theorem in [12] ensures the existence of constants $\tau \in (0,1)$ and $\bar{k}_p > 0$ such that $\|u\|_{C^{1,\tau}(\overline{\Omega})} \leq \bar{k}_p$.

Now we deal with the case $\|h\|_\infty > 1$. By Theorem 6.1 in the Appendix, there exists $x_0 \in \Omega$ such that

$$\begin{aligned}
&\int_\Omega |\nabla(\|h\|_\infty^{\frac{-1}{p^- - 1}} u)|^{p(x)-2} \nabla(\|h\|_\infty^{\frac{-1}{p^- - 1}} u) \nabla \varphi \, dx \\
&= \int_\Omega \|h\|_\infty^{\frac{-(p(x)-1)}{p^- - 1}} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx \\
&= \|h\|_\infty^{\frac{-(p(x_0)-1)}{p^- - 1}} \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = \|h\|_\infty^{\frac{-(p(x_0)-1)}{p^- - 1}} \int_\Omega h(x)\varphi \, dx \\
&\leq \|h\|_\infty^{\frac{-(p^- - 1)}{p^- - 1}} \int_\Omega h(x)\varphi \, dx = \int_\Omega \|h\|_\infty^{-1} h(x)\varphi \, dx \leq \int_\Omega \varphi \, dx.
\end{aligned}$$

Thus, in view of the previous argument, it follows that

$$\|h\|_\infty^{\frac{-1}{p^- - 1}} |u|_{1,\alpha} \leq \bar{k}_p,$$

showing that (2.4) holds true. This ends the proof. □

The case when h in (2.3) is not an L^∞-bounded function is handled in the next lemma which provides an a priori L^∞-estimate of solutions for (2.3).

Lemma 2.3. *Assume $h \in L^{p'(x)}(\Omega) \cap L^N(\Omega)$ in (2.3). Then, there exists a constant $C > 0$, depending only on N, p and Ω, such that*

$$\|u\|_\infty \leq C \|h\|_{L^N(\Omega)}^{\frac{1}{p^\pm - 1}}, \tag{2.5}$$

with

$$p^\pm := \begin{cases} p^- & \text{if } \|h\|_{L^N(\Omega)} > 1, \\ p^+ & \text{if } \|h\|_{L^N(\Omega)} \leq 1. \end{cases}$$

Proof. For each $k \in \mathbb{N}$, consider the set $A_k = \{x \in \Omega : u(x) > k\}$, where u is the solution of (2.3). Thus, to prove (2.5) amounts to show that

$$|A_k| = 0 \quad \text{for any } k > k_0,$$

where

$$k_0 := C \|h\|_{L^N(\Omega)}^{\frac{1}{p^\pm - 1}}, \tag{2.6}$$

with a constant $C > 0$ that will be chosen later on. By contradiction assume that there exists $k > k_0$ such that $|A_k| \neq 0$. Testing (2.3) with $(u - k)^+$ leads to

$$\int_{A_k} |\nabla u|^{p(x)}\, dx = \int_{A_k} h(x)\,(u-k)^+ dx \leq \|h\|_{L^N(\Omega)} \left\|(u-k)^+\right\|_{L^{N'}(A_k)}. \tag{2.7}$$

One has

$$\int_{A_k} |\nabla u|^{p(x)}\, dx \geq \|\nabla u\|_{L^{p^-}(A_k^-)}^{p^-} + \|\nabla u\|_{L^{p^+}(A_k^+)}^{p^+},$$

where $A_k^- := A_k \cap \{|\nabla u| \geq 1\}$ and $A_k^+ := A_k \cap \{|\nabla u| < 1\}$. Hölder's inequality together with classical Sobolev and Lebesgue embeddings imply

$$\begin{aligned} \left\|(u-k)^+\right\|_{L^{N'}(A_k)} &\leq C_1 \int_{A_k} |\nabla u| dx \\ &\leq C_1 \left(|A_k^-|^{\frac{p^- - 1}{p^-}} \|\nabla u\|_{L^{p^-}(A_k^-)} + |A_k^+|^{\frac{p^+ - 1}{p^+}} \|\nabla u\|_{L^{p^+}(A_k^+)} \right), \end{aligned} \tag{2.8}$$

as well as

$$\begin{aligned} \|\nabla u\|_{L^{p^\pm}(A_k^\pm)}^{p^\pm} &\geq \|\nabla u\|_{L^{p^\pm}(A_k^\pm)} (\|\nabla u\|_{L^1(A_k^\pm)} |A_k^\pm|^{-\frac{p^\pm - 1}{p^\pm}})^{p^\pm - 1} \\ &\geq \|\nabla u\|_{L^{p^\pm}(A_k^\pm)} (C_1^{-1} \|u\|_{L^{N'}(A_k^\pm)} |A_k^\pm|^{-\frac{p^\pm - 1}{p^\pm}})^{p^\pm - 1} \\ &\geq \|\nabla u\|_{L^{p^\pm}(A_k^\pm)} (C_1^{-1} k |A_k^\pm|^{\frac{1}{N'} - \frac{p^\pm - 1}{p^\pm}})^{p^\pm - 1}, \end{aligned} \tag{2.9}$$

for a certain constant $C_1 > 0$. Then, gathering (2.7)–(2.9) together, we infer that

$$\begin{aligned}
&\|\nabla u\|_{L^{p^-}(A_k^-)}\left[(C_1^{-1}k|A_k^-|^{\frac{1}{N'}-\frac{p^--1}{p^-}})^{p^--1} - C_1\,\|h\|_{L^N(\Omega)}\,|A_k^-|^{\frac{p^--1}{p^-}}\right]\\
&+\|\nabla u\|_{L^{p^+}(A_k^+)}\left[(C_1^{-1}k|A_k^+|^{\frac{1}{N'}-\frac{p^+-1}{p^+}})^{p^+-1} - C_1\,\|h\|_{L^N(\Omega)}\,|A_k^+|^{\frac{p^+-1}{p^+}}\right] \le 0.
\end{aligned} \tag{2.10}$$

Now, fix $C > 0$ in (2.6) as follows

$$C := \max\{C_1^{p^{-\prime}}, C_1^{p^{+\prime}}\}|\Omega|^{1-\frac{1}{N'}}.$$

Then, for $k > k_0$, it follows that

$$\left[(C_1^{-1}k|A_k^\pm|^{\frac{1}{N'}-\frac{p^\pm-1}{p^\pm}})^{p^\pm-1} - C_1\,\|h\|_{L^N(\Omega)}\,|A_k^\pm|^{\frac{p^\pm-1}{p^\pm}}\right] \ge 0,$$

which contradicts (2.10). This completes the proof. □

3. AN AUXILIARY SYSTEM

For each $(z_1, z_2) \in W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$, we consider the auxiliary problem

$$\begin{cases} -\Delta_{p_i(x)} u_i = f_i(x, z_1, z_2, \nabla z_1, \nabla z_2) & \text{in } \Omega \\ u_i = 0 \quad \text{on } \partial\Omega, \ i = 1, 2. \end{cases} \tag{$\mathrm{P}_{(z_1,z_2)}$}$$

Lemma 3.1. *Assume $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ holds. Suppose that*

$$z_i(x) \ge \tilde{c}d(x) \quad \text{and} \quad \|\nabla z_i\|_{p_i(x)} \le \tilde{L}, \tag{3.1}$$

for some constants $\tilde{c}, \tilde{L} > 0$ independent of z_1 and z_2. Then, for $\tilde{L}$ large enough, problem $(\mathrm{P}_{(z_1,z_2)})$ admits a unique solution (u_1, u_2) in $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$ satisfying

$$\|\nabla u_i\|_{p_i(x)} \le \tilde{L}, \ i = 1, 2. \tag{3.2}$$

Proof. First, we claim that

$$f_i(\cdot, z_1, z_2, \nabla z_1, \nabla z_2) \in L^{p_i'(x)}(\Omega) \cap L^N(\Omega), \text{ for } i = 1, 2. \tag{3.3}$$

We only show that $f_i \in L^{p_i'(x)}(\Omega)$ in (3.3) because $f_i \in L^N(\Omega)$ can be justified similarly by substituting $p_i'(\cdot)$ with N in the argument below. By (H_f), we have

$$\begin{aligned}
&\|f_i(\cdot, z_1, z_2, \nabla z_1, \nabla z_2)\|_{p_i'(x)}\\
&\le M_i\left(\|z_1^{\alpha_i(\cdot)} z_2^{\beta_i(\cdot)}\|_{p_i'(x)} + \left\||\nabla z_1|^{\gamma_i(\cdot)}\right\|_{p_i'(x)} + \left\||\nabla z_2|^{\overline{\gamma}_i(\cdot)}\right\|_{p_i'(x)}\right).
\end{aligned} \tag{3.4}$$

Lemma 2.1 together with $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ imply

$$\begin{aligned}
\left\||\nabla z_1|^{\gamma_i(\cdot)}\right\|_{p_i'(x)} + \left\||\nabla z_2|^{\overline{\gamma}_i(\cdot)}\right\|_{p_i'(x)} &= \|\nabla z_1\|_{\gamma_i(x)p_i'(x)}^{\gamma_i(x_0^i)} + \|\nabla z_2\|_{\overline{\gamma}_i(x)p_i'(x)}^{\overline{\gamma}_i(x_1^i)}\\
&\le C(\|\nabla z_1\|_{p_1(x)}^{\gamma_i(x_0^i)} + \|\nabla z_2\|_{p_2(x)}^{\overline{\gamma}_i(x_1^i)}),
\end{aligned} \tag{3.5}$$

for certain $x_0^i, x_1^i \in \Omega$ and a constant $C > 0$. From 3.1 we have

$$\|z_1^{\alpha_i(\cdot)} z_2^{\beta_i(\cdot)}\|_{p_i'(x)} \leq \begin{cases} \|(\tilde{c}d(\cdot))^{\alpha_i(\cdot)+\beta_i(\cdot)}\|_{p_i'(x)} & \text{if } \alpha_i^+, \beta_i^+ < 0, \\ \|(\tilde{c}d(\cdot))^{\alpha_i(\cdot)} z_2^{\beta_i(\cdot)}\|_{p_i'(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^-, \\ \|(\tilde{c}d(\cdot))^{\beta_i(\cdot)} z_1^{\alpha_i(\cdot)}\|_{p_i'(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^-. \end{cases}$$

By $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$, Hölder's inequality gives

$$\|z_1^{\alpha_i(\cdot)} z_2^{\beta_i(\cdot)}\|_{p_i'(x)} \leq C_0 \begin{cases} \|(\tilde{c}d(\cdot))^{\alpha_i(\cdot)+\beta_i(\cdot)}\|_{p_i'(x)} & \text{if } \alpha_i^+, \beta_i^+ < 0, \\ \big\|(\tilde{c}d(\cdot))^{\alpha_i(\cdot)}\big\|_{\frac{N'p_i'(x)}{N'-p_i'(x)\beta_i(x)}} \|z_2^{\beta_i(\cdot)}\|_{\frac{N'}{\beta_i(x)}} & \text{if } \alpha_i^+ < 0 < \beta_i^-, \\ \big\|(\tilde{c}d(\cdot))^{\beta_i(\cdot)}\big\|_{\frac{N'p_i'(x)}{N'-p_i'(x)\alpha_i(x)}} \|z_1^{\alpha_i(\cdot)}\|_{\frac{N'}{\alpha_i(x)}} & \text{if } \beta_i^+ < 0 < \alpha_i^-. \end{cases} \tag{3.6}$$

Observe that

$$\begin{aligned} &\int_\Omega d(x)^{(\alpha_i(x)+\beta_i(x))p_i'(x)}\, dx \\ &= \int_{\{d\geq 1\}} d(x)^{(\alpha_i(x)+\beta_i(x))p_i'(x)}\, dx + \int_{\{d<1\}} d(x)^{(\alpha_i(x)+\beta_i(x))p_i'(x)}\, dx \\ &\leq |\Omega| + \int_{\{d<1\}} d(x)^{(\alpha_i^+ + \beta_i^+)(p_i')^+}\, dx. \end{aligned}$$

Then, owing to [20, Lemma on page 726], which is applicable since $(\alpha_i^+ + \beta_i^+)(p_i')^+ > -1$ (see $(\mathrm{H}_{\alpha,\beta,\gamma})$), we infer that

$$\int_\Omega d(x)^{(\alpha_i(x)+\beta_i(x))p_i'(x)}\, dx < +\infty. \tag{3.7}$$

Thence, from (2.1), we derive that

$$\|(\tilde{c}d(\cdot))^{\alpha_i(\cdot)+\beta_i(\cdot)}\|_{p_i'(x)} < +\infty. \tag{3.8}$$

On account of $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ the same conclusion can be drawn for the cases $\alpha_i^+ < 0 < \beta_i^-$ and $\beta_i^+ < 0 < \alpha_i^-$. Hence, a similar argument as above produces

$$\big\|(\tilde{c}d(\cdot))^{\alpha_i(\cdot)}\big\|_{\frac{N'p_i'(x)}{N'-p_i'(x)\beta_i(x)}}, \quad \big\|(\tilde{c}d(\cdot))^{\beta_i(\cdot)}\big\|_{\frac{N'p_i'(x)}{N'-p_i'(x)\alpha_i(x)}} < +\infty. \tag{3.9}$$

Reporting (3.8)–(3.9) in (3.6), by Lemma 2.1, there exist $x_2^i, x_3^i \in \Omega$ such that

$$\begin{aligned} &\|z_1^{\alpha_i(\cdot)} z_2^{\beta_i(\cdot)}\|_{p_i'(x)} \leq C_1(1 + \|z_1^{|\alpha_i(\cdot)|}\|_{\frac{N'}{|\alpha_i(x)|}} + \|z_2^{|\beta_i(\cdot)|}\|_{\frac{N'}{|\beta_i(x)|}}) \\ &= C_1(1 + \|z_1\|_{N'}^{|\alpha_i(x_2^i)|} + \|z_2\|_{N'}^{|\beta_i(x_3^i)|}), \end{aligned}$$

where $C_1 > 0$ is a constant. The Sobolev embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{N'}(\Omega)$ together with Hölder's inequality lead to

$$\|z_1^{\alpha_i(\cdot)} z_2^{\beta_i(\cdot)}\|_{p_i'(x)} \leq \tilde{C}_1(1 + \|\nabla z_1\|_{p_1(x)}^{|\alpha_i(x_2^i)|} + \|\nabla z_2\|_{p_2(x)}^{|\beta_i(x_3^i)|}), \tag{3.10}$$

for some constant $\tilde{C}_1 > 0$. Gathering (3.4), (3.5) and (3.10) together it follows that

$$\begin{aligned}&\|f_i\,(\cdot, z_1, z_2, \nabla z_1, \nabla z_2)\|_{p_i'(x)} \\ &\leq C_2(1 + \|\nabla z_1\|_{p_1(x)}^{|\alpha_i(x_2^i)|} + \|\nabla z_2\|_{p_2(x)}^{|\beta_i(x_3^i)|}\, \|\nabla z_1\|_{p_1(x)}^{\gamma_i\left(x_0^i\right)} + \|\nabla z_2\|_{p_2(x)}^{\overline{\gamma}_i\left(x_1^i\right)}),\end{aligned} \tag{3.11}$$

for certain constant $C_2 > 0$. Repeating the argument above by starting in (3.4) with N instead of $p_i'(\cdot)$ and by using $(\mathrm{H}_{\alpha,\beta,\gamma})$, we get

$$\begin{aligned}&\|f_i\,(\cdot, z_1, z_2, \nabla z_1, \nabla z_2)\|_N \\ &\leq \tilde{C}_2(1 + \|\nabla z_1\|_{p_1(x)}^{|\alpha_i(\hat{x}_2^i)|} + \|\nabla z_2\|_{p_2(x)}^{|\beta_i(\hat{x}_3^i)|} + \|\nabla z_1\|_{p_1(x)}^{\gamma_i(\hat{x}_0^i)} + \|\nabla z_2\|_{p_2(x)}^{\overline{\gamma}_i(\hat{x}_1^i)}) \\ &\leq \tilde{C}_2(1 + \|\nabla z_1\|_{p_1(x)}^{\max\{|\alpha_i(\hat{x}_2^i)|, \gamma_i(\hat{x}_0^i)\}} + \|\nabla z_2\|_{p_2(x)}^{\max\{|\beta_i(\hat{x}_3^i)|, \overline{\gamma}_i(\hat{x}_1^i)\}}),\end{aligned} \tag{3.12}$$

where $\tilde{C}_2 > 0$ is a constant. Hence, on the basis of (3.1), the claim follows.

Consequently, the unique solvability of $(\mathrm{P}_{(z_1,z_2)})$ comes directly from the Browder–Minty Theorem (see, e.g., [5]).

The task is now to show that the estimate (3.2) holds true. Thanks to Lemma 2.1, there exist $x_4^i \in \Omega$ such that

$$\|\nabla u_i\|_{p_i(x)}^{p_i\left(x_4^i\right)} = \int_\Omega |\nabla u_i|^{p_i(x)} dx. \tag{3.13}$$

Testing $(\mathrm{P}_{(z_1,z_2)})$ with (u_1, u_2), Hölder's inequality and the embedding

$$W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{N'}(\Omega)$$

entail

$$\begin{aligned}&\int_\Omega |\nabla u_i|^{p_i(x)} dx \leq \int_\Omega f_i\,(x, z_1, z_2, \nabla z_1, \nabla z_2)\, u_1 dx \\ &\leq C_0\, \|f_i\,(\cdot, z_1, z_2, \nabla z_1, \nabla z_2)\|_N\, \|\nabla u_i\|_{p_i(x)},\end{aligned} \tag{3.14}$$

for a constant $C_0 > 0$. Combining (3.13)–(3.14) with (3.12) and bearing in mind (3.1), one derives that

$$\begin{aligned}&\|\nabla u_i\|_{p_i(x)} \leq [C_0\, \|f_i\,(., z_1, z_2, \nabla z_1, \nabla z_2)\|_N]^{\frac{1}{p_i(x_4^i)-1}} \\ &\leq \tilde{C}_3(1 + \|\nabla z_1\|_{p_1(x)}^{\max\{|\alpha_i(\hat{x}_2^i)|, \gamma_i(\hat{x}_0^i)\}} + \|\nabla z_2\|_{p_2(x)}^{\max\{|\beta_i(\hat{x}_3^i)|, \overline{\gamma}_i(\hat{x}_1^i)\}})^{\frac{1}{p_i(x_4^i)-1}} \\ &\leq \tilde{C}_3(1 + \tilde{L}^{\max\{|\alpha_i(\hat{x}_2^i)|, \gamma_i(\hat{x}_0^i)\}} + \tilde{L}^{\max\{|\beta_i(\hat{x}_3^i)|, \overline{\gamma}_i(\hat{x}_1^i)\}})^{\frac{1}{p_i(x_4^i)-1}} \leq \tilde{L},\end{aligned}$$

provided that $\tilde{L} > 0$ is sufficiently large, where $\tilde{C}_3 > 0$ is a constant independent of z_i. This is possible because, according to $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$, one has

$$\max\{|\alpha_i(\hat{x}_2^i)|, |\beta_i(\hat{x}_3^i)|, \gamma_i(\hat{x}_0^i), \overline{\gamma}_i(\hat{x}_1^i)\} < p_i(x_4^i) - 1.$$

This completes the proof. □

Lemma 3.2. *Under assumptions* (H_f) *and* $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$, *for* (z_1, z_2) *satisfying* (3.1), *there exists a constant* $L > 1$ *independent of* z_i *such that every solution* (u_1, u_2) *of* $(\mathrm{P}_{(z_1,z_2)})$ *belongs to* $L^\infty(\Omega) \times L^\infty(\Omega)$ *and satisfies the estimate*

$$\|u_i\|_\infty < L. \tag{3.15}$$

Proof. It is a direct consequence of Lemma 2.3 where (3.12) as well as $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ and (3.1) are used. □

4. COMPARISON RESULTS

Let $\xi_i, \xi_{i,\delta} \in C^{1,\tau}(\overline{\Omega})$, $\tau \in (0,1)$, be the solutions of the Dirichlet problems

$$-\Delta_{p_i(x)}\xi_i(x) = 1 \text{ in } \Omega, \quad \xi_i(x) = 0 \text{ on } \partial\Omega \tag{4.1}$$

and

$$-\Delta_{p_i}\xi_{i,\delta}(x) = \begin{cases} 1 & \text{in } \Omega \backslash \overline{\Omega}_\delta \\ -1 & \text{in } \Omega_\delta \end{cases}, \quad \xi_{i,\delta}(x) = 0 \text{ on } \partial\Omega, \tag{4.2}$$

where

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\},$$

with a fixed $\delta > 0$ sufficiently small.

Lemma 4.1. *There are constants* $\tau > 0$ *and* $c_1, k_{p_1}, k_{p_2} > 1 > c_0$ *such that*

$$c_0 d(x) \le \xi_{i,\delta}(x) \le \xi_i(x) \le c_1 d(x) \quad \textit{for all } x \in \Omega \tag{4.3}$$

and

$$\|\xi_{i,\delta}\|_{C^{1,\tau}(\overline{\Omega})}, \|\xi_i\|_{C^{1,\tau}(\overline{\Omega})} \le k_{p_i},\ i = 1, 2. \tag{4.4}$$

Proof. From (4.1) and (4.2), it is readily seen that $\xi_{i,\delta}(x) \le \xi_i(x)$ for all $x \in \Omega$, for $i = 1, 2$. The Strong Maximum Principle together with [1, Lemma 3] entail $\xi_{i,\delta}(x) \ge c_0 d(x)$ in Ω, for $\delta > 0$ sufficiently small in (4.2) while, invoking Lemma 2.2, we infer that (4.4) holds true. Moreover, using (4.4), a similar argument to that in the proof of [9, Lemma 3.1] shows that the last inequality in (4.3) is verified. This ends the proof. □

For a constant $C > 1$ set

$$\underline{u}_i = C^{-1}\xi_{i,\delta} \quad \text{and} \quad \overline{u}_i = C\xi_i. \tag{4.5}$$

We claim that $\overline{u}_i \geq \underline{u}_i$ in $\overline{\Omega}$. Indeed, observe, from (4.1) and (4.2), that the integrals

$$\int\limits_{\Omega\setminus\overline{\Omega}_\delta} |\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i\ dx, \tag{4.6}$$

$$-\int\limits_{\Omega_\delta} |\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i\ dx, \tag{4.7}$$

$$\int\limits_{\Omega} |\nabla\xi_{i}|^{p_i(x)-2}\nabla\xi_{i}\nabla\varphi_i\ dx, \tag{4.8}$$

are positive for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$. This is crucial so that Theorem 6.1 in the Appendix is applicable. By (4.5) and thanks to Theorem 6.1, there exist $x_i^1, x_i^2 \in \Omega$ such that

$$\begin{aligned}
&\int\limits_{\Omega} |\nabla\underline{u}_i|^{p_i(x)-2}\nabla\underline{u}_i\nabla\varphi_i\ dx \\
&= \int\limits_{\Omega} C^{-(p_i(x)-1)}|\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i\ dx \\
&= \int\limits_{\Omega\setminus\overline{\Omega}_\delta} C^{-(p_i(x)-1)}|\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i\ dx \\
&\quad - \int\limits_{\Omega_\delta} (-C^{-(p_i(x)-1)}|\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i)\ dx \\
&= C^{-(p_i(x_i^1)-1)} \int\limits_{\Omega\setminus\overline{\Omega}_\delta} |\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i\ dx \\
&\quad - C^{-(p_i(x_i^2)-1)} \int\limits_{\Omega_\delta} (-|\nabla\xi_{i,\delta}|^{p_i(x)-2}\nabla\xi_{i,\delta}\nabla\varphi_i)\ dx.
\end{aligned} \tag{4.9}$$

Using (4.2) we obtain

$$\begin{aligned}
\int\limits_{\Omega} |\nabla\underline{u}_i|^{p_i(x)-2}\nabla\underline{u}_i\nabla\varphi_i\ dx &= C^{-(p_i(x_i^1)-1)} \int\limits_{\Omega\setminus\overline{\Omega}_\delta} \varphi_i\ dx - C^{-(p_i(x_i^2)-1)} \int\limits_{\Omega_\delta} \varphi_i\ dx \\
&\leq C^{-(p_i^- -1)} \int\limits_{\Omega\setminus\overline{\Omega}_\delta} \varphi_i\ dx - C^{-(p_i^+ -1)} \int\limits_{\Omega_\delta} \varphi_i\ dx.
\end{aligned} \tag{4.10}$$

Again, Theorem 6.1 and (4.1) imply

$$\begin{aligned}&\int_\Omega |\nabla \overline{u}_i|^{p_i(x)-2}\nabla \overline{u}_i \nabla \varphi_i \; dx = \int_\Omega C^{p_i(x)-1}|\nabla \xi_i|^{p_i(x)-2}\nabla \xi_i \nabla \varphi_i \; dx \\ &= C^{p_i(x_i^0)-1}\int_\Omega |\nabla \xi_i|^{p_i(x)-2}\nabla \xi_i \nabla \varphi_i \; dx \\ &\geq C^{p_i^- -1}\int_\Omega |\nabla \xi_i|^{p_i(x)-2}\nabla \xi_i \nabla \varphi_i \; dx = C^{p_i^- -1}\int_\Omega \varphi_i \; dx,\end{aligned} \tag{4.11}$$

for certain $x_i^0 \in \Omega$. Then, combining (4.10)–(4.11) together implies

$$\int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2}\nabla \underline{u}_i \nabla \varphi_i \; dx \leq \int_\Omega |\nabla \overline{u}_i|^{p_i(x)-2}\nabla \overline{u}_i \nabla \varphi_i \; dx,$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, provided that $C > 0$ is large enough. This proves the claim.

Set

$$R := \max_{i=1,2}\{1, k_{p_i}\}, \tag{4.12}$$

where k_{p_1} and k_{p_2} are given by (4.4). The following results allow us to achieve useful comparison properties.

Proposition 4.2. *Assume* $(\mathrm{H}_{\alpha,\beta,\gamma})$ *is fulfilled with* $\alpha_i^{\mp} + \beta_i^{\mp} > 0$ $(i = 1, 2)$. *Then, for* $C > 0$ *large enough in* (4.5), *it holds*

$$-\Delta_{p_i(x)}\underline{u}_i \leq m_i \begin{cases} \underline{u}_1^{\alpha_i(x)}\underline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^-, \beta_i^- > 0 \\ \overline{u}_1^{\alpha_i(x)}\underline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ \underline{u}_1^{\alpha_i(x)}\overline{u}_2^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^- \end{cases} \text{in } \Omega, \tag{4.13}$$

$$-\Delta_{p_i(x)}\overline{u}_i \geq 2M_i(RC)^{\max\{\gamma_i^+, \bar{\gamma}_i^+\}} + M_i \begin{cases} \overline{u}_1^{\alpha_i(x)}\overline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^-, \beta_i^- > 0 \\ \underline{u}_1^{\alpha_i(x)}\overline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ \overline{u}_1^{\alpha_i(x)}\underline{u}_2^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^- \end{cases} \text{in } \Omega, \tag{4.14}$$

where $R > 0$ *is provided in* (4.12), *for* $i = 1, 2$.

Proof. Assume $\alpha_i^-, \beta_i^- > 0$. From (4.5) and Lemma 4.1, we have

$$
\begin{aligned}
&m_i \int_\Omega \underline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} \varphi_i \,\mathrm{d}x = m_i \int_\Omega C^{-(\alpha_i(x)+\beta_i(x))} \xi_{1,\delta}^{\alpha_i(x)} \xi_{2,\delta}^{\beta_i(x)} \varphi_i \,\mathrm{d}x \\
&\geq m_i \int_\Omega (Cc_0^{-1})^{-(\alpha_i(x)+\beta_i(x))} d(x)^{\alpha_i(x)+\beta_i(x)} \varphi_i \,\mathrm{d}x \\
&\geq m_i (Cc_0^{-1})^{-(\alpha_i^+ + \beta_i^+)} \left(\delta^{\alpha_i^+ + \beta_i^+} \int_{\Omega \setminus \overline{\Omega}_\delta} \varphi_i \,\mathrm{d}x + \int_{\Omega_\delta} d(x)^{\alpha_i(x)+\beta_i(x)} \varphi_i \,\mathrm{d}x \right) \\
&\geq C^{-(p_i^- - 1)} \int_{\Omega \setminus \overline{\Omega}_\delta} \varphi_i \,\mathrm{d}x - C^{-(p_i^+ - 1)} \int_{\Omega_\delta} \varphi_i \,\mathrm{d}x,
\end{aligned}
\tag{4.15}
$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, $i = 1, 2$, and for $C > 0$ large enough. Thus, combining (4.10) together with (4.15), we infer that

$$
\int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2} \nabla \underline{u}_i \nabla \varphi_i \,\mathrm{d}x \leq m_i \int_\Omega \underline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} \varphi_i \,\mathrm{d}x,
$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, for $i = 1, 2$. This proves the first case in (4.13).

Next, we show (4.14) for $\alpha_i^-, \beta_i^- > 0$. Using (4.5), (4.12), (4.1) and (4.11), it follows that

$$
\begin{aligned}
&M_i \int_\Omega (\overline{u}_1^{\alpha_i(x)} \overline{u}_2^{\beta_i(x)} + 2(RC)^{\max\{\gamma_i^+, \bar{\gamma}_i^+\}}) \varphi_i \,\mathrm{d}x \\
&= M_i \int_\Omega (C^{\alpha_i(x)+\beta_i(x)} \xi_1^{\alpha_i(x)} \xi_2^{\beta_i(x)} + 2(RC)^{\max\{\gamma_i^+, \bar{\gamma}_i^+\}}) \varphi_i \,\mathrm{d}x \\
&\leq M_i \int_\Omega (C^{\alpha_i^+ + \beta_i^+} R^{\alpha_i^+ + \beta_i^+} + 2(RC)^{\max\{\gamma_i^+, \bar{\gamma}_i^+\}}) \varphi_i \,\mathrm{d}x \\
&\leq \tilde{M}_R \max\{C^{\alpha_i^+ + \beta_i^+}, C^{\max\{\gamma_i^+, \bar{\gamma}_i^+\}}\} \int_\Omega \varphi_i \,\mathrm{d}x \\
&\leq C^{p_i^- - 1} \int_\Omega \varphi_i \,\mathrm{d}x \leq \int_\Omega |\nabla \overline{u}_i|^{p_i(x)-2} \nabla \overline{u}_i \nabla \varphi_i \,\mathrm{d}x,
\end{aligned}
$$

for $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, $i = 1, 2$, and for $C > 0$ large enough.

Now, we deal with the other cases in (4.13) and (4.14) with respect to the sign of the exponents. We only prove the inequalities corresponding to the case $\alpha_i^+ < 0 < \beta_i^-$

because the complementary situation $\beta_i^+ < 0 < \alpha_i^-$ is carried out in a similar way. So assume $\alpha_i^+ < 0 < \beta_i^-$. On account of Lemma 4.1 and $(\mathrm{H}_{\alpha,\beta,\gamma})$ one has

$$\begin{aligned}
&\int_\Omega C^{\alpha_i(x)-\beta_i(x)}\xi_1^{\alpha_i(x)}\xi_{2,\delta}^{\beta_i(x)}\varphi_i \,\mathrm{d}x \\
&\geq C^{\alpha_i^- -\beta_i^+}R^{\alpha_i^-}(c_0\delta)^{\beta_i^+}\int_{\Omega\setminus\overline{\Omega}_\delta}\varphi_i \,\mathrm{d}x + C^{\alpha_i^- -\beta_i^+}c_0^{\beta_i^+}c_1^{\alpha_i^-}\int_{\Omega_\delta} d(x)^{\alpha_i(x)+\beta_i(x)}\varphi_i \,\mathrm{d}x \\
&\geq C^{-(p_i^- -1)}\int_{\Omega\setminus\overline{\Omega}_\delta}\varphi_i \,\mathrm{d}x - C^{-(p_i^+ -1)}\int_{\Omega_\delta}\varphi_i \,\mathrm{d}x,
\end{aligned}$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, provided that $C > 0$ is large enough. Then, on the basis of 4.5, (4.1), (4.2) and (4.10), one gets

$$m_i\int_\Omega \overline{u}_1^{\alpha_i(x)}\underline{u}_2^{\beta_i(x)}\varphi_i \,\mathrm{d}x \geq \int_\Omega |\nabla\underline{u}_i|^{p_i(x)-2}\nabla\underline{u}_i\nabla\varphi_i \,\mathrm{d}x.$$

Next, we show (4.14) when $\alpha_i^+ < 0 < \beta_i^-$. By (4.12), $(\mathrm{H}_{\alpha,\beta})$, (3.15), (4.3) and Lemma 4.1, it follows that

$$\begin{aligned}
&M_i\int_\Omega (\underline{u}_1^{\alpha_i(x)}\overline{u}_2^{\beta_i(x)} + 2(RC)^{\max\{\gamma_i^+,\bar{\gamma}_i^+\}})\varphi_i \,\mathrm{d}x \\
&= M_i\int_\Omega (C^{-\alpha_i(x)+\beta_i(x)}\xi_{1,\delta}^{\alpha_i(x)}\xi_2^{\beta_i(x)} + 2(RC)^{\max\{\gamma_i^+,\bar{\gamma}_i^+\}})\varphi_i \,\mathrm{d}x \\
&\leq M_i C^{-\alpha_i^- +\beta_i^+}\left[(c_0\delta)^{\alpha_i^-}R^{\beta_i^+}\int_{\Omega\setminus\overline{\Omega}_\delta}\varphi_i \,\mathrm{d}x + c_1^{\beta_i^+}c_0^{\alpha_i^-}\int_{\Omega_\delta} d(x)^{\alpha_i^- +\beta_i^+}\varphi_i \,\mathrm{d}x\right] \\
&\quad + 2(RC)^{\max\{\gamma_i^+,\bar{\gamma}_i^+\}})\int_\Omega \varphi_i \,\mathrm{d}x \leq C^{p_i^- -1}\int_\Omega \varphi_i \,\mathrm{d}x,
\end{aligned} \tag{4.16}$$

for $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, provided that $C > 0$ is large enough. Thus, gathering (4.11)–(4.16) together yields

$$\int_\Omega |\nabla\overline{u}_i|^{p_i(x)-2}\nabla\overline{u}_i\nabla\varphi_i \,\mathrm{d}x \geq M_1\int_\Omega (\underline{u}_1^{\alpha_i(x)}\overline{u}_2^{\beta_i(x)} + 2(RC)^{\max\{\gamma_i^+,\bar{\gamma}_i^+\}})\varphi_i \mathrm{d}x. \qquad \square$$

Proposition 4.3. *Assume $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ is fulfilled. Then, for $C > 0$ large enough in* (4.12)*, it holds*

$$-\Delta_{p_i(x)}\underline{u}_i \leq m_i\begin{cases} L^{\alpha_i^-}\,\underline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ L^{\beta_i^-}\,\underline{u}_1^{\alpha_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^- \\ L^{\alpha_i^- +\beta_i^-} & \text{if } \alpha_i^+,\beta_i^+ < 0 \end{cases} \text{ in } \Omega, \text{ for } i = 1,2,$$

where the constant $L > 1$ is provided by Lemma 3.2.

Proof. Assume $\alpha_i^+ < 0 < \beta_i^-$. The case $\beta_i^+ < 0 < \alpha_i^-$ can be handled in much the same way. By (4.5), (4.3) and (4.10), one has

$$\begin{aligned}
m_i \int_\Omega L^{\alpha_i^-} \underline{u}_2^{\beta_i(x)} \varphi_i dx &= m_i L^{\alpha_i^-} \int_\Omega (C^{-1}\xi_{2,\delta})^{\beta_i(x)} \varphi_i dx \\
&\geq m_i L^{\alpha_i^-} C^{-\beta_i^+} \int_\Omega (c_0 d(x))^{\beta_i(x)} \varphi_i dx \\
&\geq m_i L^{\alpha_i^-} C^{-\beta_i^+} ((c_0\delta)^{\beta_i^+} \int_{\Omega\setminus\overline{\Omega}_\delta} \varphi_i dx + \int_{\Omega_\delta} (c_0 d(x))^{\beta_i(x)} \varphi_i dx) \\
&\geq \int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2} \nabla \underline{u}_i \nabla \varphi_i dx,
\end{aligned}$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, and for $C > 0$ large enough. If $\alpha_i^+, \beta_i^+ < 0$, from (4.10), it follows that

$$\begin{aligned}
m_i \int_\Omega L^{\alpha_i^- + \beta_i^-} \varphi_i dx &= m_i L^{\alpha_i^- + \beta_i^-} \left(\int_{\Omega\setminus\overline{\Omega}_\delta} \varphi_i dx + \int_{\Omega_\delta} \varphi_i dx \right) \\
&\geq \int_\Omega |\nabla \underline{u}_i|^{p_i(x)-2} \nabla \underline{u}_i \nabla \phi_i dx,
\end{aligned}$$

for all $\varphi_i \in W_0^{1,p_i(x)}(\Omega)$ with $\varphi_i \geq 0$, provided $C > 0$ is sufficiently large. This ends the proof. □

5. PROOF OF THE MAIN RESULT

5.1. CASE $\alpha_i^\mp + \beta_i^\mp > 0$

Using the functions in (4.5) as well as the constant $R > 0$ in (4.12), we introduce the closed, bounded and convex set

$$\mathcal{K}_C = \left\{ (y_1, y_2) \in C_0^1(\overline{\Omega})^2 : \underline{u}_i \leq y_i \leq \overline{u}_i \text{ in } \Omega \text{ and } \|\nabla y_i\|_\infty \leq CR \right\}.$$

Define the map

$$\mathcal{T} : \mathcal{K}_C \to C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}), \quad (z_1, z_2) \mapsto \mathcal{T}(z_1, z_2) = (u_1, u_2)_{(z_1, z_2)},$$

where (u_1, u_2) is required to satisfy $(\mathrm{P}_{(z_1,z_2)})$. It is worth noting that solutions of problem $(\mathrm{P}_{(z_1,z_2)})$ coincide with the fixed points of the operator $\mathcal{T}$. To reach the desired conclusion, we shall apply Schauder's fixed point theorem.

For $(z_1, z_2) \in \mathcal{K}_C$ we have

$$z_1^{\alpha_i(x)} z_2^{\beta_i(x)} \leq \begin{cases} \overline{u}_1^{\alpha_i(x)} \overline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^-, \beta_i^- > 0, \\ \underline{u}_1^{\alpha_i(x)} \overline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^-, \\ \overline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^-. \end{cases}$$

In Ω, using (4.5) together with Lemma 4.1, we obtain

$$\begin{aligned} z_1^{\alpha_i(x)} z_2^{\beta_i(x)} &\leq \begin{cases} C^{\alpha_i^+ + \beta_i^+} d(x)^{\alpha_i(x)+\beta_i(x)} & \text{if } \alpha_i^-, \beta_i^- > 0 \\ C^{-\alpha_i^- + \beta_i^+} d(x)^{\alpha_i(x)+\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ C^{\alpha_i^+ - \beta_i^-} d(x)^{\alpha_i(x)+\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^- \end{cases} \\ &\leq \begin{cases} C^{\alpha_i^+ + \beta_i^+} \operatorname{diam}(\Omega)^{\alpha_i(x)+\beta_i(x)} & \text{if } \alpha_i^-, \beta_i^- > 0 \\ C^{-\alpha_i^- + \beta_i^+} \operatorname{diam}(\Omega)^{\alpha_i(x)+\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ C^{\alpha_i^+ - \beta_i^-} \operatorname{diam}(\Omega)^{\alpha_i(x)+\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^-. \end{cases} \end{aligned}$$

Thus, we derive from (H_f), (4.5) and Lemma 4.1 the estimate

$$\begin{aligned} &|f_i(x, z_1, z_2, \nabla z_1, \nabla z_2)| \leq M_i(z_1^{\alpha_i(x)} z_2^{\beta_i(x)} + |\nabla z_1|^{\gamma_i(x)} + |\nabla z_2|^{\bar{\gamma}_i(x)}) \\ &\leq M_i C^{|\alpha_i^\pm| + |\beta_i^\pm|} L_0 + 2M_i (CR)^{\max\{\gamma_i^+, \bar{\gamma}_i^+\}} \text{ in } \Omega, \end{aligned} \tag{5.1}$$

where constant $L_0 > 0$ is independent of C.

Consequently, the unique solvability of (u_1, u_2) in ($\mathrm{P}_{(z_1,z_2)}$), which is readily derived from Minty Browder's Theorem (see, e.g., [5]), guarantees that $\mathcal{T}$ is well defined. Moreover, the regularity theory up to the boundary in [12] yields $(u_1, u_2) \in C_0^{1,\tau}(\overline{\Omega})^2$ for certain $\tau \in (0, 1)$ and a constant $\hat{R} > 0$ such that it holds

$$\|u_i\|_{C^{1,\tau}(\overline{\Omega})} < \hat{R}. \tag{5.2}$$

Proposition 5.1. *$\mathcal{K}_C$ is invariant by the operator $\mathcal{T}$.*

Proof. Using the fact that $z_1, z_2 \in \mathcal{K}_C$, it follows that

$$z_1^{\alpha_i(x)} z_2^{\beta_i(x)} \geq \begin{cases} \underline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^-, \beta_i^- > 0 \\ \overline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ \underline{u}_1^{\alpha_i(x)} \overline{u}_2^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^- \end{cases} \text{ in } \Omega.$$

Then, bearing in mind (H_f) and Proposition 4.2, the weak comparison principle entails

$$\underline{u}_1 \leq y_1 \leq \overline{u}_1 \text{ and } \underline{u}_2 \leq u_2 \leq \overline{u}_2 \text{ in } \Omega. \tag{5.3}$$

On the other hand, since $\max\{|\alpha_i^\pm| + |\beta_i^\pm|, \gamma_i^+, \bar{\gamma}_i^+\} < p_i^- - 1$, it follows from (5.1) that

$$|f_i(x, z_1, z_2, \nabla z_1, \nabla z_2)| \leq (CR)^{p_i^- - 1},$$

provided that C is sufficiently large. Hence, thanks to Lemma 2.2, we infer that

$$\|\nabla u_1\|_\infty, \|\nabla u_2\|_\infty \leq CR. \tag{5.4}$$

Consequently, gathering (5.2)–(5.4) together yields $(u_1, u_2) \in \mathcal{K}_C$, showing that $\mathcal{T}(\mathcal{K}_C) \subset \mathcal{K}_C$. $\square$

Proposition 5.2. *$\mathcal{T}$ is compact and continuous.*

Proof. On the basis of (5.2) and the compactness of the embedding $C^{1,\tau}(\overline{\Omega}) \subset C_0^1(\overline{\Omega})$ we infer that $\mathcal{T}(C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}))$ is a relatively compact subset of $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. This shows the compactness of the operator $\mathcal{T}$.

Next, we prove that $\mathcal{T}$ is continuous with respect to the topology of $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. Let $(z_{1,n}, z_{2,n}) \to (z_1, z_2)$ in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ for all n. Denoting $(u_{1,n}, u_{2,n}) = \mathcal{T}(z_{1,n}, z_{2,n})$, we have from (5.2) that $(u_{1,n}, u_{2,n}) \in C^{1,\tau}(\overline{\Omega}) \times C^{1,\tau}(\overline{\Omega})$. By the Ascoli–Arzelà Theorem, there holds

$$(u_{1,n}, u_{2,n}) \to (u_1, u_2) \text{ in } C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}).$$

On the other hand, for $z_1, z_2 \in \mathcal{K}_C$ one has

$$f_i(x, z_{1,n}, z_{2,n}, \nabla z_{1,n}, \nabla z_{2,n}) \to f_i(x, z_1, z_2, \nabla z_1, \nabla z_2) \in W^{-1,p_i'(x)}(\Omega).$$

Thus, we conclude that $\mathcal{T}$ is continuous. $\square$

5.2. CASE $\alpha_i^{\mp} + \beta_i^{\mp} \leq 0$

Using the functions $\underline{u}_i$ $(i = 1, 2)$ in (4.5), the $W^{1,p_i(x)}$- gradient estimate $\tilde{L}$ in Lemma 3.1 as well as the L^∞-bound L in Lemma 3.2, we introduce the set

$$\tilde{\mathcal{K}}_{\tilde{L}} = \left\{ (y_1, y_2) \in \prod_{i=1,2} W_0^{1,p_i(x)}(\Omega) : \underline{u}_i \leq y_i \leq L \text{ in } \Omega \text{ and } \|\nabla y_i\|_{p_i(x)} \leq \tilde{L} \right\},$$

which is closed, bounded and convex in $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$. Define the operator

$$\tilde{\mathcal{T}} : \tilde{\mathcal{K}}_{\tilde{L}} \to W_0^{1,p_1(x)}(\overline{\Omega}) \times W_0^{1,p_2(x)}(\overline{\Omega}), \quad (z_1, z_2) \mapsto \tilde{\mathcal{T}}(z_1, z_2) = (u_1, u_2),$$

where (u_1, u_2) is required to satisfy $(\mathrm{P}_{(z_1,z_2)})$. On account of (4.3), (4.5) and Lemma 3.1, we deduce that (u_1, u_2) is the unique solution of problem $(\mathrm{P}_{(z_1,z_2)})$. Then, the map $\tilde{\mathcal{T}}$ is well defined.

Proposition 5.3. *The set $\tilde{\mathcal{K}}_{\tilde{L}}$ is invariant by the operator $\tilde{\mathcal{T}}$.*

Proof. For any $(z_1, z_2) \in \tilde{\mathcal{K}}_{\tilde{L}}$, combining (H_f) with Proposition 4.3 we derive that $u_i \geq \underline{u}_i$ in Ω $(i = 1, 2)$. Moreover, Lemma 3.2 implies that $u_i \leq L$ while Lemma 3.1 ensures that there exists a large constant $\tilde{L} > 0$ such that $\|\nabla y_i\|_{p_i(x)} \leq \tilde{L}$. Hence, $u_i \in \tilde{\mathcal{K}}_{\tilde{L}}$ establishing that $\tilde{\mathcal{T}}(\tilde{\mathcal{K}}_{\tilde{L}}) \subset \tilde{\mathcal{K}}_{\tilde{L}}$. $\square$

Proposition 5.4. *The map $\tilde{\mathcal{T}}$ is compact and continuous.*

Proof. Let $(z_{1,n}, z_{2,n}) \to (z_1, z_2)$ in $W_0^{1,p_1(x)}(\Omega) \times W_0^{1,p_2(x)}(\Omega)$, that is,

$$(z_{1,n}, z_{2,n}) \to (z_1, z_2) \text{ in } L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega)$$

and

$$(\nabla z_{1,n}, \nabla z_{2,n}) \to (\nabla z_1, \nabla z_2) \text{ in } \left(L^{p_1(x)}(\Omega)\right)^N \times \left(L^{p_2(x)}(\Omega)\right)^N,$$

with $(z_{1,n}, z_{2,n}) \in \tilde{\mathcal{K}}_{\tilde{L}}$. According to [14, Theorem 2.4], the whole sequences $(z_{1,n}, z_{2,n})$ and $(\nabla z_{1,n}, \nabla z_{2,n})$ converge in measure to (z_1, z_2) and $(\nabla z_1, \nabla z_2)$, respectively. Consequently, given that the function f_i is of Carathéodory type, one can write

$$\begin{aligned} &f_i\left(x, z_{1,n}(x), z_{2,n}(x), \nabla z_{1,n}(x), \nabla z_{2,n}(x)\right) \\ &\qquad \longrightarrow f_i\left(x, z_1(x), z_2(x), \nabla z_1(x), \nabla z_2(x)\right) \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Again, [14, Theorem 2.4] ensures that $(\nabla z_{1,n}, \nabla z_{2,n})$ converges to $(\nabla z_1, \nabla z_2)$ in modular, that is

$$\lim_{n\to\infty} \rho_{p_i(x)}\left(\nabla z_{i,n} - \nabla z_i\right) = 0,$$

or equivalently

$$\left(|\nabla z_{1,n} - \nabla z_1|^{p_1(x)}, |\nabla z_{2,n} - \nabla z_2|^{p_2(x)}\right) \to 0 \ \text{ in } L^1(\Omega) \times L^1(\Omega).$$

Then, there exists a subsequence $(|\nabla z_{1,n_k} - \nabla z_1|^{p_1(x)}, |\nabla z_{2,n_k} - \nabla z_1|^{p_2(x)})$ and positive measurable functions $(g_1, g_2) \in L^1(\Omega) \times L^1(\Omega)$ such that

$$\begin{cases} |\nabla z_{1,n_k}(x) - \nabla z_1(x)|^{p_1(x)} \le g_1(x) \\ |\nabla z_{2,n_k}(x) - \nabla z_2(x)|^{p_2(x)} \le g_2(x) \end{cases} \quad \text{for a.e. } x \in \Omega. \tag{5.5}$$

Here, it is worth noting that $z_i \in [\underline{u}_i, L]$ since $z_{i,n} \in [\underline{u}_i, L]$. Moreover, from Lemma 4.1 and (4.5) it holds

$$\begin{aligned} z_{1,n}^{\alpha_i(x)} z_{2,n}^{\beta_i(x)} &\le \begin{cases} \underline{u}_1^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} & \text{if } \alpha_i^+, \beta_i^+ < 0 \\ \underline{u}_1^{\alpha_i(x)} L^{\beta_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ L^{\alpha_i(x)} \underline{u}_2^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^-. \end{cases} \\ &\le M_0 \begin{cases} d(x)^{\alpha_i(x)+\beta_i(x)} & \text{if } \alpha_i^+, \beta_i^+ < 0 \\ d(x)^{\alpha_i(x)} & \text{if } \alpha_i^+ < 0 < \beta_i^- \\ d(x)^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^-, \end{cases} \end{aligned} \tag{5.6}$$

for a certain positive constant $M_0 := M_0(L, C, c_0, \alpha_i, \beta_i)$.

Assume that $\alpha_i^\pm + \beta_i^\pm < 0$. On account of (H_f) and (5.6), we deduce from (5.5) that

$$\begin{aligned} &|f_i\left(x, z_{1,n}, z_{2,n}, \nabla z_{1,n}, \nabla z_{2,n}\right)| \le M_i(z_{1,n}^{\alpha_i(x)} z_{2,n}^{\beta_i(x)} + |\nabla z_{1,n}|^{\gamma_i(x)} + |\nabla z_{2,n}|^{\overline{\gamma}_i(x)}) \\ &\le M_i\Big(K(x) + \Big\{g_1(x)^{\frac{1}{p_1(x)}} + |\nabla z_1|\Big\}^{\gamma_i(x)} + \Big\{g_2(x)^{\frac{1}{p_2(x)}} + |\nabla z_2|\Big\}^{\overline{\gamma}_i(x)}\Big). \end{aligned}$$

where

$$K(x) = \begin{cases} (\tilde{c}d(x))^{\alpha_i(x)+\beta_i(x)} & \text{if } \alpha_i^+, \beta_i^+ < 0, \\ (\tilde{c}d(x))^{\alpha_i(x)} \left\| z_{2,n}^{\beta_i(x)} \right\|_\infty & \text{if } \alpha_i^+ < 0 < \beta_i^-, \\ \left\| z_{1,n}^{\alpha_i(x)} \right\|_\infty (\tilde{c}d(x))^{\beta_i(x)} & \text{if } \beta_i^+ < 0 < \alpha_i^-. \end{cases}$$

Setting

$$G_i(x) = M_i\Big(K(x) + \Big\{g_1(x)^{\frac{1}{p_1(x)}} + |\nabla z_1|\Big\}^{\gamma_i(x)} + \Big\{g_2(x)^{\frac{1}{p_2(x)}} + |\nabla z_2|\Big\}^{\overline{\gamma}_i(x)}\Big),$$

we claim that $G_i \in L^{p_i'(x)}(\Omega)$. Indeed, since $g_1, g_2 \in L^1(\Omega)$, it is readily seen that

$$\Big\{g_1(x)^{\frac{1}{p_1(x)}} + |\nabla z_1|\Big\}^{\gamma_i(x)} \in L^{\frac{p_1(x)}{\gamma_i(x)}}(\Omega) \tag{5.7}$$

and

$$\Big\{g_2(x)^{\frac{1}{p_2(x)}} + |\nabla z_2|\Big\}^{\overline{\gamma}_i(x)} \in L^{\frac{p_2(x)}{\overline{\gamma}_i(x)}}(\Omega). \tag{5.8}$$

Assumption $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ ensures that the embeddings

$$L^{\frac{p_1(x)}{\gamma_i(x)}}(\Omega) \hookrightarrow L^{p_i'(x)}(\Omega) \quad \text{and} \quad L^{\frac{p_2(x)}{\overline{\gamma}_i(x)}}(\Omega) \hookrightarrow L^{p_i'(x)}(\Omega)$$

hold true, while $(\tilde{\mathrm{H}}_{\alpha,\beta,\gamma})$ together with an argument similar to (3.7) guarantee that

$$\int_\Omega K(x)^{p_i'(x)} dx < \infty. \tag{5.9}$$

Then, gathering (5.7)–(5.9) together we conclude that $G_i(x) \in L^{p_i'(x)}(\Omega)$, showing the claim.

The generalized Lebesgue's Dominated Convergence Theorem (see [10, Lemma 3.2.8]) implies that

$$f_i\left(x, z_{1,n_k}, z_{2,n_k}, \nabla z_{1,n_k}, \nabla z_{2,n_k}\right) \to f_i\left(x, z_1, z_2, \nabla z_1, \nabla z_2\right) \quad \text{in } L^{p_i'(x)}(\Omega).$$

The convergence principle implies that the entire sequence $(f_i\left(x, z_{1,n}, z_{2,n}, \nabla z_{1,n}, \nabla z_{2,n}\right))$ converges to $f_i\left(x, z_1, z_2, \nabla z_1, \nabla z_2\right)$ in $L^{p_i'(x)}(\Omega) \hookrightarrow W^{-1,p_i'(x)}(\Omega)$, showing the continuity of $\tilde{\mathcal{T}}$.

Furthermore, it is worth noting that the operator $\tilde{\mathcal{T}}$ can be written as

$$\tilde{\mathcal{T}} \coloneqq \left(\mathcal{L}_1^{-1} \circ \Phi_1,\ \mathcal{L}_2^{-1} \circ \Phi_2\right),$$

where $\mathcal{L}_i = -\Delta_{p_i(x)}$ and $\Phi_i(z_1, z_2) = f_i\left(x, z_1, z_2, \nabla z_1, \nabla z_2\right)$ for all $(z_1, z_2) \in \tilde{\mathcal{K}}_{\tilde{L}}$. Thus, the compactness of the embedding $L^{p_i'(x)}(\Omega) \hookrightarrow W^{-1,p_i'(x)}(\Omega)$ implies that $\Phi_i\left(\mathcal{K}_{\tilde{R}}\right)$ is a relatively compact subset of $W^{-1,p_i'(x)}(\Omega)$, hence the compactness of Φ_i. Therefore, the boundedness of $\mathcal{L}_i^{-1}$ (see [19, Theorem 3.2]), leads to the compactness of $\tilde{\mathcal{T}}$. The proof is completed. □

5.3. PROOF OF THEOREM 1.1

By virtue of Propositions 5.1 and 5.2 (resp. Propositions 5.3 and 5.4), we are in a position to apply Schauder's fixed point theorem (see, e.g., [24]) to the set $\mathcal{K}_C$

(resp. $\tilde{\mathcal{K}}_{\tilde{L}}$) and the map $\mathcal{T} : \mathcal{K}_C \to \mathcal{K}_C$ (resp. $\tilde{\mathcal{T}} : \tilde{\mathcal{K}}_{\tilde{L}} \to \tilde{\mathcal{K}}_{\tilde{L}}$). This ensures the existence of $(u_1, u_2) \in \mathcal{K}_C$ (resp. $(u_1, u_2) \in \tilde{\mathcal{K}}_{\tilde{L}}$ satisfying $(u_1, u_2) = \mathcal{T}(u_1, u_2)$ (resp. $(u_1, u_2) = \tilde{\mathcal{T}}(u_1, u_2)$). Taking into account the definition of $\mathcal{T}$ (resp. $\tilde{\mathcal{T}}$), it turns out that $(u_1, u_2) \in C^{1,\tau}(\overline{\Omega}) \times C^{1,\tau}(\overline{\Omega})$ for certain $\tau \in (0, 1)$ (resp. $(u_1, u_2) \in (W_0^{1,p_1(x)}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,p_2(x)}(\Omega) \cap L^\infty(\Omega)))$ is a (positive) solution of problem (P). Moreover, because the solution (u_1, u_2) lies in $\mathcal{K}_C$ (resp. $\tilde{\mathcal{K}}_{\tilde{L}}$), Lemma 4.1 implies that (1.3) is fulfilled. This completes the proof.

6. APPENDIX

Denoting B_1 the unit ball and S_1 the unit sphere of $\mathbb{R}^N$, let $\Psi_\epsilon, \Pi_\epsilon : \mathbb{R}^N \to \mathbb{R}$ be sequences of mollifiers defined for $\epsilon > 0$ by

$$\Psi_\epsilon(x) = \frac{1}{\epsilon^N} \Psi\left(\frac{x}{\epsilon}\right) \quad \text{and} \quad \Pi_\epsilon(x) = \frac{1}{\epsilon^N} \Pi\left(\frac{x}{\epsilon}\right), \tag{6.1}$$

with $\Psi : B_1 \to \mathbb{R}$ a bump function satisfying

$$\Psi(x) = C^{-1} e^{-\frac{1}{1-|x|^2}}, \quad C = \int\limits_{B_1} e^{-\frac{1}{1-|x|^2}} dx, \quad \text{and} \quad \Pi = \overline{C}^{-1} \pi,$$

where $\pi \in C_0^1(B_1)$ is the solution of the problem

$$\begin{cases} -\Delta\pi = 1 & \text{in } B_1, \\ \pi = 0 & \text{on } S_1, \end{cases}$$

and $\overline{C} = \int_{B_1} \pi(x)dx$. Both the functions Ψ and Π satisfy

$$\int\limits_{\mathbb{R}^N} \Psi(x)dx = \int\limits_{\mathbb{R}^N} \Pi(x)dx = 1. \tag{6.2}$$

The Mean Value Theorem is stated as follows.

Theorem 6.1. *Let $u \in W_0^{1,p(x)}(\Omega)$ be the solution of a nonlinear elliptic equation of the form*

$$-\Delta_{p(x)} u = h(x) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega, \tag{6.3}$$

where h is a sign-constant function. Let $f : \Omega \to \mathbb{R}$ be a Lipschitz continuous function satisfying $-\infty < m \le f(x) \le M < \infty$ for some constants m, M. Then, for any sign-constant function $\phi \in W_0^{1,p(x)}(\Omega)$, there exists a real $\gamma \in [m, M]$, depending on ϕ, such that

$$\int\limits_\Omega f(x)|\nabla u|^{p(x)-2} \nabla u \nabla\phi dx = \gamma \int\limits_\Omega h(x)\phi dx. \tag{6.4}$$

Remark 6.2. Actually, equality (6.4) shall be proved for any nonnegative function $\phi \in C_0^\infty(\Omega)$, and the generalization to $W_0^{1,p(x)}(\Omega)$ is deduced by a density argument. Also, without loss of generality, we will assume that h is nonnegative, because otherwise, instead of (6.3), we consider the problem

$$-\Delta_{p(x)} u = -h(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

whose solution is $\hat{u} = -u$. Similarly, we will assume that ϕ is nonnegative.

Proof of Theorem 6.1. Inspired by [4], the proof of (6.4) is divided in four steps.

Step 1. For any functions $f, g : \mathbb{R}^N \to \mathbb{R}$, denote $\star$ the classical convolution product defined by

$$f \star g(x) = \int_{\mathbb{R}^N} f(x-z)g(z)dz, \text{ for any } x \in \Omega,$$

and set

$$\begin{aligned}
&\phi_\epsilon = \phi \star \Pi_\epsilon, \\
&\Omega(y) = \{x \in \Omega \,|\, f(x) \le y\}, \quad y \in [m, M], \\
&\Omega_\epsilon = \{x \in \Omega \,|\, d(x, \partial\Omega) \ge \epsilon\}, \\
&a_{\epsilon,\Omega(y)} = \mathbb{1}_{\Omega(y)\cap\Omega_\epsilon} \star \Psi_\epsilon.
\end{aligned} \tag{6.5}$$

Let $\breve{F}_\epsilon, F_\epsilon : [m, M] \to \mathbb{R}$ ($\epsilon > 0$) be the functionals defined for any nonnegative function $\phi \in C_0^\infty(\Omega)$ by

$$\breve{F}_\epsilon(y) = \int_\Omega a_{\epsilon,\Omega(y)}(x)\, f(x) |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx,$$

$$F_\epsilon(y) = \int_\Omega a_{\epsilon,\Omega(y)}(x)\, |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx,$$

where u refers to the solution of problem (6.3). We claim that

$$\lim_{\epsilon\to 0} \left| \int_m^M d\breve{F}_\epsilon(y) - \int_m^M y dF_\epsilon(y) \right| = 0. \tag{6.6}$$

Indeed, let $P_n = \{m = y_0 < \ldots < y_n = M\}$ be a partition of the interval $[m, M]$ such that $\mathcal{N} = y_k - y_{k-1} = \delta < \eta$, and let $y'_k \in [y_{k-1}, y_k]$. We check that

$$\begin{aligned}
&\left| \sum_{k=1}^n \left[\breve{F}_\epsilon(y_k) - \breve{F}_\epsilon(y_{k-1})\right] - \sum_{k=1}^n y'_k \left[F_\epsilon(y_k) - F_\epsilon(y_{k-1})\right] \right| \\
&= \left| \sum_{k=1}^n \int_\Omega (f(x) - y'_k) a_{\epsilon,\Omega_k}(x) |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx \right|.
\end{aligned} \tag{6.7}$$

with

$$\Omega_k = \{x \in \Omega \,|\, y_{k-1} < f(x) \leq y_k\} = \Omega(y_k) \backslash \Omega(y_{k-1})$$

and

$$a_{\epsilon,\Omega_k} = a_{\epsilon,\Omega(y_k)} - a_{\epsilon,\Omega(y_{k-1})}.$$

For any $x \in \Omega_k \cap \Omega_\epsilon + B_\epsilon(0)$, there exists $(y, s) \in \Omega_k \cap \Omega_\epsilon \times B_\epsilon(0)$, such that $x = y + s$. From the Mean Value Theorem, and the C-Lipschitz regularity of f, we infer that

$$|f(x) - y'_k| \leq |f(y+s) - f(y)| + |f(y) - y'_k| \leq C|s| + |y_k - y_{k-1}| \leq C\epsilon + \eta.$$

Since $a_{\epsilon,\Omega(y_0)} = a_{\epsilon,\Omega(m)} = 0$, and $a_{\epsilon,\Omega(y_n)} = a_{\epsilon,\Omega(M)} = a_{\epsilon,\Omega}$, it holds

$$\sum_{k=1}^{n} a_{\epsilon,\Omega_k}(x) = a_{\epsilon,\Omega}(x). \tag{6.8}$$

Gathering (6.7)–(6.8) together leads to

$$\begin{aligned} &\left| \sum_{k=1}^{n} \left[\breve{F}_\epsilon(y_k) - \breve{F}_\epsilon(y_{k-1}) \right] - \sum_{k=1}^{n} y'_k \left[F_\epsilon(y_k) - F_\epsilon(y_{k-1}) \right] \right| \\ &\leq (C\epsilon + \eta) \int\limits_{\Omega} a_{\epsilon,\Omega} |\nabla u|^{p(x)-1} \, |\nabla \phi_\epsilon| \, dx. \end{aligned} \tag{6.9}$$

Observe that

$$\lim_{\epsilon \to 0} a_{\epsilon,\Omega(y)} = \mathbb{1}_{\Omega(y)} \, (y \in [m, M]) \quad \text{and} \quad \lim_{\epsilon \to 0} \nabla \phi_\epsilon = \nabla \phi \quad \text{a.e. in } \Omega, \tag{6.10}$$

see for example [11, Theorem 7, page 714]. Thus, (6.9)–(6.10) together imply

$$\lim_{\epsilon \to 0} \lim_{\eta \to 0} \left| \sum_{k=1}^{n} \left[\breve{F}_\epsilon(y_k) - \breve{F}_\epsilon(y_{k-1}) \right] - \sum_{k=1}^{n} y'_k \left[F_\epsilon(y_k) - F_\epsilon(y_{k-1}) \right] \right| = 0.$$

This achieves the proof of (6.6).

Step 2. After integrating by parts, we may write that

$$\int\limits_{m}^{M} y dF_\epsilon(y) = [M F_\epsilon(M) - m F_\epsilon(m)] - \int\limits_{m}^{M} F_\epsilon(y) dy,$$

which implies, according to (6.6),

$$\lim_{\epsilon \to 0} \left| \left(\breve{F}_\epsilon(M) - \breve{F}_\epsilon(m) \right) - \left(M F_\epsilon(M) - m F_\epsilon(m) - \int\limits_{m}^{M} F_\epsilon(y) dy \right) \right| = 0. \tag{6.11}$$

By definition of F_ϵ and $\breve{F}_\epsilon$, (6.11) becomes

$$\begin{aligned}
\lim_{\epsilon\to 0}\Bigg| &\int_\Omega a_{\epsilon,\Omega}(x)\, f(x)|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx \\
&- M\int_\Omega a_{\epsilon,\Omega}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx \\
&+\int_m^M\left[\int_\Omega a_{\epsilon,\Omega(y)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx\right]dy\Bigg| = 0.
\end{aligned}\tag{6.12}$$

By the way, $a_{\epsilon,\Omega(y)} = a_{\epsilon,\Omega} - a_{\epsilon,\Omega\backslash\Omega(y)}$, so from (6.12) we obtain

$$\begin{aligned}
\lim_{\epsilon\to 0}\Bigg| &\int_\Omega a_{\epsilon,\Omega}(x)\, f(x)|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx \\
&- m\int_\Omega a_{\epsilon,\Omega}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx \\
&-\int_m^M\left[\int_\Omega a_{\epsilon,\Omega\backslash\Omega(y)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx\right]dy\Bigg| = 0.
\end{aligned}\tag{6.13}$$

Step 3. Set $A(y) = \Omega(y)$ (resp. $A(y) = \Omega\backslash\Omega(y)$). Assume that $\phi \geq 0$ is constant in $A(y)$. Then from the Dominated Convergence Theorem and (6.10) we get

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_m^M\left[\int_\Omega a_{\epsilon,A(y)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\phi_\epsilon dx\right]dy \\
&=\int_m^M\left[\int_\Omega \mathbb{1}_{A(y)}(x)\,|\nabla u|^{p(x)-2}\nabla u\nabla\phi dx\right]dy = 0.
\end{aligned}\tag{6.14}$$

Now suppose that ϕ is not constant in $A(y)$. For $(\epsilon, y, \phi) \in \mathbb{R}_+^\star \times [m, M] \times (C_0^\infty(\Omega))_+$ fixed, we claim that there exists a nonnegative function $\Phi_{\epsilon,y} \in W_0^{1,p(x)}(\Omega)$ such that

$$\nabla\Phi_{\epsilon,y} = a_{\epsilon,A(y)}\nabla\phi_\epsilon \quad \text{a.e. in } \Omega. \tag{6.15}$$

Indeed, consider the singular quasilinear elliptic problem

$$\begin{cases}
-\Delta v = -|a_{\epsilon,A(y)}\nabla\phi_\epsilon|^2 v^{-1} - 2\mathrm{div}\left(a_{\epsilon,A(y)}\nabla\phi_\epsilon\right) & \text{in } \Omega,\\
v > 0 & \text{in } \Omega,\\
v = 0 & \text{on } \partial\Omega.
\end{cases}\tag{6.16}$$

Here, the divergence of the vector $a_{\epsilon,A(y)}\nabla\phi_\epsilon$ must be interpreted in the classical sense, which is possible since the convolution ϕ_ϵ (resp. $a_{\epsilon,A(y)}$) involves the function ϕ (resp. Ψ_ϵ) which is infinitely differentiable. We are going to prove the existence of at least one solution for (6.16) in the sense of distributions, i.e., there exists $v_\epsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla v_\epsilon \nabla \xi dx = -\int_\Omega |a_{\epsilon,A(y)}\nabla\phi_\epsilon|^2 v_\epsilon^{-1}\xi dx - 2\int_\Omega \operatorname{div}\left(a_{\epsilon,A(y)}\nabla\phi_\epsilon\right)\xi dx$$

for every $\xi \in H_0^1(\Omega)$.
Existence of a solution for (6.16). Define

$$\begin{aligned}
&a_\epsilon(x) = |a_{\epsilon,A(y)}\nabla\phi_\epsilon|,\\
&b_\epsilon(x) = \operatorname{div}\left(a_{\epsilon,A(y)}\nabla\phi_\epsilon\right),\\
&K_\epsilon = \left\{w \in C_0^1(\Omega) : \underline{v}_\epsilon \le w \le \overline{v}_\epsilon\right\},\\
&\underline{v}_\epsilon = \lambda\epsilon^{-1/2}\psi,\\
&\overline{v}_\epsilon = \max\{\lambda\epsilon^{-1/2}\|\psi\|_\infty, \|b_\epsilon\|_\infty\|\chi\|_\infty\},\\
&f_\epsilon(x,z) = -a_\epsilon^2(x)z^{-1} - 2b_\epsilon(x), \quad z \in K_\epsilon,\\
&\hat{\phi}(x) = \mathbb{1}_{A(y)}(x)\min_{\epsilon\in(0,1)}\int_{B_1(0)}\phi(x-\epsilon y)dy,
\end{aligned}$$

where $\psi, \chi \in C_0^1(\overline{\Omega})$ are respectively the solutions of the problems

$$\begin{cases} -\Delta\psi = \hat{\phi} & \text{in } \Omega,\\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\chi = 1 & \text{in } \Omega,\\ \chi = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.17}$$

see [22, Proposition 2.1]. The fact that $\phi \ge 0$ is not constant in $A(y)$ implies that $\hat{\phi}$ is a positive nontrivial function, hence the existence of $\lambda > 0$ such that

$$d(x) \le \lambda\psi(x) \quad \text{for all } x \in \Omega, \tag{6.18}$$

with $d(x) := d(x, \partial\Omega)$, see [23, Theorems 1 and 2]. Let $z \in K_\epsilon$, consider the auxiliary problem

$$\begin{cases} -\Delta v = f_\epsilon(x,z) & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.19}$$

Following the ideas of [22], we prove that $f_\epsilon \in L^q(\Omega)$. Indeed, due to the classical Mean Value Theorem, the estimation $a_{\epsilon,A(y)}(x) \le \|\nabla a_{\epsilon,A(y)}\|_\infty d(x)$ with (6.18) imply

$$a_\epsilon^2 z^{-1} \le a_\epsilon^2 \underline{v}_\epsilon^{-1} = |a_{\epsilon,A(y)}\nabla\phi_\epsilon|^2(\lambda\epsilon^{-1/2}\psi)^{-1} \le \epsilon^{1/2}\|\nabla a_{\epsilon,A(y)}\|_\infty|\nabla\phi_\epsilon|^2, \tag{6.20}$$

hence $a_\epsilon^2 z^{-1} \in L^q(\Omega)$ for any $q > N$. Consequently, $f_\epsilon \in L^q(\Omega)$, and from [22, Proposition 2.1] there exists $\check{v}_\epsilon \equiv Az \in C_0^1(\Omega)$ satisfying (6.19). Since the mapping

$S : K_\epsilon \to L^q(\Omega)$, defined by $Sz = f_\epsilon(x,z)$, is continuous and bounded, it follows from [17, Lemma 2.1] that $A : K_\epsilon \to C_0^1(\Omega)$ is compact. It remains to prove that $A : K_\epsilon \to K_\epsilon$. Indeed, let $z \in K_\epsilon$ and $\check{v}_\epsilon = Az$. Then $-\Delta\check{v}_\epsilon \leq \|b_\epsilon\|_\infty = -\Delta(\|b_\epsilon\|_\infty \chi)$, χ defined in (6.17), which implies according to the weak comparison principle that $\check{v}_\epsilon \leq \|b_\epsilon\|_\infty \chi \leq \overline{v}_\epsilon$.

Now to prove that $\underline{v}_\epsilon \leq \check{v}_\epsilon$, we proceed as in [17]. Let

$$B_s = \{x \in \Omega : f_\epsilon(x,\cdot) < \lambda s\hat{\phi}(x)\} \quad (s > 0)$$

and let $v_{s,\epsilon}$ be the solution of the problem

$$-\Delta v = s^{-1} f_{s,\epsilon}(x, \underline{v}_\epsilon) \equiv s^{-1} \begin{cases} f_\epsilon(x, \underline{v}_\epsilon) & \text{in } B_s \\ \lambda s\hat{\phi}(x) & \text{in } \Omega\backslash B_s \end{cases}, \quad v = 0 \text{ on } \partial\Omega.$$

Notice that $-\Delta\check{v}_\epsilon = f_\epsilon(x,z) \geq f_{s,\epsilon}(x,\underline{v}_\epsilon) = -\Delta\left(sv_{s,\epsilon}\right)$, and from the weak comparison principle we have $\check{v}_\epsilon \geq sv_{s,\epsilon}$. Also, setting $v_0 = \lambda\psi$, ψ defined in (6.17), then v_0 satisfies $-\Delta v_0 = \lambda\hat{\phi}$. From [17, Lemma 2.1], and using (6.18) and (6.20), there exists $C > 0$ independent of $v_{s,\epsilon}$ and v_0 such that

$$\begin{aligned} |v_{s,\epsilon} - v_0|_{C^1} &\leq C \left\| s^{-1} f_{s,\epsilon} - \lambda\hat{\phi} \right\|_q \\ &= C \left(\int_{B_s} \left| s^{-1}[-a_\epsilon^2(x)\underline{v}_\epsilon^{-1} - 2b_\epsilon(x)] - \lambda\hat{\phi}(x) \right|^q dx \right)^{1/q} \\ &\leq C|B_s|^{1/q} \left(s^{-1} \left[\epsilon^{1/2} \|\nabla a_{\epsilon,A(y)}\|_\infty \|\nabla\phi_\epsilon\|_\infty^2 + 2\|b_\epsilon\|_\infty \right] + \lambda\|\hat{\phi}\|_\infty \right) \\ &\leq C|B_s|^{1/q} M_{s,\epsilon}, \end{aligned} \tag{6.21}$$

where

$$M_{s,\epsilon} = s^{-1}[\epsilon^{-1/2}\|\nabla\psi\|_\infty \|\nabla\phi\|_\infty^2 + 2\epsilon^{-1}\|\nabla\psi\|_\infty\|\nabla\phi\|_\infty + 2\|\Delta\phi\|_\infty] + \lambda\|\phi\|_\infty|B_1|.$$

Also, from (6.18), we check that

$$\check{v}_\epsilon \geq sv_{s,\epsilon} \geq \lambda s\left(\lambda^{-1}v_0 - \lambda^{-1}|v_{s,\epsilon} - v_0|_{C^1} d(x)\right) \geq \lambda s(1 - |v_{s,\epsilon} - v_0|_{C^1})\psi. \tag{6.22}$$

Now, consider the set

$$G = \{x \in \Omega : d(x, \partial[A(y) \cap \Omega_\epsilon]) \geq \epsilon\}.$$

If $x \in G$, $b_\epsilon(x) = \mathbb{1}_{A(y)}(x)\Delta\phi_\epsilon(x)$, and by definition of ϕ_ϵ it occurs

$$-2b_\epsilon(x) = 2\mathbb{1}_{A(y)}(x)\phi \star (-\Delta\Pi_\epsilon)(x) = 2\epsilon^{-2}\overline{C}^{-1}\mathbb{1}_{A(y)}(x) \int_{B_1(0)} \phi(x - \epsilon y)dy. \tag{6.23}$$

Assume that $\epsilon > 0$ is small enough so that $\epsilon^{-1/2} \geq \lambda\overline{C} + 2^{-1}N\|\nabla\phi\|_\infty\|\nabla\pi\|_\infty$. From (6.23) we deduce that

$$-2b_\epsilon(x) \geq (2\lambda + \overline{C}^{-1}N\|\nabla\phi\|_\infty\|\nabla\pi\|_\infty)\epsilon^{-3/2}\mathbb{1}_{A(y)}(x)\int_{B_1(0)}\phi(x-\epsilon y)dy. \tag{6.24}$$

Besides, considering that $\lambda > 0$ satisfies (6.18), for any $(x,z) \in G \times K_\epsilon$ we get

$$\begin{aligned}-a_\epsilon^2(x)z^{-1} &\geq -|a_{\epsilon,A(y)}\nabla\phi_\epsilon|^2\underline{v}_\epsilon^{-1} \geq -|a_{\epsilon,A(y)}\nabla\phi_\epsilon|^2(\epsilon^{-1/2}d(x))^{-1}\\ &\geq -\overline{C}^{-1}N\|\nabla\phi\|_\infty\|\nabla\pi\|_\infty\epsilon^{-3/2}\mathbb{1}_{A(y)}(x)\int_{B_1(0)}\phi(x-\epsilon y)dy.\end{aligned} \tag{6.25}$$

Therefore, combining (6.24) and (6.25), in the set G we end up with

$$f_\epsilon(x,z) \geq 2\lambda\epsilon^{-3/2}\mathbb{1}_{A(y)}(x)\int_{B_1(0)}\phi(x-\epsilon y)dy \geq \lambda s\hat{\phi}(x),$$

with $s = 2\epsilon^{-1/2}$. Consequently $B_s \subset \Omega\backslash G$, where $B_s = \{x \in \Omega : f_\epsilon(x,.) < \lambda s\hat{\phi}(x)\}$, and so $|B_s| \leq |\Omega\backslash G| < k_N\epsilon^N$ for some constant $k_N > 0$. Applying (6.21) with $s = 2\epsilon^{-1/2}$ and $q \in (N,(4/3)N)$, we may write, for $\epsilon > 0$ small enough,

$$|v_{s,\epsilon} - v_0|_{C^1} \leq C\epsilon^{3/4}M_{s,\epsilon} < \epsilon^{1/4}\overline{M} < 1/2, \tag{6.26}$$

where $\overline{M} > 0$ is some constant independent of ϵ. We deduce from (6.22) and (6.26) that

$$\check{v}_\epsilon \geq \lambda(s/2)\psi = \lambda\epsilon^{-1/2}\psi = \underline{v}_\epsilon.$$

Therefore $\check{v}_\epsilon \in K_\epsilon$, and the existence of a solution of (6.16) follows from Schauder's fixed point theorem.
Proof of (6.15). Let $v_\epsilon \in K_\epsilon$ be a solution of (6.16). We show that

$$\int_\Omega \left|\nabla v_\epsilon - a_{\epsilon,A(y)}\nabla\phi_\epsilon\right|^2 dx = 0.$$

Indeed, multiplying (6.16) by v_ϵ and integrating over Ω, we may write

$$\begin{aligned}0 &= \int_\Omega \left[|\nabla v_\epsilon|^2 + \left|a_{\epsilon,A(y)}\nabla\phi_\epsilon\right|^2 + 2\mathrm{div}(a_{\epsilon,A(y)}\nabla\phi_\epsilon)v_\epsilon\right]dx\\ &= \int_\Omega \left[|\nabla v_\epsilon|^2 - 2a_{\epsilon,A(y)}\nabla\phi_\epsilon.\nabla v_\epsilon + \left|a_{\epsilon,A(y)}\nabla\phi_\epsilon\right|^2\right]dx\\ &= \int_\Omega \left|\nabla v_\epsilon - a_{\epsilon,A(y)}\nabla\phi_\epsilon\right|^2 dx.\end{aligned}$$

As a consequence, we define $\Phi_{\epsilon,y} \equiv v_\epsilon$ in (6.15).

Step 4. Since u is a solution of problem (6.3), and thanks to the positivity of the functions h and $\Phi_{\epsilon,y}$, from (6.15) we have

$$\begin{aligned} I_\epsilon &= \int_m^M \left[\int_\Omega a_{\epsilon,A(y)}(x)\, |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx \right] dy \\ &= \int_m^M \left[\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \Phi_{\epsilon,y} dx \right] dy \\ &= \int_m^M \left[\int_\Omega h(x) \Phi_{\epsilon,y} dx \right] dy \geq 0. \end{aligned} \tag{6.27}$$

Besides, using the same argument as in (6.14), we prove that the sequence I_ϵ converges as $\epsilon \to 0$. Therefore from (6.27) we deduce that $\lim\limits_{\epsilon\to 0} I_\epsilon \geq 0$, and combining (6.12) and (6.13), we end up with

$$\begin{aligned} &m \lim_{\epsilon\to 0} \int_\Omega a_{\epsilon,\Omega}(x)\, |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx \\ &\leq \lim_{\epsilon\to 0} \int_\Omega a_{\epsilon,\Omega}(x)\, f(x) |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx \\ &\leq M \lim_{\epsilon\to 0} \int_\Omega a_{\epsilon,\Omega}(x)\, |\nabla u|^{p(x)-2} \nabla u \nabla \phi_\epsilon dx. \end{aligned}$$

The Dominated Convergence Theorem and (6.10) imply that

$$\begin{aligned} &m \int_\Omega h(x)\phi dx = m \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx \\ &\leq \int_\Omega f(x) |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx \\ &\leq M \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx = M \int_\Omega h(x) \phi dx. \end{aligned}$$

This leads to the existence of $\gamma \in [m, M]$, depending on ϕ, such that

$$\int_\Omega f(x) |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx = \gamma \int_\Omega h(x) \phi dx.$$

This completes the proof. □

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