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Original Research Article

OPTIMAL MODELLING OF STEEL MULTI-SPAN BEAMS USING THE GRADIENT-ITERATIVE METHOD

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A b s t r a c t

The article describes the gradient-iterative optimization method and outlines the method's basic assumptions and illustrates its general use. The method's implementation was illustrated based on a steel I-beam. The described calculation example concerns the optimization of the height of the web of a multi-span beam. The method enables finding an optimal solution with the use of simple and commonly available software.
To illustrate the effectiveness of the optimization method, multiple calculations were performed for beams with various spans and various load conditions.

Keywords: structure optimization, gradient-iterative method, multi-span beams

1. INTRODUCTION

Modern engineering of construction elements places great emphasis on the economy of implemented solutions. Designed elements must fulfil all bearing capacity and serviceability requirements and must simultaneously meet certain economic demands. In most cases weight of an optimally-designed element will be as small as possible as this minimizes consumption of materials, thus also minimizing production costs.

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The article outlines optimal modelling of multi-span steel beams. The analysis of the problem was formulated in accordance with European building standards. The new proposition of gradient-iterative method was used in order to determine an optimal solution. Proposed method allows rapid selection of optimal solutions.

2. THE MOTIVATION FOR THE NEW CALCULATION METHOD

Various construction optimization research base on the use of mathematical methods of optimal control [1, 2, 5]. Methods based on refined control theory make it possible to find optimal solutions satisfying posed boundary conditions as well as chosen set of inequality and equality constraints. Optimization based on the maximum principle gives good results but has two significant, from the perspective of design, restrictions:

- The necessity to formulate the problem of optimization in the mathematical framework of the method, namely as a multipoint boundary value problem for a system of the first order ordinary differential equations, subject to certain constraints due to code requirements. Stating such a problem can be automatized only for a certain class of problems and in general must be performed analytically in a way which is not familiar to the structural designers.
- Lack of user-friendly software which would enable finding solutions to complex problems formulated in categories of optimal control methods.

The method combines the gradient descent method and an iterative solution method to the formulated optimization problem. The method can be outlined with six steps:

1. Stating a mathematical form of functions describing the assessed optimization task.
2. Determining the objective function and decision variables.
3. Determining optimization restrictions.
4. Determining the optimization starting point and the direction of the sought solution.
5. A description of the increment function.
6. Iteratively obtaining a solution which fulfils optimization criteria.

The described method can be used to solve various optimization problems. It enables finding optimal solutions with the help of commonly available software and considerably faster than in the case of some other optimization methods, which often require the use of multipoint boundary value problems numeric solvers, which are often not commonly available.

Paired with the finite-element method it enables finding optimal solutions for statically indeterminate constructions.
Simple functions which allow for a thorough description of the optimization problem are used when formulating a task for the gradient-iterative method. Thanks to uncomplicated mathematical formulas, numerical calculations can be carried out quickly.
When stating the problem, special attention should be paid to determine the increment function in a correct way. Incorrect input can lengthen calculations considerably, and in the worst case can even prevent receiving a reliable result.

3. A GENERAL DESCRIPTION OF THE PROBLEM

The problem concerns the optimal modelling of steel multi-span I-beams. The analysis is constrained to chosen three-, four- and five-span structural systems. Fig. 1÷3 below show a static diagram and assumed load phases.

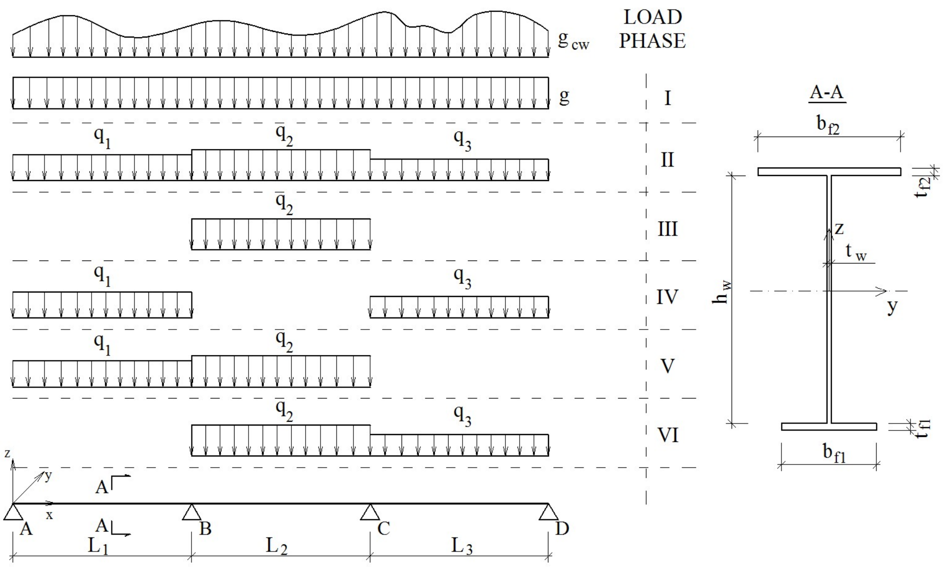

Fig. 1. Static diagram, cross-section and configuration of external forces for a three-span beam

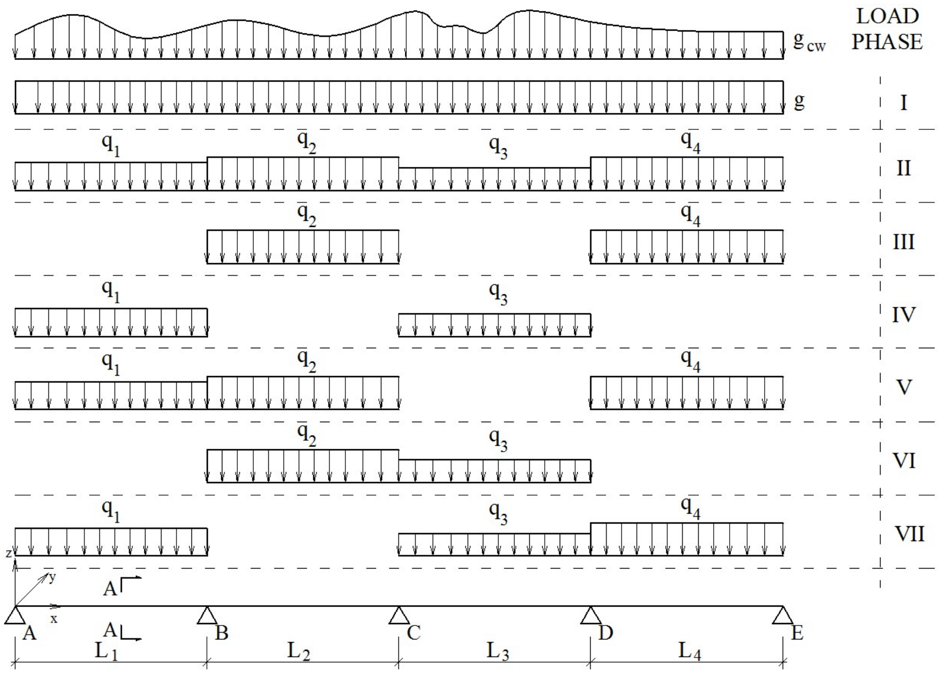

Fig. 2. Static diagram and configuration of external forces for a four-span beam

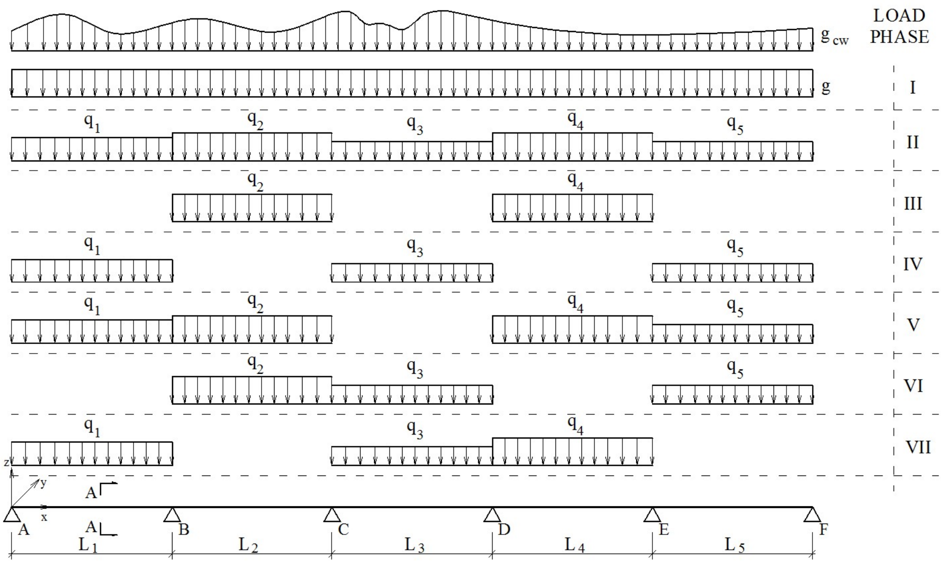

Fig. 3. Static diagram and configuration of external forces for a five-span beam

Load phase 1 takes under consideration the element's self-weight g_{cw} and dead load g. Phases II÷VII illustrate live load in various calculated cases.
Linear steady load g and dynamic load q were transferred to the upper flange of girder by purlins in spacing L_p in accordance with fig. 4.

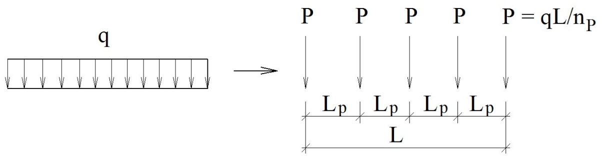

Fig. 4. Transformation of linear load on the concentrated forces

where:
n_p – number of concentrated forces occurring within one span section.

The analysed example accounts for 6 (for three-span beam) and 7 (for four and five-span beam) design combinations of each load phase:

$$
\begin{aligned}
C1 &= F_I \\
C2 &= F_I + F_{II} \\
C3 &= F_I + F_{III} \\
C4 &= F_I + F_{IV} \\
C5 &= F_I + F_V \\
C6 &= F_I + F_{VI} \\
C7 &= F_I + F_{VII}
\end{aligned}
\tag{3.1}
$$

where:
$F_{I \div VIII}$ – load phase.

4. CALCULATION PROCEDURE

It is necessary to establish optimal cross-section dimensions which will minimize the assumed objective function which is the element's volume:

$$V = \int_0^L A(x)dx \tag{4.1}$$

where:
A(x) – the area of cross-section considered a function of a coordinate measured along the beams axis,
L – element's overall length.

Optimization requires selecting the height h_w variable along the girder's axis which minimizes the set objective function (4.1) and fulfils all assumed optimization restrictions [7].
The load was divided into three phases (fig. 1÷3). A continuous linear load was assumed.

4.1. Finite-element method in the analysed example:

The beam was discretized into finite elements with variable stiffness *EI(h)* and fixed length L_{ES} (fig. 5).

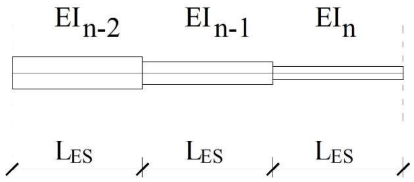

Fig. 5. Beam discretization

The stiffness matrix was defined (4.2) and a Boolean matrix was constructed for *n* finite elements (4.3).

$$k(EI,L)=\begin{pmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \tag{4.2}$$

$$B_i=\left| \begin{array}{l} B_{(1,2\cdot(top^i-1)+1)}=1 \\ B_{(2,2\cdot(top^i-1)+2)}=1 \\ B_{(3,2\cdot(top^i-1)+3)}=1 \\ B_{(4,2\cdot(top^i-1)+4)}=1 \end{array} \right. \tag{4.3}$$

where:
top^i – component of incidence matrix corresponding with *i*-th finite element.

Function (4) describes the incidence matrix for *n* finite elements.

$$top=\left| \begin{array}{l} \text{for } i\in 1\dots n \\ \quad top_{i,1}=i \\ \quad top_{i,2}=i+1 \end{array} \right. \tag{4.4}$$

The above matrices (4.3) define the Boolean matrix for every i-th finite element, enabling later automatic calculations for any number of elements. The overall size of the Boolean matrix for the discussed task is 4 x (2n+2). Function (4.3) shows only non-zero elements of the matrix. Boolean matrix for every i-th finite element is shown below:

$$B_i = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \underbrace{0 \quad \dots \quad 0}_{2i-2} & & & \underbrace{0 \quad 0 \quad 0 \quad 1}_{4} & & & & \underbrace{0 \quad \dots \quad 0}_{(2n+2)-(2i-6)} & & \end{pmatrix} \tag{4.5}$$

Stiffness matrix aggregation:

$$K = \sum_{i=1}^{n} B_i^T K_{e.i} B_i \tag{4.6}$$

where:
$K_{e.i}$ – stiffness matrix calculated in accordance with equation (3) for i-th finite element,
B_i – Boolean matrix for i-th finite element.

Definition of vectors corresponding to uniformly distributed and to point load:

$$Z_c(q, L_{ES}) = \begin{pmatrix} \dfrac{qL_{ES}}{2} \\ \dfrac{qL_{ES}{}^2}{12} \\ \dfrac{qL_{ES}}{2} \\ \dfrac{-qL_{ES}{}^2}{12} \end{pmatrix} \tag{4.7}$$

$$Z_s(P, L_{ES}) = \begin{pmatrix} \dfrac{P}{2} \\ \dfrac{PL_{ES}}{8} \\ \dfrac{P}{2} \\ \dfrac{-PL_{ES}}{8} \end{pmatrix} \tag{4.8}$$

where:
q – uniformly distributed load,
P – point load,
L_{ES} – length of finite element.

Aggregation of load vectors:

$$Z = \sum_{i=1}^{n} B_i^T Z_{e.i} \tag{4.9}$$

where:
$Z_{e.i}$ – load vector corresponding to uniformly distributed load for every i-th finite element.

Dead load was approximated with a load uniformly distributed along a single finite element. Approximation of dead load with a piecewise-constant load is exact enough in the case of minimum 5 finite elements within one span section. Boundary conditions for a three-span beam were described by vector:

$$w = \begin{vmatrix} w_1 = 1 \\ w_2 = 0 \\ \text{for } i \in 2 \ldots (n+1) \\ w_{(2i-1)} = \begin{vmatrix} 1 \text{ if } \big((i-1)L_{ES} = L_1\big) \vee ((i-1)L_{ES} = L_1 + L_2) \vee \\ \big((i-1)L_{ES} = L_1 + L_2 + L_3\big) \\ 0 \text{ otherwise} \end{vmatrix} \\ w_{2n+2} = 0 \end{vmatrix} \tag{4.10}$$

The boundary conditions vector was established analogically for four and five-span beams. The solution to the set of equations and calculation of displacement vectors for finite elements:

$$Q = K_{wb}^{-1} S_{wb} \tag{4.11}$$

$$R = K_{wb} Q - S_{wb} \tag{4.12}$$

where:
Q – nods displacements vector,
R – reaction vector,
K_{wb} – stiffness matrix incorporating boundary conditions,
S_{wb} – node load vector with boundary conditions.

Calculation of node forces in elements:

$$f_{e.i} = K_{wb.i} B_i Q - Z_{e.i} \tag{4.13}$$

where:
$f_{e.i}$ – node forces vector for i-th finite element.

Equations (4.2÷4.12) were used to formulate function *MES(EI,x)*, which enables determining the value of internal forces as well as vertical and angular displacement.

4.2. Ultimate limit state for analysed cross-sections:

All guidelines included in the following norms were implemented in order to determine the bearing capacity of cross-sections:
EN-1990, EN-1991-1, EN 1993-1-1, EN 1993-1-5.

The following issues were verified according to the above standards:
- Bearing capacity in bending, accounting for element's critical moment
- Bearing capacity under shear stress
- Interaction between transverse force and bending moment
- Bearing capacity under concentrated load
- Interaction between concentrated load and bending moment
- Resistance in relation to web slenderness.

During the verification of the above ultimate limit states, effective flange width and plate buckling effects should be taken under consideration when assuming an effective cross-section field. Due to the above the calculation procedure was additionally implemented with a function to determine reduced geometric dimensions for any given cross-section.
Calculations assumed protection against loss of stability in purlin resting points. Accordingly, the buckling length in the analysed beam is equal to L_p.

4.3. Incremental function and optimization loop

The starting point of the optimization process was obtaining minimal dimensions of the cross-section due to set geometric restrictions [8].
A stepwise increment of the decision variable was assumed within one calculation loop. The direction of the increment Δh_w depends on fulfilling bearing capacity conditions and is determined in relation to the result of the cross-section verification result. Additionally, the increment value Δh_w decreases with subsequent calculation phases.
The result of a calculation loop is the optimal height of the cross-section of one finite element. What follows, the total amount of calculation loops within one iteration is equal to n+1.

where:
n – number of finite elements.

Calculations are performed as follows:
- Finding the optimal cross-section height for each finite element.
- Cross-section fulfils all determined bearing conditions and minimizes determined objective function (4.1). Calculations are performed for internal forces determined via the finite element method for initial values.
- FEM calculations and updating value of cross-section forces and linear displacement.
- Re-determining optimal cross-section height along girder's length.
- Verification of boundary nodes' displacement.
- Iterative calculations (FEM calculations are carried out for each iteration, an optimal solution is determined for defined internal forces and boundary nodes' displacement is verified).
- Iterative calculations are stopped upon receiving an expected iterative convergence.

5. CALCULATION RESULTS

Optimization task was performed for the below input data:
- steel class: S235,
- dead external load: $g = 10.5 \cdot 10^3\ N/m$,
- live load: $q_1 = 35 \cdot 10^3\ N/m,\ q_2 = q_3 = q_4 = q_5 = q_1$,
- fixed geometric dimensions: $t_{f1} = t_{f2} = 0.02\ m;\ t_w = 0.008\ m$,
- span lengths: $L_1 = 15\ m,\ L_2 = L_3 = L_4 = L_5 = L_1$,
- length between purlins: $L_p = 3\ m$,
- length of a finite element: $L_{ES} = 1.0\ m$.

The first calculation phase concerned solving a problem with three control variables of the following admissible values:
- web height h_w: <0.30 ; 1.50> [m],
- bottom flange width b_{f1}: <0.15 ; 0.30> [m],
- top flange height b_{f2}: <0.15 ; 0.30> [m].

The illustrations below show: optimal height of optimized girder (fig. 6), optimal flange width (fig. 7, 8) as well as the envelope of nodal vertical displacements (fig. 9).

Maximum vertical displacement was equal to 0.041 m. Limit admissible deflection for considered girder was equal to L/250 = 0.060 m.

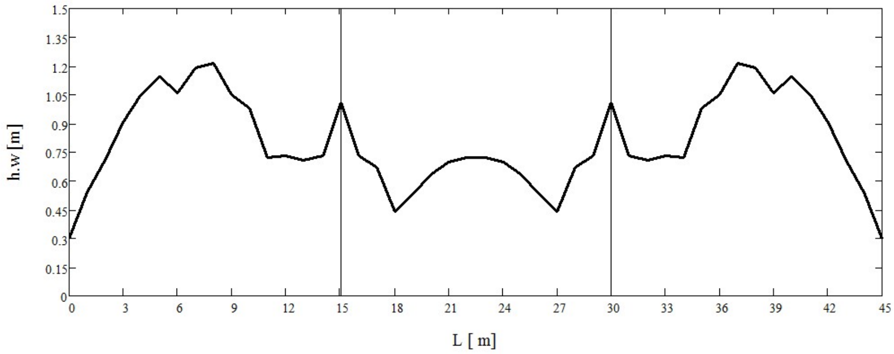

Fig. 6. Optimal height h_w (three-span beam)

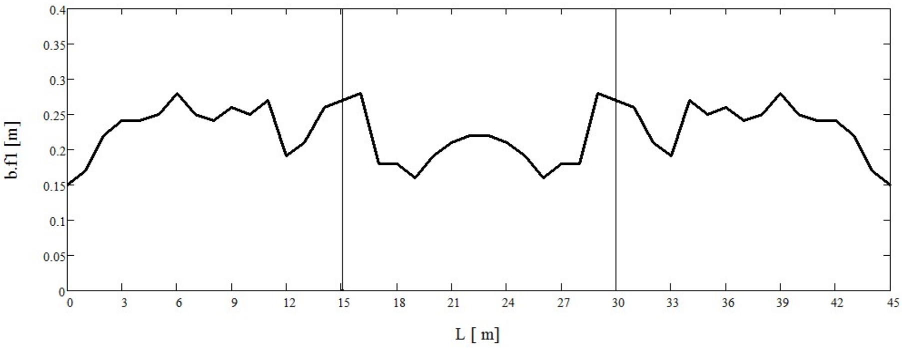

Fig. 7. Optimal width b_{f1} (three-span beam)

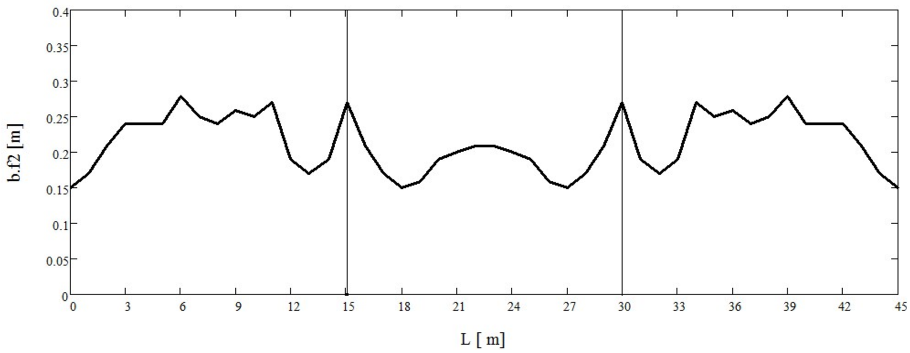

Fig. 8. Optimal width b_{f2} (three-span beam)

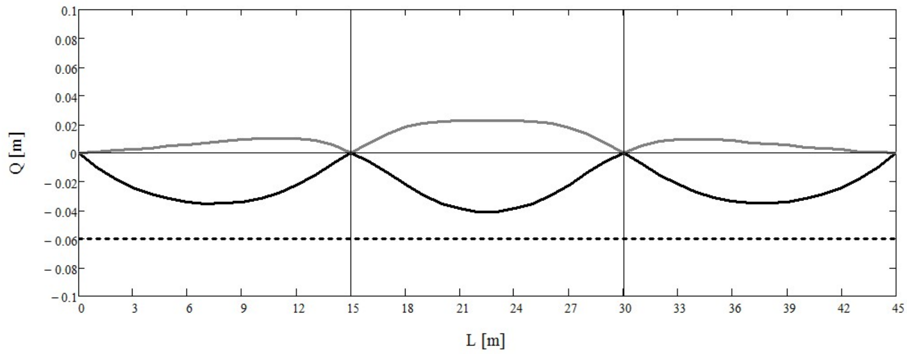

Fig. 9. Envelope of nodal vertical displacements (three-span beam)

The value of the objective function (4.1) for the obtained solution is equal to V = $763.5 \cdot 10^{-3}$ m^3.

In practice, obtaining an optimal shape for the girder presented on fig. 6, 7 and 8 is impossible. This is why a designed height is determined as a simplified envelope of the optimal height distribution. Calculations were carried out once more.

The second calculation phase concerned solving a problem with fixed dimensions: $b_{f1} = b_{f2} = 0.25\ m$ and one control variable with the following admissible values:

- height h_w: <0.30; 1.50> [m].

Fig. 10 shows a girder of an optimal shape. Additionally, the illustration contains a thin line showing the optimal calculated height of the web for the beam under consideration.

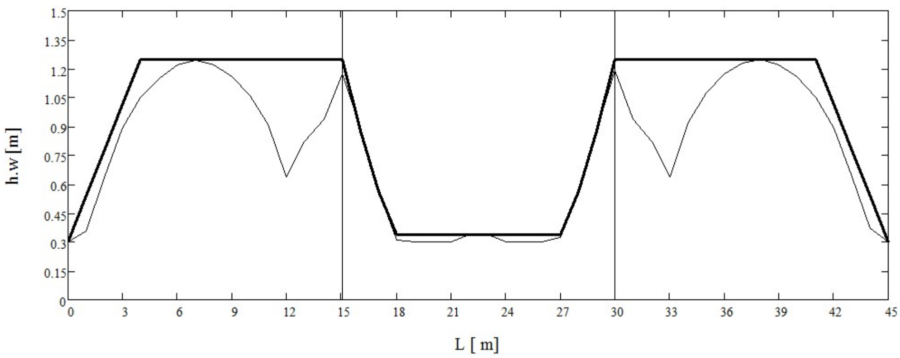

Fig. 10. Optimal height h_w (three-span beam)

The value of the objective function (4.1) for the result possible to construct in practice is equal to V = $858.2 \cdot 10^{-3}$ m^3.

Fig. 11, 13 show the optimal girder height for four and five-span beams. The calculations assumed that the optimal result will be approximated by a function allowing a beam shape constructible in practice. Fig. 12, 14 show nodal vertical displacement for obtained optimal results.

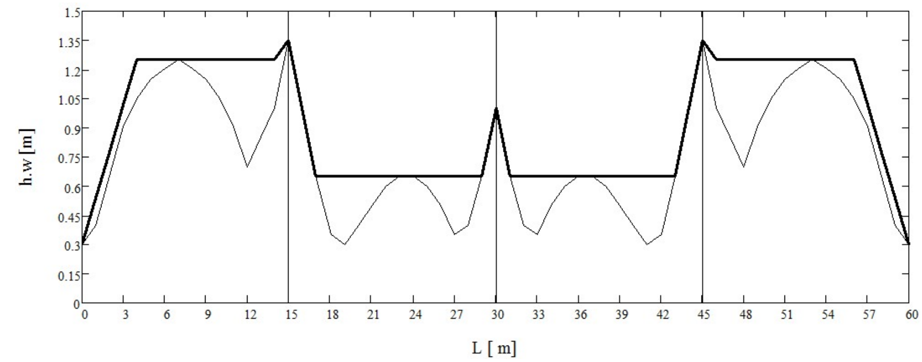

Fig. 11. Optimal height h_w (four-span beam)

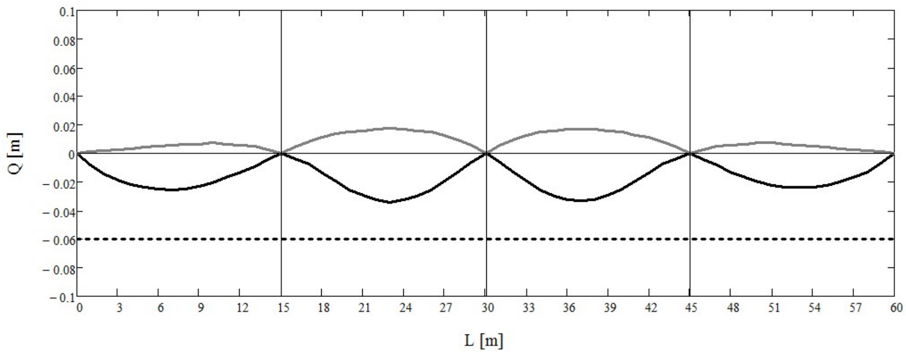

Fig. 12. Envelope of nodal vertical displacement (four-span beam)

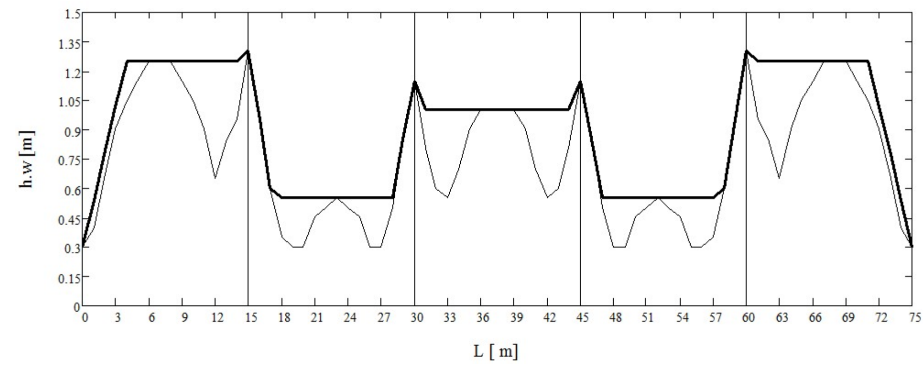

Fig. 13. Optimal height h_w (five-span beam)

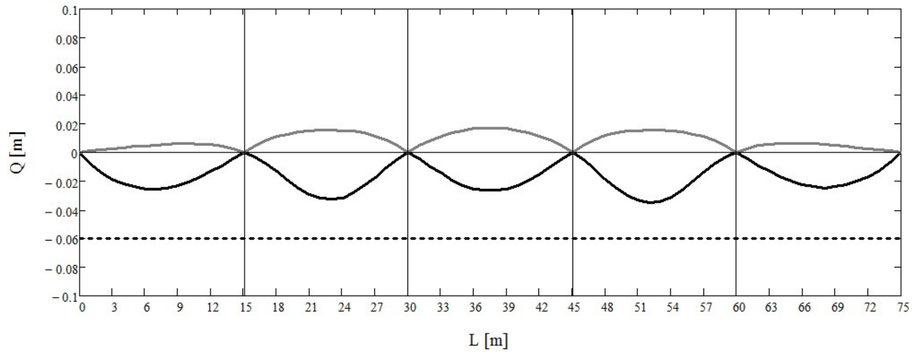

Fig. 14. Envelope of nodal vertical displacement (five-span beam)

Additional calculations were performed to illustrate the effectiveness of the calculation method being described.

Fig. 15÷17 show the optimal height of a girder for the material parameters and load specification as described above but for various span lengths.

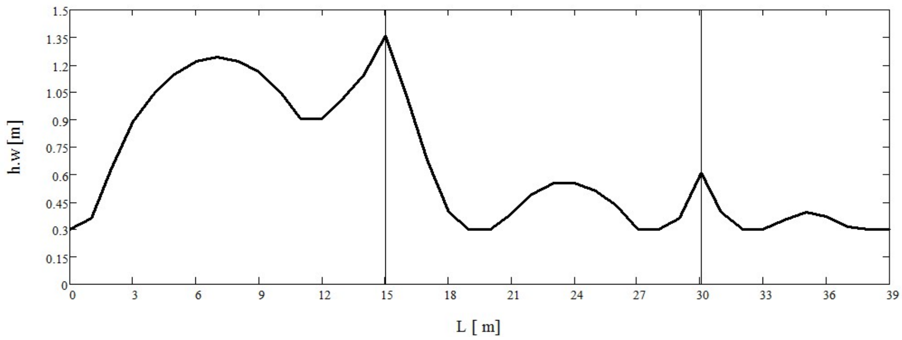

Fig. 15. Envelope of nodal vertical displacement
(three-span beam, $L_1 = 15\ m, L_2 = 15\ m, L_3 = 9\ m$)

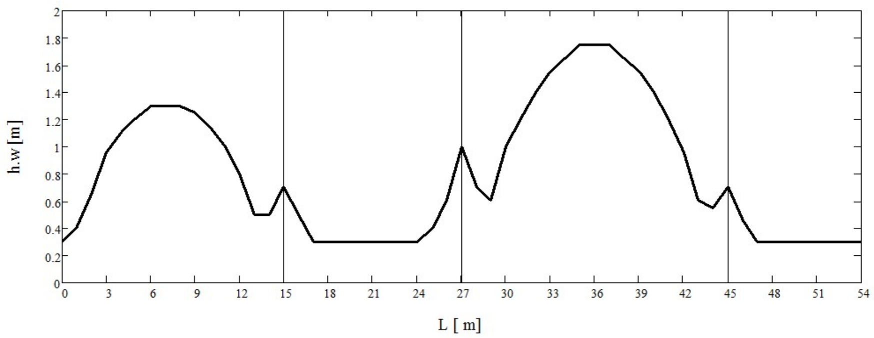

Fig. 16. Optimal height h_w (four-span beam, L_1 = 15 m, L_2 = 12 m, L_3 = 18 m, L_4 = 9 m)

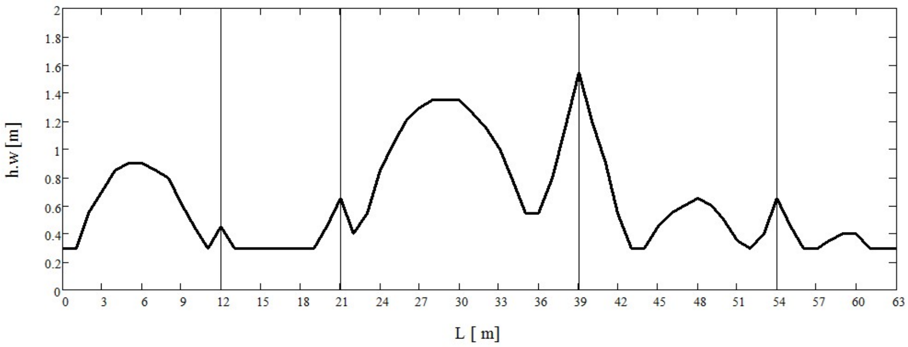

Fig. 17. Optimal height h_w
(five-span beam, L_1 = 12 m, L_2 = 9 m, L_3 = 18 m, L_4 = 15 m, L_5 = 9 m)

6. CONCLUSIONS

In the case of complex structural configurations, the problem of finding an optimal solution with the use of the maximum principle requires solving a multi-point boundary value problem with additional set of constraints, what requires analytical statement of the problem and solving it numerically. Such a solution cannot be obtained with the use of simple numerical algorithms – existing software enable solving it e.g. with the use of collocation method however the software itself is difficult in use.

The gradient-iterative method makes it possible to find solutions even to complex problems in a relatively simpler way. By formulating the task with the help of simple functions and carrying out calculation loops, the set of solutions contains an optimal result which fulfils all predefined optimization criteria.

The gradient-iterative method paired with the FEM algorithm offers considerable benefits to structural designers. As the method takes relatively small time it can be used in construction design offices to optimize various structural elements.

The example described in the article of a statically indeterminate steel beam demonstrates the effectiveness of the calculation methodology. Using the gradient-iterative method it was possible to perform multiple optimization calculations and specify design recommendations for the optimal modelling of three, four and five-span beams.

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