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$(1,2)$-PDS IN GRAPHS WITH THE SMALL NUMBER OF VERTICES OF LARGE DEGREES

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Abstract. We define and study a perfect $(1,2)$-dominating set which is a special case of a $(1,2)$-dominating set. We discuss the existence of a perfect $(1,2)$-dominating set in graphs with at most two vertices of maximum degree. In particular, we present a complete solution if the maximum degree equals $n-1$ or $n-2$.

Keywords: domination, secondary domination, neighborhoods, maximum degree.

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1. INTRODUCTION

For concepts not defined here, see [4]. Let $G=(V(G),E(G))$ be an undirected, simple graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $x\in V(G)$ is denoted by $d_G(x)$ and the maximum degree of G is denoted by $\Delta(G)$. The set of all vertices of maximum degree is denoted by $S_{\Delta(G)}$. The number $\delta(G)$ is the minimum degree of G. A vertex of degree 1 is a leaf and $L(G)=\{x\in V(G): d_G(x)=1\}$. By $N_G(v)$ and $N_G[v]=N_G(v)\cup\{v\}$ we mean an *open neighborhood* and a *closed neighborhood* of a vertex v, respectively. The distance between vertices x,y in G is denoted by $d_G(x,y)$. Let $V_0\subset V$, then for $x\in V\setminus V_0$ the distance is defined as $d_G(x,V_0)=\min\{d_G(x,y): y\in V_0\}$.

A subset $D\subseteq V(G)$ is a *dominating* set of G if for every vertex $x\in V(G)\setminus D$ there is $y\in D$ such that $xy\in E(G)$. In particular, $V(G)$ is also dominating. To simplify the notation, we will sometimes write DS instead of the dominating set. The idea of a DS arose in chessboard problems described by de Jaenish [9] in 1862 and was formalized a hundred years later by Berge [1] and Ore [16] in 1962. Ore first used the name dominating set and dominating number, followed in 1977 by Cockayne and Hedetniemi [3] published a survey of domination results, where they used the symbol $\gamma(G)$ to denote the domination number. Since then, a lot of results related to domination have been written. Looking through the literature we see that dominating sets have been studied in various directions in recent decades, in particular many different variants and generalizations can be found.

The great interest in domination theory results from its potential applications. For example, recently Raczek in the publication [18] mentioned many applications of domination parameters. It just cite examples of applications related to: ranging from environmental sensors, emergency vehicle communication, road safety, health, home, peer-to-peer messaging, disaster rescue operations, aviation/land/naval defense, weapons and robots. In 2008 Hedetniemi *et al.* introduced the concept of $(1, k)$-dominating sets being a weaker version of dominating sets, see [7]. Let $k \in \mathbb{N}$. A subset $D \subseteq V(G)$ is a $(1, k)$-*dominating set* (shortly $(1, k)$-DS) if each vertex v of $V \setminus D$ has a neighbor in D and there is another vertex of D at a distance at most k from v. In particular $V(G)$ is also a $(1, k)$-DS for every $k \geq 1$.

Note that if $k = 1$, then we obtain a $(1, 1)$-DS known in the literature as 2-dominating sets or double dominating sets, see for example [2, 5, 6, 8]. In this paper, to unify the symbolism and nomenclature, we will use the term $(1, 1)$-DS instead of 2-dominating or double dominating sets. From the definition of a $(1, k)$-DS, it follows that the parameter k is not defined clearly.

Theorem 1.1 ([7])**.** *Let $k, l \in \mathbb{N}$ and $k > l$. Every $(1, l)$-DS is a $(1, k)$-DS.*

In [7] it was also proved that studying of a $(1, k)$-DS makes sense only for $k \in \{1, 2, 3, 4\}$.

Theorem 1.2 ([7])**.** *Every DS of cardinality at least 2 in a connected graph G with $\gamma(G) \geq 2$ is a $(1, 4)$-DS.*

Because $(1, 1)$-DS are rather good described, it is natural to next consider the case of $(1, 2)$-dominating sets. We can observe that a $(1, 2)$-DS is a weaker version of a $(1, 1)$-DS because a vertex outside the $(1, 2)$-DS must have only one adjacent vertex in this set and the other one should be sufficiently close, i.e. at the distance of at most 2.

The concept of a $(1, k)$-DS and outlined research directions initiated the interest of mathematicians in studying of a $(1, 2)$-DS. Since 2008, some interesting results have appeared, for example Michalski et al. in [15] and Kayathri and Vallirani in [10] studied parameters of $(1, 2)$-domination, Raczek in [17, 18] considered computational complexity of it. Independent dominating sets in graphs and their product were also studied, see for example [7, 13, 14]. From the definition of a $(1, 2)$-DS, it immediately follows that every $(1, 1)$-DS is a $(1, 2)$-DS. Consequently, Michalski introduced in [12] and then studied in [11, 15] a proper $(1, 2)$-dominating set.

A subset $D \subset V(G)$ is a *proper* $(1, 2)$-*DS* if D is a $(1, 2)$-DS and is not a $(1, 1)$-DS, i.e. D is $(1, 2)$-dominating and there is a vertex $x \in V(G) \setminus D$ such that x has exactly one neighbor in the set D and there is a vertex $u \in D$ such that $d_G(x, u) = 2$.

Since the set $V(G)$ is a $(1, 1)$-DS, it is not the proper $(1, 2)$-DS. Firstly, the problem of the existence of a proper $(1, 2)$-DS in graphs was solved.

Theorem 1.3 ([15])**.** *A connected graph G has a proper $(1, 2)$-DS if and only if G is not complete.*

Motivated by the above we continue studying $(1, 2)$-dominating sets and introduce a perfect $(1, 2)$-DS being the special case of the proper $(1, 2)$-DS.

A subset $D \subset V(G)$ is a *perfect* $(1,2)$-*dominating set* of G (shortly $(1,2)$-PDS) if every $x \in V(G) \setminus D$ has exactly one neighbor in the set D and there is a vertex $u \in D$ such that $d_G(x,u) = 2$. Then we say that x is perfect $(1,2)$-dominated by D (shortly x is $(1,2)$-PD).

It is obvious that if D is a perfect $(1,2)$-dominating set, then it is also a proper $(1,2)$-DS and $V(G)$ is not a perfect $(1,2)$-dominating set. Studying the $(1,2)$-PDS firstly we have to solve the problem of the existence of the $(1,2)$-PDS in graphs. In this paper we consider the class of graphs with at most two vertices of maximum degree.

2. (1,2)-PDS IN GRAPHS WITH $\Delta(G) = n - 1$

In this section we study the problem of the existence of a $(1,2)$-PDS in graphs of the maximum degree $\Delta(G) = n-1$ for $n \geq 3$. We give the complete characterization of graphs with $\Delta(G) = n-1$ which have a $(1,2)$-PDS.

Theorem 2.1. *Let G be a connected n-vertex graph of the maximum degree $\Delta(G) = n-1$, $n \geq 3$. If G has a $(1,2)$-PDS, then G has the unique vertex of maximum degree $\Delta(G)$.*

Proof. Let $\Delta(G) = n-1, n \geq 3$ and $V(G) = S_{\Delta(G)} \cup S$, where $S = \{x \in V(G) : d_G(x) < \Delta(G)\}$. Suppose that D is a $(1,2)$-PDS of the graph G. Clearly, $|D| \geq 2$. Let, for a contradiction, $S_{\Delta(G)} = \{x_1, \ldots, x_t\}$, $2 \leq t \leq n$ be the set of vertices of the maximum degree $\Delta(G)$. Then $S = \{u_1, \ldots, u_{n-t}\}$ and observe that it can be empty. Let $y \in S$ and consider the following possibilities.

(i) $x_i, x_j \in D$, for $1 \leq i, j \leq t$.
Since $d_G(x_i) = d_G(x_j) = n-1$, y is $(1,1)$-dominated by D, a contradiction to perfectness of D.

(ii) $x_i \in D$ and $u_j \in D$, for $1 \leq i \leq t, 1 \leq j \leq n-t$.
Because $|S_{\Delta(G)}| \geq 2$, there is $x_p \in S_{\Delta(G)}$ such that $p \neq i$ and x_p is $(1,1)$-dominated by D, a contradiction.

(iii) $u_j, u_p \in D$, for $1 \leq j, p \leq n-t$.
Then every $x_i \in S_{\Delta(G)}$ is $(1,1)$-dominated by D, a contradiction.

From above cases the theorem follows. □

Theorem 2.2. *Let G be a connected n-vertex graph, $n \geq 3$ with the unique vertex of maximum degree $\Delta(G) = n-1$. G has a $(1,2)$-PDS if and only if the vertex of maximum degree is a cutvertex.*

Proof. Let $S_{\Delta(G)} = \{x\}$ and suppose that x is a cutvertex. Then $V(G) \setminus \{x\} = \bigcup_{i=1}^{r} V_i$, where $r \geq 2$, for each $i, j \in \{1, \ldots, r\}, i \neq j$ holds $V_i \cap V_j = \emptyset$ and each V_i induces a connected subgraph. We will show that, for each $i \in \{1, \ldots, r\}$, the set $D_i = V_i \cup \{x\}$ is a $(1,2)$-PDS. Let $y \in V(G) \setminus D_i$, $1 \leq i \leq r$. Since $d_G(x) = n-1$, so y is adjacent to x and consequently y is dominated by D_i. Moreover, $V_i \cap V_j = \emptyset$ for all $j \in \{1, \ldots, r\} \setminus \{i\}$, then $d_G(y, V_i) \geq 2$. Because x is adjacent to every vertex of V_i, so there is a path $y - x - u$, where $u \in V_i$ and consequently $d_G(y, V_i) = 2$. Hence, D_i is a $(1,2)$-PDS.

Conversely, suppose that G has a $(1,2)$-PDS and let x be the unique vertex of maximum degree $\Delta(G) = n-1$. We will show that x is a cutvertex. Let D be a $(1,2)$-PDS of G. Then $x \in D$, otherwise there are at least two vertices $u, v \in D$ adjacent to x and x is $(1,1)$-dominated, a contradiction to perfectness of a $(1,2)$-PDS. Clearly, $V(G) \setminus D \neq \emptyset$ because of $V(G)$ is not a $(1,2)$-PDS. Let $y \in V(G) \setminus D$. Then y is adjacent to exactly one vertex from D, otherwise y is $(1,1)$-dominated, a contradiction to perfectness of a $(1,2)$-PDS. By the assumption, x is the unique vertex of the maximum degree $\Delta(G) = n-1$, so $N_G(y) \cap D = \{x\}$. Consequently, for each $y \in V(G) \setminus D$ and for $u \in D \setminus \{x\}$ we have u and y are not adjacent, so $V(G) \setminus \{x\}$ contains at least two components: $D \setminus \{x\}$ and $V(G) \setminus D$ which shows that x is a cutvertex of G. Thus, the theorem is proved. □

3. (1,2)-PDS IN GRAPHS WITH $\Delta(G) = n-2$

In this section, we give necessary and sufficient conditions for the existence of a $(1,2)$-PDS in graphs with one or two vertices of maximum degree $\Delta(G) = n-2$.

Theorem 3.1. *Let G be a connected n-vertex graph, $n \geq 3$ with $L(G) \neq \emptyset$. Then G has a $(1,2)$-PDS.*

Proof. Let $x \in L(G)$ be an arbitrary leaf of the graph G. Then the set $V(G) \setminus \{x\}$ is a $(1,2)$-PDS. Thus, the theorem is proved. □

Therefore, in future considerations, we assume that $\delta(G) \geq 2$.

First we consider graphs with $|S_{\Delta(G)}| = 1$.

Theorem 3.2. *Let G be a connected n-vertex graph, $n \geq 6$, $\delta(G) \geq 2$, $\Delta(G) = n-2$, $S_{\Delta(G)} = \{x\}$ and y is the unique vertex nonadjacent to x. Let $N(x)$ induce a connected subgraph. A graph G has a $(1,2) - PDS$ if and only if there is $u \in N(x)$ such that:*

(i) *$N(u) \cap N(y) = \emptyset$ and*
(ii) *$N(x) \setminus N(u) = N(y) \cup \{u\}$ and*
(iii) *every vertex from $N(u) \setminus \{x, y\}$ is adjacent to a vertex from $N(y)$.*

Proof. Suppose that there is $u \in N(x)$ such that conditions (i), (ii), (iii) hold. We shall show that $D = \{u, y\}$ is a $(1,2)$-PDS of G. Clearly, the vertex x is dominated by u and by the connectivity of G there is $u' \in N(x)$ adjacent to y. Hence, there is a path $x - u' - y$ in graph G, and therefore x is $(1,2)$-PD by D. Let $u'' \in N(x) \setminus \{u\}$. Then, by connectivity of $N(x)$ and from (ii), it follows that u'' is dominated either by u or y. If $u'' \in N(u)$, then from (iii), it follows that there is a vertex $v \in N(y)$ adjacent to u'' and there is a path $u'' - v - y$. If $u'' \in N(y) \setminus \{u\}$, then $u'' \notin N(u)$ and there is a path $u'' - x - u$. Moreover, from (i), the vertex u'' is not $(1,1)$-dominated, so u'' is $(1,2)$-PD by D.

Assume now that G has a $(1,2)$-PDS, say D. We shall show that (i), (ii) and (iii) hold. To prove it firstly we show the following claims.

(a) $D \neq \{x,y\}$.

Assume that $D = \{x,y\}$ is a $(1,2)$-PDS. Then, by the connectivity of G, there is a vertex $u \in N(x) \cap N(y)$ such that u is $(1,1)$-dominated by D, a contradiction.

(b) $D \neq \{x,u\}$ for an arbitrary $u \in N(x)$.

By the connectivity of $N(x)$, it follows that there is $u' \in N(x)$ adjacent to u, so u' is $(1,1)$-dominated by D, a contradiction.

(c) $|D \cap N(x)| = 1$.

If $D \cap N(x) = \emptyset$, then $D = \{x,y\}$, a contradiction with (a). Suppose that $|D \cap N(x)| \geq 2$. Then there exists $N(x) \supseteq D_p = \{u_1,\ldots,u_p\}$, $2 \leq p \leq n-2$ such that $D_p \subseteq D$. Clearly, $x \in D$, otherwise x is $(1,1)$-dominated by D. If $p < n-2$, then there is $u' \in N(x) \setminus D_p$ adjacent to some vertex of D_p and by $x \in D$ the vertex u' is $(1,1)$-dominated, a contradiction. So $D_p = N(x)$. By $\delta(G) \geq 2$, it follows that the vertex y is adjacent to at least two vertices of $N(x)$, so y is $(1,1)$-dominated, a contradiction.

Above claims imply that $D = \{u,y\}$ for some $u \in N(x)$. We show that the vertex u satisfies (i), (ii) and (iii). Since D is a $(1,2)$-PDS, so every vertex $u' \in N(x) \setminus \{u\}$ is adjacent either to u or y, otherwise it is $(1,1)$-dominated. Consequently, (i) and (ii) hold. Because G is connected and $\delta(G) \geq 2$, so the set $N(x) \setminus \{u\}$ is partitioned into nonempty and disjoint subsets $N(u) \setminus \{x\}$ and $N(y) \setminus \{u\}$. Moreover, by $(1,2)$-PD of D we obtain that every vertex from $N(u) \setminus \{x,y\}$ has to be adjacent to some vertex in $N(y)$, so the condition (iii) holds. Thus, the theorem is proved. □

Theorem 3.3. *Let G be a connected graph of order n, $n \geq 6$, $\delta(G) \geq 2$, $\Delta(G) = n-2$, $S_{\Delta(G)} = \{x\}$ and y is the unique vertex nonadjacent to x. Let $N(x)$ induce a disconnected subgraph with components N_i, $i \in \{1,2,\ldots,p\}$, $p \geq 2$. A graph G has a $(1,2)$-PDS if and only if there is $1 \leq i \leq p$ such that either $|V(N_i) \cap N(y)| \leq 1$ or there is $u \in V(N_i)$ such that*

(i) *$N(u) \cap N(y) = \emptyset$ and*
(ii) *$V(N_i) \setminus N(u) = [N(y) \cap V(N_i)] \cup \{u\}$ and*
(iii) *every vertex from $N(u) \setminus \{x,y\}$ is adjacent to a vertex from $N(y)$ and*
(iv) *$\bigcup_{j=1, j\neq i}^{p} V(N_j) \subseteq N(y)$.*

Proof. Suppose that there is $1 \leq i \leq p$ such that $|V(N_i) \cap N(y)| \leq 1$. We shall show that G has a $(1,2)$-PDS. If $|V(N_i) \cap N(y)| = 0$, then the set $D = V(G) \setminus V(N_i)$ is the $(1,2)$-PDS of G. If $|V(N_i) \cap N(y)| = 1$, then the set $D = V(N_i) \cup \{x\}$ is a $(1,2)$-PDS of G.

Suppose that for any $i \in \{1,\ldots,p\}$ we have $|V(N_i) \cap N(y)| \geq 2$ and there is $u \in V(N_i)$ such that conditions (i)–(iv) hold. Proving analogously as in Theorem 3.2 we show that the set $D = \{u,y\}$ is a $(1,2)$-PDS.

Conversely, let us assume that a graph G has a $(1,2)$-PDS, say D. Proving analogously as (a) in Theorem 3.2 we obtain that $D \neq \{x,y\}$. Moreover, we show the following claims.

(d) If $x, y \in D$, then $N(y) \subset D$.

Suppose on contrary that there is $u' \in N(y)$ such that $u' \notin D$. Then u is dominated by x and y simultaneously, so a vertex u' is $(1,1)$-dominated by D, a contradiction.

(e) For an arbitrary $i \in \{1, \ldots, p\}$ holds $V(N_i) \subset D$ or $|V(N_i) \cap D| \leq 1$.

If $|V(N_i)| = 1$, then (e) holds. Suppose that there is $D' \subseteq V(N_i)$, $D' \subset D$ and $|D'| \geq 2$. Clearly, $x \in D$ otherwise x is $(1,1)$-dominated by D. If $D' = V(N_i)$, then (e) holds. If $D' \subset V(N_i)$, then by the connectivity of N_i, it follows that there is $u' \in V(N_i)$ adjacent to $u'' \in D'$, so u' is dominated by u'' and x simultaneously. Consequently, u' is $(1,1)$-dominated, a contradiction.

From the fact that $D = \{x, y\}$, it follows that there exists a vertex $u \in V(N_i)$ such that $u \in D$. Then, based on (e), we must consider two cases: either $V(Ni) \subset D$ or exactly one vertex u belongs to D. In the first case, by (d), we obtain that $|V(N_i) \cap N(y)| \leq 1$. In the second case, when u is a unique vertex from $V(N_i)$ that belongs to D, we obtain that a set $\{u, y\}$ is a $(1,2)$-PDS and conditions (i)–(iv) are satisfied. Thus, the theorem is proved. □

Now we will consider cases when $|S_{\Delta(G)}| = 2$. We will distinguish cases when vertices x and y are adjacent or not and their neighborhoods are the same or different.

Theorem 3.4. *Let G be an n-vertex connected graph, $n \geq 5$ and $\delta(G) \geq 2$, $\Delta(G) = n - 2$, such that $S_{\Delta(G)} = \{x, y\}$ and x, y are not adjacent. A graph G has a $(1,2)$-PDS if and only if there is a vertex of degree 2 in G.*

Proof. From the fact that x and y are not adjacent, it follows that $N(x) = N(y)$. Let $n \geq 5$ and suppose that there is $u \in V(G)$ such that $d_G(u) = 2$. Clearly, $u \in V(G) \setminus S_{\Delta(G)}$. Then $D_1 = \{x, u\}$ and $D_2 = \{y, u\}$ are $(1,2)$-PDS of a graph G.

Suppose now that G has a $(1,2)$-PDS, say D. We shall show that there is the vertex $u \in V(G)$ such that $d_G(u) = 2$. Because D is a $(1,2)$-PDS, so $|D| \geq 2$ and either $x \notin D$ or $y \notin D$, otherwise D is $(1,1)$-dominating, a contradiction with the perfectness of G. Moreover, $|(V(G) \setminus \{x, y\}) \cap D| \leq 1$, otherwise vertices x and y are $(1,1)$-dominated by D, a contradiction. Without loose of the generality, suppose that $y \notin D$. Then $x \in D$ and there is a vertex $u \in V(G) \setminus \{x, y\}$ such that $u \in D$. If $d_G(u) \geq 3$, then there is $v \in V(G) \setminus \{x, y\}$ adjacent to u. Consequently, v is $(1,1)$-dominated by D, a contradiction. From the above, it follows that there is a vertex $u \in V(G) \setminus \{x, y\}$ such that $d_G(u) = 2$, which ends the proof. □

Theorem 3.5. *Let G be an n-vertex connected graph, $n \geq 6$ and $\delta(G) \geq 2$, $\Delta(G) = n - 2$, such that $S_{\Delta(G)} = \{x, y\}$, x, y are adjacent, $N[x] = N[y]$ and $R = N(x) \setminus \{y\}$. Let u be the unique vertex that is neither adjacent to x nor to y. A graph G has a $(1,2)$-PDS if and only if there is a vertex $v \in R$ such that every $v' \in R \setminus \{v\}$ is adjacent to either u or v and $N(u) \cap N(v) = \emptyset$ and*

(i) *$uv \in E(G)$ and $|N(u)| \geq 3$ or*
(ii) *$uv \notin E(G)$ and every vertex from $N(v) \setminus \{x, y\}$ is adjacent to a vertex from $N(u)$.*

Proof. Let G be an n-vertex connected graph, $n \geq 6$ and $\Delta(G) = n - 2$, such that $S_{\Delta(G)} = \{x, y\}$, x, y are adjacent and $N[x] = N[y]$. Let u be a unique vertex that is neither adjacent to x nor to y and $R = N(x) \setminus \{y\}$. Then there is no $v' \in R$ adjacent to every $v'' \in R \setminus \{v'\}$, otherwise $d_G(v') = n-2$ which is a contradiction with the cardinality of $S_{\Delta(G)}$. Assume that $v \in R$ such that every $v' \in R \setminus \{v\}$ is adjacent to either u or v and $N(u) \cap N(v) = \emptyset$ and (i) or (ii) are valid. We will prove that a set $D = \{u, v\}$ is a $(1,2)$-PDS of a graph G. Vertices x, y are dominated by v and for $u \in D$ we have $d_G(x,u) = d_G(y,u) = 2$, so x, y are $(1,2)$-PD by D. If a vertex $v' \in N(u)$, then v' is dominated by u and by connectivity of G there exists a path $v' - x - v$ in a graph G. Hence, v' is $(1,2)$-PD by D. If $v' \in N(v)$, then v' is dominated by v. If a condition (i) is valid, then there exists a path $v' - v - u$ in G and v' is $(1,2)$-PD by D. It is obvious that $d_G(v) \leq n-3$ because $|N(u)| \geq 3$ and $N(u) \cap N(v) = \emptyset$.

If a condition (ii) is valid, then there exists a vertex $v'' \in N(u)$ such that $v'v'' \in E(G)$. Hence, there is a path $v' - v'' - u$ in G and v' is $(1,2)$-PD by D. Because of the fact that $N(u) \cap N(v) = \emptyset$ vertices from the set $R \setminus \{v\}$ are not $(1,1)$-dominated.

Conversely, assume that G has a $(1,2)$-PDS, and we will denote it by D. Firstly we prove the following claims.

(1) $D \neq \{x, y\}$.

Assume that $D = \{x, y\}$. Then, by connectivity of G and $N[x] = N[y]$, every vertex from the set R is $(1,1)$-dominated, a contradiction.

(2) $u \in D$.

Assume that $u \notin D$. Then vertices x, y are not $(1,2)$-PD by D, because a vertex u is the only vertex in G such that $d_G(x,u) = d_G(y,u) = 2$, a contradiction.

(3) $x \notin D$ and $y \notin D$.

Assume that $x \in D$ and, by (2), $u \in D$. Then every vertex from the set $N(x) \setminus \{y\}$ is dominated by x. On the other hand there exists a vertex $v' \in R$ adjacent to $u \in D$. Hence, v' is $(1,1)$-dominated, a contradiction. Analogously we can prove that $y \notin D$.

(4) $|D \cap R| = 1$.

If $D \cap R = \emptyset$, then $D = \{x, y\}$ or $D = \{x, u\}$ or $D = \{y, u\}$, a contradiction with (1) and (3). If $|D \cap R| \geq 2$, then vertices x and y are $(1,1)$-dominated by $N[x] = N[y]$, a contradiction.

Above claims (1)–(4) imply that there exists a vertex $v \in R$ such that $v \in D$ and $d_G(v) \leq n-3$ because $v \notin S_{\Delta(G)}$. Therefore, $D = \{u, v\}$. It is obvious that vertices x and y are $(1,2)$-PD by D. The fact that D is a $(1,2)$-PDS follows that every vertex from $R \setminus \{v\}$ is adjacent to either u or v, otherwise there exists a vertex w such that either w is not dominated by D or w is $(1,1)$-dominated by D. Hence, $N(u) \cap N(v) = \emptyset$. By assumption that G is a connected graph and $\delta(G) \geq 2$, then a vertex u is adjacent to at least two vertices from the set R. If one of them is a vertex v, then u has to be adjacent to three vertices since $v \notin S_\Delta(G)$. Hence, the condition (i) is valid. If u is not adjacent to v, then every vertex from $N(v)$ has to be adjacent to a vertex from $N(u)$, by the fact that $D = \{u, v\}$ is the $(1,2)$-PDS. Consequently, the condition (ii) is valid. Thus, the theorem is proved. □

Theorem 3.6. *Let G be an n-vertex connected graph, $n \geq 5$ and $\delta(G) \geq 2$, $\Delta(G) = n-2$, such that $S_{\Delta(G)} = \{x, y\}$, x, y are adjacent and $N[x] \neq N[y]$. Let $u_1, u_2 \in V(G)$ are vertices such that $u_1 \in N(x) \setminus N[y]$ and $u_2 \in N(y) \setminus N[x]$. Let $R = V(G) \setminus \{x, y, u_1, u_2\}$ and no vertex in R is adjacent to all other vertices in this set. A graph G has a $(1,2)$-PDS if and only if*

(i) *every vertex* $v \in R$ *is adjacent to either* u_1 *or* u_2 *and* $N(u_1) \cap N(u_2) = \emptyset$ *or*
(ii) $N(u_1) = \{x, u_2\}$ *and* u_2 *is not adjacent to every* $u' \in R$ *or* $N(u_2) = \{y, u_1\}$ *and* u_1 *is not adjacent to every* $u' \in R$.

Proof. Let G be an n-vertex connected graph, $n \geq 5$ and $\Delta(G) = n - 2$, such that $S_{\Delta(G)} = \{x, y\}$, x, y are adjacent and $N[x] \neq N[y]$. Let $u_1, u_2 \in V(G)$ are vertices such that $u_1 \in N(x) \setminus N[y]$ and $u_2 \in N(y) \setminus N[x]$. Let $R = V(G) \setminus \{x, y, u_1, u_2\}$ and no vertex in R is adjacent to all other vertices in this set. Assume that the condition (i) or (ii) is valid. We shall show that a graph G has a $(1, 2)$-PDS. Assume that the condition (i) holds. We will prove that $D = \{u_1, u_2\}$ is a $(1, 2)$-PDS of G. Every vertex $v \in R$ is dominated by u_1 or u_2 and is not $(1, 1)$-dominated because $N(u_1) \cap N(u_2) = \emptyset$. If $v \in R$ is adjacent to u_1, then there exists a path $v - y - u_2$. Thus, v is $(1, 2)$-PD by D. It is obvious that x, y are $(1, 2)$-PD by D because x is adjacent to u_1 and y is adjacent to u_2. From the fact that $xy \in E(G)$ there exist paths $x - y - u_2$ and $y - x - u_1$. Now, assume that the condition (ii) holds. Then $u_1u_2 \in E(G)$. We will show that $D_1 = \{x, u_1\}$ or $D_2 = \{y, u_2\}$ is a $(1, 2)$-PDS of G. If $N(u_1) = \{x, u_2\}$, then every $v \in N(x) \setminus \{u_1\}$ is dominated by x and $d_G(v, u_1) = 2$. A vertex $u_2 \notin N(x)$ is dominated by u_1 and $d_G(u_2, x) = 2$. A vertex u_2 is not adjacent to every vertex from R, hence $d_G(u_2) \leq n - 3$. Analogously we can prove if $D_2 = \{y, u_2\}$ is a $(1, 2)$-PDS.

Conversely, assume that G has a $(1, 2)$-PDS. We will prove that conditions (i) or (ii) are valid. Let us denote by D a $(1, 2)$-PDS of G. Because of the assumption that $\delta(G) \geq 2$, it follows that $d_G(u_1) \geq 2$ and $d_G(u_2) \geq 2$. Hence, there exist vertices $v_1, v_2 \in V(G)$ such that $u_1v_1 \in E(G)$ and $u_2v_2 \in E(G)$. We have to consider two possibilities:

1. $v_1 = u_2$ and u_1 is adjacent only to two vertices x and u_2. Hence, $N(u_1) = \{x, u_2\}$. Assume that $N(u_2) \neq \{y, u_1\}$, hence there exists at least one vertex $v \in R$ adjacent to u_2. From the fact that $u_2 \notin S_{\Delta(G)}$, it follows that u_2 is not adjacent to every vertex from the set R. Firstly we prove necessary claims.

(5) $y \notin D$.

Assume that $y \in D$. Therefore, $x \notin D$, otherwise vertices from R are $(1, 1)$-dominated. It implies that $u_2 \in D$ because it is the only vertex such that $d_G(x, u_2) = 2$. But there exists a vertex $u^* \in R$ adjacent to y and u_2 simultaneously. Hence, u^* is $(1, 1)$-dominated, a contradiction.

(6) $u_1 \in D$.

A vertex u_1 is the only vertex in a graph G such that $d_G(u_1, y) = 2$. If $u_1 \notin D$, then y is not $(1, 2)$-PD by D, a contradiction.

(7) $|R \cap D| = 0$.

Assume that $|R \cap D| \neq 0$. Therefore, a vertex x is $(1, 1)$-dominated because x is adjacent to vertices from R and to $u_1 \in D$ by (6), a contradiction.

(8) $u_2 \notin D$.

If $u_2 \in D$, then $x \notin D$ otherwise y is $(1, 1)$-dominated. Because $u_2 \notin S_{\Delta(G)}$, then there exists a vertex $v^* \in R$ nonadjacent to u_2. So v^* is not dominated, a contradiction.

From above claims it follows that the only $(1, 2)$-PDS is a set $D_1 = \{u_1, x\}$. Analogously we can prove that $D_2 = \{u_2, y\}$ is the $(1, 2)$-PDS if $N(u_2) = \{u_1, y\}$ and u_1 is not adjacent to every vertex from R. Hence, condition (ii) is valid.

2. $N(u_1) \cap R \neq \emptyset$ and $N(u_2) \cap R \neq \emptyset$.

It is obvious that u_1 and u_2 are adjacent to vertices from the set R. Then $D \neq \{x, y\}$, otherwise $u' \in R$ is $(1,1)$-dominated.

(9) $D \neq \{x, v\}$ for $v \in N(x) \cup \{u_2\}$ and $D \neq \{y, z\}$ for $z \in N(y) \cup \{u_1\}$.

Assume that $D = \{x, v\}$ for $v \in N(x) \cup \{u_2\}$. By connectivity of G and the fact that $x \in S_{\Delta(G)}$ there exists a vertex $v' \in N(x)$ adjacent to v, so v' is $(1,1)$-dominated. If $v = u_2$, then by $N(u_2) \cap R \neq \emptyset$ there also exists a vertex $v^* \in R$ adjacent to u_2 and x, a contradiction. Analogously we can prove that $D \neq \{y, z\}$ for $z \in N(y) \cup \{u_1\}$.

From above considerations it is obvious that $D = \{u_1, u_2\}$. Vertices x, y are adjacent and $x, y \in S_{\Delta(G)}$ hence $d_G(x) = n - 2$ and $d_G(y) = n - 2$. It is clear that $x, y \notin D$, otherwise there is a vertex in R which is $(1,1)$-dominated. Therefore, $D = \{u_1, u_2\}$ is the only possible a $(1,2)$-PDS. Vertices x, y are $(1,2)$-PD because x is adjacent to u_1 and $d_G(x, u_2) = 2$ and y is adjacent to u_2 and $d_G(y, u_1) = 2$. Moreover, every vertex $u' \in R$ has to be adjacent either to u_1 or u_2 and $N(u_1) \cap N(u_2) = \emptyset$, otherwise there exists $u'' \in R$ which is $(1,1)$-dominated. Thus, the theorem is proved. □

CONCLUDING REMARKS

In future considerations, we can study cases when the maximum degrees of vertices will be $n - 3$ or when we increase the cardinality of the set $S_{\Delta(G)}$. Case studies show that solving these problems will not be immediate.

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