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POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH NONLINEAR NONLOCAL BOUNDARY CONDITIONS

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Abstract. We consider the existence of at least three positive solutions of a nonlinear first order problem with a nonlinear nonlocal boundary condition given by

$$x'(t) = r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)), \quad t \in [0,1],$$

$$\lambda x(0) = x(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0,1],$$

where $r : [0,1] \to [0,\infty)$ is continuous; the nonlocal points satisfy $0 \leq \tau_1 < \tau_2 < \ldots < \tau_n \leq 1$, the nonlinear function f_i and τ_j are continuous mappings from $[0,1] \times [0,\infty) \to [0,\infty)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ respectively, and $\lambda > 0$ is a positive parameter.

Keywords: positive solutions, Leggett-Williams fixed point theorem, nonlinear boundary conditions.

Mathematics Subject Classification: 34B08, 34B18, 34B15, 34B10.

1. INTRODUCTION

Consider the first order boundary value problem with a nonlinear nonlocal boundary condition

$$x'(t) = r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)), \quad t \in [0,1], \tag{1.1}$$

$$\lambda x(0) = x(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0,1], \tag{1.2}$$

where $r : [0,1] \to [0,\infty)$ is continuous, $f_i : [0,1]\times[0,\infty) \to [0,\infty)$, $\tau_j : [0,1]\times[0,\infty) \to [0,\infty)$ are continuous, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$, the nonlocal points satisfy $0 \le \tau_1 < \tau_2 < \ldots < \tau_n \le 1$, and the scalar λ satisfies

$$\lambda > \exp\Bigg(\int_0^1 r(\eta)d\eta\Bigg). \tag{1.3}$$

The motivation of this present paper has come from a recent paper due to Anderson [1], who used the Leggett-Williams multiple fixed point theorem [8] to establish three positive solutions of the boundary value problem (1.1) and (1.2). For completeness, we state here the statement of the theorem.

Assume that the nonlinear functions Λ_j satisfy

$$0 \le x\psi_j(t,x) \le \Lambda_j(t,x) \le x\Psi_j(t,x), \quad t \in [0,1],\ x \in [0,\infty) \tag{1.4}$$

for some positive continuous functions $\psi_j, \Psi_j : [0,1]\times[0,\infty) \to [0,\infty)$. Set

$$\beta_j = \max_{[0,1]\times[0,c]} \Psi_j(t,x) \quad \text{and} \quad \alpha_j = \min_{[0,1]\times[0,d]} \psi_j(t,x) \tag{1.5}$$

for some real constants c and d. Then, using the Leggett-Williams multiple fixed point theorem, Anderson proved the following theorem.

Theorem 1.1. *Suppose that* (1.4) *holds and the scalar* λ *satisfies*

$$\lambda > \Bigg(1 + \sum_{j=1}^{m} \beta_j\Bigg) \exp\Bigg(\int_0^1 r(\eta)d\eta\Bigg) > 1. \tag{1.6}$$

Further, suppose that there exist constants $0 < c_1 < c_2 < \lambda c_2 \le c_4$ *such that*

(F_1) $f_i(t,x) \le \frac{Mc_4}{m}$ *for* $t \in [0,1]$ *and* $x \in [0,c_4]$*;*

(F_2) $f_i(t,x) > \frac{Nc_2}{m}$ *for* $t \in [0,1]$ *and* $x \in [c_2,\lambda c_2]$*;*

(F_3) $f_i(t,x) < \frac{Mc_1}{m}$ *for* $t \in [0,1]$ *and* $x \in [0,c_1]$

for $i = 1,2,\ldots,m$*, where*

$$M = \frac{1}{\int_0^1 G(1,s)ds}\left[1 - \frac{\exp\Big(\int_0^1 r(\eta)d\eta\Big)\sum_{j=1}^{n}\beta_j}{\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)}\right] \tag{1.7}$$

and

$$N = \frac{1}{\int_0^1 G(0,s)ds}\left[1 - \frac{\sum_{j=1}^{n}\alpha_j}{\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)}\right]. \tag{1.8}$$

Then the boundary value problem (1.1) *and* (1.2) *has at least three positive solutions* x_1, x_2 *and* x_3 *satisfying* $\|x_1\| = x_1(1) < c_1$, $c_2 < \psi(x_2) = x_2(0)$, $\|x_3\| = x_3(1) > c_1$ *with* $\psi(x_3) = x_3(0) < c_2$.

We consider a Banach space $X = C[0,1]$ endowed with the sup norm. Define a cone K on X by

$$K = \{x \in X; x \geq 0 \text{ and } x \text{ increasing}\}, \tag{1.9}$$

and a nonnegative concave continuous functional ψ on K by

$$\psi(x) = \min_{t\in[0,1]} x(t) = x(0),\ x \in K. \tag{1.10}$$

Now, we state the main result of this paper.

Theorem 1.2. *Assume that* (1.3) *holds. Suppose that there exist three constants* $0 < c_1 < c_2 < \lambda c_2 \leq c_4$ *such that*

$$(H_1)\quad \lambda\sum_{i=1}^{m}\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) f_i(s,x(s))ds+\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j)) \leq \frac{\left(\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)\right)c_4}{\exp\left(\int_0^1 r(\eta)d\eta\right)}$$

for $\tau_j \in [0,1]$, $j = 1,2,\ldots,n$ *and* $x \in [0,c_4]$*;*

$$(H_2)\quad \sum_{i=1}^{m}\int_0^1 \exp\left(\int_s^1 r(\eta)d\eta\right) f_i(s,x(s))ds+\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j)) > c_2\left(\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)\right)$$

for $\tau_j \in [0,1]$, $j = 1,2,\ldots,n$ *and* $x \in [c_2,\lambda c_2]$*;*
and

$$(H_3)\quad \lambda\sum_{i=1}^{m}\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) f_i(s,x(s))ds+\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j)) < \frac{\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)c_1}{\exp\left(\int_0^1 r(\eta)d\eta\right)}$$

for $\tau_j \in [0,1]$, $j = 1,2,\ldots,n$ *and* $x \in [0,c_4]$ *hold. Then the boundary value problem* (1.1) *and* (1.2) *has at least three positive solutions* x_1, x_2 *and* x_3 *satisfying* $\|x_1\| = x_1(1) < c_1$, $c_2 < \psi(x_2) = x_2(0)$ *and* $\|x_3\| = x_3(1) > c_1$ *with* $\psi(x_3) = x_3(0) < c_4$.

We, now show that the conditions of Theorem 1.1 imply the conditions of Theorem 1.2. In fact, (1.6) implies (1.3). First, suppose that (1.4) and (F_1) hold, that is, $f_i(t,x) \leq \frac{Mc_4}{m}$ for $t \in [0,1]$ and $x \in [0,c_4]$. Then,

$$\begin{aligned}
&\lambda\sum_{i=1}^{m}\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) f_i(s,x(s))ds + \sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))\\
&\quad\leq \lambda\frac{Mc_4}{m}\sum_{i=1}^{m}\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) ds + c_4\sum_{j=1}^{n}\beta_j\\
&\quad\leq \lambda Mc_4\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) ds + c_4\sum_{j=1}^{n}\beta_j\\
&\quad= c_4\left[\lambda M\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) ds + \sum_{j=1}^{n}\beta_j\right].
\end{aligned}$$

Using the value of M, which is given in Theorem 1.1, the above inequality gives

$$\lambda\sum_{i=1}^{m}\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) f_i(s,x(s))ds+\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))$$

$$= c_4\left[\lambda\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) ds\frac{1}{\int_0^1 G(1,s)ds}\left(1-\frac{\exp\left(\int_0^1 r(\eta)d\eta\right)\sum_{j=1}^n\beta_j}{\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)}\right)+\sum_{j=1}^{n}\beta_j\right]$$

$$= c_4\left[\lambda\int_0^1 \exp\left(-\int_0^s r(\eta)d\eta\right) ds\frac{\left(\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)\right)}{\lambda\int_0^1\exp\left(\int_s^1 r(\eta)d\eta\right)ds}\right.$$

$$\left.\times\left(1-\frac{\exp\left(\int_0^1 r(\eta)d\eta\right)\sum_{j=1}^n\beta_j}{\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)}\right)+\sum_{j=1}^{n}\beta_j\right]$$

$$= c_4\left[\frac{\int_0^1\exp\left(-\int_0^s r(\eta)d\eta\right)ds}{\int_0^1\exp\left(\int_s^1 r(\eta)d\eta\right)ds}\left(\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)-\exp\left(\int_0^1 r(\eta)d\eta\right)\sum_{j=1}^{n}\beta_j\right)+\sum_{j=1}^{n}\beta_j\right]$$

$$= c_4\left[\frac{\int_0^1\exp\left(-\int_0^s r(\eta)d\eta\right)ds\exp\left(\int_0^1 r(\eta)d\eta\right)}{\int_0^1\exp\left(\int_s^1 r(\eta)d\eta\right)ds}\left(\frac{\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)}{\exp\left(\int_0^1 r(\eta)d\eta\right)}-\sum_{j=1}^{n}\beta_j\right)+\sum_{j=1}^{n}\beta_j\right]$$

$$= \frac{\lambda-\exp\left(\int_0^1 r(\eta)d\eta\right)}{\exp\left(\int_0^1 r(\eta)d\eta\right)}c_4,$$

which implies that (H_1) holds. In a similar way, we can show that if (1.4) and (F_3) hold, then (H_3) holds.

Finally, suppose that (F_2) holds. In addition, suppose that (1.4) holds. Then for $c_2 \le x \le \lambda c_2$, $t \in [0,1]$, using the value of N, given in Theorem 1.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{m} \int_0^1 \exp\left(\int_s^1 r(\eta)d\eta\right) f_i(s,x(s))ds + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)) \\
&> \frac{Nc_2}{m} \sum_{i=1}^{m} \int_0^1 \exp\left(\int_s^1 r(\eta)d\eta\right)ds + c_2 \sum_{j=1}^{n} \alpha_j \\
&= c_2\left[N \int_0^1 \exp\left(\int_s^1 r(\eta)d\eta\right) ds + \sum_{j=1}^{n} \alpha_j\right] \\
&= c_2\left[\int_0^1 \exp\left(\int_s^1 r(\eta)d\eta\right) ds \frac{1}{\int_0^1 G(0,s)ds}\left(1 - \frac{\sum_{j=1}^{n}\alpha_j}{\alpha - \exp\left(\int_0^1 r(\eta)d\eta\right)}\right) + \sum_{j=1}^{n} \alpha_j\right] \\
&= c_2\left[\frac{\int_0^1 \exp\left(\int_s^1 r(\eta)d\eta\right) ds \left(\lambda - \exp(\int_0^1 r(\eta)d\eta)\right)}{\int_0^1 \exp\left(\int_s^0 r(\eta)d\eta\right) \exp\left(\int_0^1 r(\eta)d\eta\right) ds}\left(1 - \frac{\sum_{j=1}^{n}\alpha_j}{\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)}\right)\right. \\
&\qquad \left. + \sum_{j=1}^{n} \alpha_j\right] \\
&= c_2\left[\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right) - \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{n}\alpha_j\right] \\
&= c_2\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right).
\end{aligned}$$

This shows that condition (1.4) and (F_2) implies condition (H_2). Thus, Theorem 1.2 provides a better condition than the condition given in Theorem 1.1.

Leggett-Williams multiple fixed point theorem has played an important role in establishing multiple positive periodic solutions of functional differential equations. For example, one may refer to [2, 4–7, 9–15] for first order ordinary and delay differential equations and refer to [3] for higher order equations. Once the problem is transformed into an equivalent integral operator, then it becomes easy to study the existence of fixed points of the operator using the Leggett-Williams multiple fixed point theorem, which is equivalent to establish the existence of positive solutions of the concerned problem. In a recent work, Padhi *et al.* [11], provide some simple way of applying the Leggett-Williams multiple fixed point theorem to a system of first order functional differential equations. In this paper, we have applied the same technique, and obtained Theorem 1.2. The detailed proof of Theorem 1.2 is given in Section 3.

2. PRELIMINARIES

Let X be a Banach space, K be a cone in X, and ψ be a nonnegative continuous functional on K. Further, let $a, b, c > 0$ be constants. Define

$$K_a = \{x \in K; \|x\| < a\}$$

and

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b, \|x\| \leq c\}.$$

Theorem 2.1 (Leggett-Williams multiple fixed point theorem, [8])**.** *Let $X = (X, \|\cdot\|)$ be a Banach space and $K \subset X$ be a cone, and $c_4 > 0$ be a constant. Suppose that there exists a concave nonnegative continuous function ψ on K with $\psi(x) \leq \|x\|$ for $x \in \overline{K}_{c_4}$ and let $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*

(i) *$\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$ and $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_3)$;*
(ii) *$\|Ax\| < c_1$ for all $x \in K_{c_1}$;*
(iii) *$\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3$.*

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{K}_{c_4}$. Furthermore, we have $x_1 \in \overline{K}_{c_1}$, $x_2 \in \{x \in K(\psi, c_2, c_4) : \psi(x) > c_2\}$, and $x_3 \in \overline{K}_{c_4} \setminus \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

3. PROOF OF THEOREM 1.2

Consider an operator $A : K \to X$ by

$$Ax(t) = \sum_{i=1}^{m} \int_0^1 G(t,s) f_i(s, x(s)) ds + \frac{\exp\left(\int_0^t r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)}, \tag{3.1}$$

where $G(t,s)$ is the Green's Kernel, given by

$$G(t,s) = \frac{\exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \times \begin{cases} \lambda & \text{if } 0 \leq s \leq t \leq 1, \\ \exp(\int_0^1 r(\eta) d\eta) & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2}$$

The operator A in (3.1) can be rewritten as

$$
\begin{aligned}
Ax(t) = & \sum_{i=1}^{m} \int_0^t \frac{\lambda \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_s^t r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
& + \sum_{i=1}^{m} \int_t^1 \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \exp\left(\int_s^t r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
& + \frac{\exp\left(\int_0^t r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)}.
\end{aligned} \tag{3.3}
$$

Let $x \in K$. Then the fact that f_i, $i = 1, 2, \ldots, m$ and Λ_j, $j = 1, 2, \ldots, n$ are positive imply that $Ax \geq 0$ for all $t \in [0, 1]$. We claim that fixed points of the operator A are the solutions of the boundary value problem (1.1) and (1.2). In fact, if $x = Ax$, then from (3.3), we have

$$
\begin{aligned}
\lambda x(0) - x(1) = & \sum_{i=1}^{m} \int_0^1 \frac{\lambda \exp\left(\int_s^1 r(s) ds\right)}{\left(\lambda - \exp\left(\int_0^1 r(s) ds\right)\right)} f_i(s, x(s)) ds \\
& + \lambda \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \\
& - \sum_{i=1}^{m} \int_0^1 \frac{\lambda \exp\left(\int_s^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} f_i(s, x(s)) ds \\
& - \frac{\exp\left(\int_0^1 r(\eta) d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)\right)} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1],\ j = 1, 2, \ldots, n.
\end{aligned}
$$

Hence, the boundary condition (1.2) is satisfied. Again differentiating (1.9) with respect to t, with $Ax = x$, we obtain

$$\begin{aligned}
x'(t) &= \sum_{i=1}^{m} \int_0^t \frac{\lambda r(t) \exp\left(\int_s^t r(\eta)d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} f_i(s, x(s))ds \\
&\quad + \sum_{i=1}^{m} \frac{\lambda}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} f_i(t, x(t)) \\
&\quad + \sum_{i=1}^{m} \int_t^1 \frac{r(t) \exp\left(\int_0^1 r(\eta)d\eta\right) \exp\left(\int_s^t r(\eta)d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} f_i(s, x(s))ds \\
&\quad - \sum_{i=1}^{m} \frac{\exp\left(\int_0^1 r(\eta)d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} f_i(t, x(t)) \\
&\quad + \frac{\exp\left(\int_0^t r(\eta)d\eta\right) r(t)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)) \\
&= r(t)\left[\sum_{i=1}^{m} \int_0^t \lambda \frac{\exp\left(\int_s^t r(\eta)d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} f_i(s, x(s))ds\right] \\
&\quad + r(t)\left[\sum_{i=1}^{m} \int_t^1 \frac{\exp\left(\int_0^1 r(\eta)d\eta\right) \exp\left(\int_s^1 r(\eta)d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} f_i(s, x(s))ds\right] \\
&\quad + r(t)\left[\frac{\exp\left(\int_0^t r(\eta)d\eta\right)}{\left(\lambda - \exp\left(\int_0^1 r(\eta)d\eta\right)\right)} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))\right] \\
&\quad + \sum_{i=1}^{m} f_i(t, x(t)) \\
&= r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)),
\end{aligned}$$

which shows that $x(t)$ satisfies (1.1). Moreover,

$$x'(t) = (Ax)'(t) = r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)), \quad t \in [0,1]$$

implies that x is increasing and $A : K \to K$. Further, one may verify that A is completely continuous. We, now show that all conditions of Theorem 2.1 are satisfied. In order to use Theorem 2.1, we use λc_2 in place of c_3. For $x \in K$, we have $x(0) = \psi(x) \leq \|x\| = x(1)$, since $x \in K$. Let $x \in \overline{K}_{c_4}$, that is, $\|x\| \leq c_4$. Then we obtain, using (H_1),

$$\begin{aligned}
&\|Ax\| = Ax(1)\\
&= \sum_{i=1}^{m} \int_0^1 G(1,s) f_i(s, x(s))ds + \frac{\exp\Big(\int_0^1 r(\eta)d\eta\Big) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{\Big(\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)\Big)}\\
&= \frac{\exp\Big(\int_0^1 r(\eta)d\eta\Big)}{\Big(\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)\Big)} \left[\lambda \sum_{i=1}^{m} \int_0^1 \exp\Big(-\int_0^s r(\eta)d\eta\Big) f_i(s, x(s))ds + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))\right]\\
&\leq c_4,
\end{aligned}$$

that is, $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$. In a similar way, using (H_3), one can prove condition (ii) of Theorem 2.1, that $A : \overline{K}_{c_1} \to K_{c_1}$.

In order to verify condition (i) of Theorem 2.1, we choose $x_k(t) = \lambda c_2$ for $t \in [0,1]$. Since $\psi(x_k(t)) = \min_{t\in[0,1]} x_k(t) = \lambda c_2 > c_2$; $c_2 \leq \psi(x)$, $\|x\| = \lambda c_2$, then the set $\{x \in K; c_2 \leq \psi(x), \|x\| \leq \lambda c_2\} \neq \emptyset$. Let $x \in K(\psi, c_2, c_3)$. Then $c_2 \leq \psi(x) \leq x \leq \|x\| = x(1) = \lambda c_2$ for $t \in [0,1]$, that is, $c_2 \leq x(t) \leq \lambda c_2$ for $t \in [0,1]$ holds. Hence

$$\begin{aligned}
&\psi(Ax) = Ax(0)\\
&= \sum_{i=1}^{m} \int_0^1 G(0,s) f_i(s, x(s))ds + \frac{\sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{\Big(\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)\Big)}\\
&= \sum_{i=1}^{m} \int_0^1 \frac{\exp\Big(\int_s^0 r(\eta)d\eta\Big) \exp\Big(\int_0^1 r(\eta)d\eta\Big)}{\Big(\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)\Big)} f_i(s, x(s))ds + \sum_{j=1}^{n} \frac{\Lambda_j(\tau_j, x(\tau_j))}{\Big(\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)\Big)}\\
&= \frac{1}{\Big(\lambda - \exp\Big(\int_0^1 r(\eta)d\eta\Big)\Big)} \left[\sum_{i=1}^{m} \int_0^1 \exp\Big(\int_s^1 r(\eta)d\eta\Big) f_i(s, x(s))ds + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))\right]
\end{aligned}$$

holds. Then, using (H_2), the above inequality yields that

$$\psi(Ax) > c_2 \text{ for } x \in \psi(K, c_2, c_3),\ t \in [0,1],$$

that is, condition (i) of Theorem 2.1 is satisfied.

Finally, suppose that $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > \lambda c_2$. Then

$$\psi(Ax) = Ax(0) \geq \frac{1}{\lambda} Ax(1) = \frac{\|Ax\|}{\lambda} = \frac{\lambda c_2}{\lambda} = c_2$$

implies that condition (iii) of Theorem 2.1 is satisfied. Consequently, the boundary value problem (1.1) and (1.2) has at least three positive solutions. This completes the proof of the theorem.

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