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New optical solitary waves for unstable Schrödinger equation in nonlinear medium

Qin ZHOU[1*], Hadi REZAZADEH[2*], Alper KORKMAZ[3], Mostafa ESLAMI[4], Mohammad MIRZAZADEH[5], Mohammadreza REZAZADEH[6]

[1]School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan, 430212, P.R. China

[2]Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran

[3]Department of Mathematics, Çankırı Karatekin University, Çankırı, Turkey

[4]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

[5]Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar-Vajargah, Iran

[6]Department of Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran

[*]Corresponding author: Q. Zhou – qinzhou@whu.edu.cn, H. Rezazadeh – rezazadehadi1363@gmail.com

In this paper and for the first time, we describe and introduce a new extended direct algebraic method which is a new method for solving nonlinear partial differential equations arising in nonlinear optics and nonlinear science. By applying this method, we have constructed new solitary wave solutions for the unstable Schrödinger equation. A large family of traveling wave type exact solutions covering exponential, generalized trigonometric, rational and generalized hyperbolic functions to this equation is determined. The solutions are expressed in explicit forms.

Keywords: solitons, new extended direct algebraic method, unstable Schrödinger equation.

1. Introduction

The difficulties to integrate the nonlinear partial differential equations (PDEs) force researcher to develop new techniques to determine solutions to them. From the bright idea that accepts $A\exp(mx)$ as a predicted solution and tries to determine A and m by substituting the predicted solution into the target equation and algebra manipulations (method of characteristics while solving ordinary differential equations, ODEs), various methods of solving the nonlinear PDEs.

One of the first techniques to be implemented to determine exact solutions of the nonlinear PDEs is a hyperbolic tangent method [1]. In this method, the predicted solution is assumed as a finite power series of some hyperbolic tangent function [2]. Determining the degree of the polynomial of the predicted solution is followed by substitution of it into the target equation. Then, the procedure continues in algebraic methods to find the relations among the parameters. The extended form of the hyperbolic tangent function method is also one of the pioneer approaches to solve nonlinear PDEs [3–5].

In the exp-function method, the predicted solution is assumed as a rational function of two finite expressions of exponential functions [6]. The numerator and the denominator are both some finite series. Periodic solutions are also determined by using the exp-function method [7].

Another efficient approach to solve the nonlinear PDEs is a method of generalized unified solutions. By this method the solutions are classified to be polynomials or rational functions [8–11]. We suppose that multi-wave polynomial solutions represent direct nonlinear interactions of basic waves, which are solutions of the auxiliary equations, while multi-waves rational solutions describe indirect nonlinear interactions of multi-waves.

The $\exp(-\Phi(\xi))$-expansion approach is one of recent methods to determine a large family of the solutions to nonlinear PDEs [12–14]. In the method, a formal solution in series forms of a particular exponential function satisfying some auxiliary ODE is used as a predicted solution. The procedure consists in homogeneous balance, the substitution of the predicted solution into the target equation and determining the relation among the parameters.

The method of tangent function expansion is also a significant tool to set the solutions of nonlinear PDEs [15]. In the method, the predicted solution is assumed as a finite power series of tangent function. In the related literature, variation techniques based on that method can be observed to solve plenty of nonlinear PDEs [16].

In this paper, first we describe and introduce a new method for solving nonlinear partial differential equations. We called it a new extended direct algebraic method. This method is an extended and general approach to solve nonlinear PDEs. In the method, the predicted solution is assumed as a finite powers series. Finally, after completing the implementation steps, plenty of exact solutions in various function families are determined explicitly. In fact, the methods summarized above can be reached by the appropriate choice of parameters in the predicted solution defined in a new extended direct algebraic method. The details are summarized in the next sections.

Then aim of this paper is to construct a traveling wave solution of the following unstable nonlinear Schrödinger equation [17, 18]

$$iu_t + u_{xx} + 2\lambda|u|^2 u - 2\gamma u = 0 \tag{1}$$

with the new extended direct algebraic method, which describes time evolution of disturbances in marginally stable or unstable media. MANAFIAN [17] have obtained more families of new exact solutions which contain soliton solutions, periodic solutions and rational solutions based on the tan($\Phi(\xi)/2$)-expansion method. The modified extended direct algebraic method was used by TALA-TEBUE *et al*. [18] to obtain dark solitons, bright solitons, solitary wave, periodic solitary wave and elliptic function solutions. DIANCHEN LU *et al*. [19] also obtained some new exact solutions using the exponential rational function and the new Jacobi elliptic function rational expansion method. ARSHAD *et al*. [20] have successfully proposed a modified extended mapping method and implemented it to construct the exact soliton and elliptic function solutions of the unstable nonlinear Schrödinger's equation.

2. The new extended direct algebraic method

In this section, we will outline the main steps of the new extended direct algebraic method [21].

Step 1. At first we consider a nonlinear partial differential equation of the form

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \tag{2}$$

By using the wave transformation $u(x, t) = U(\xi)$, $\xi = x - \theta t$, which can be converted to an ODE as the following form

$$G(U, U', U'', \ldots) = 0 \tag{3}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $Q(\xi)$ as follows

$$U(\xi) = \sum_{j=0}^{n} b_j Q^j(\xi), \qquad b_n \neq 0 \tag{4}$$

where b_j $(0 \leq j \leq n)$ are constant coefficients to be determined later and $Q(\xi)$ satisfies the ODE in the form

$$Q'(\xi) = \ln(A)\left[\alpha + \beta Q(\xi) + \sigma Q^2(\xi)\right], \qquad A \neq 0, 1 \tag{5}$$

the solutions of ODE (5) are as follows.

Family 1: when $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$Q_1(\xi) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \tan_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)$$

$$Q_2(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \cot_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)$$

$$Q_3(\xi) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\left[\tan_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right) \pm \sqrt{pq}\,\sec_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right)\right]$$

$$Q_4(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\left[\cot_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right) \pm \sqrt{pq}\,\csc_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\,\xi\right)\right]$$

$$Q_5(\xi) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4\sigma}\left[\tan_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4}\,\xi\right) - \cot_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4}\xi\right)\right]$$

Family 2: when $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$Q_6(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\tanh_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\,\xi\right)$$

$$Q_7(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\coth_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\,\xi\right)$$

$$Q_8(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\left[\tanh_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right) \pm i\sqrt{pq}\,\mathrm{sech}_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)\right]$$

$$Q_9(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\left[\coth_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right) \pm \sqrt{pq}\,\mathrm{csch}_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\,\xi\right)\right]$$

$$Q_{10}(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4\sigma}\left[\tanh_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}\,\xi\right) + \coth_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}\xi\right)\right]$$

Family 3: when $\alpha\sigma > 0$ and $\beta = 0$, then

$$Q_{11}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\,\tan_A\left(\sqrt{\alpha\sigma}\,\xi\right)$$

$$Q_{12}(\xi) = -\sqrt{\frac{\alpha}{\sigma}}\,\cot_A\left(\sqrt{\alpha\sigma}\,\xi\right)$$

$$Q_{13}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\left[\tan_A\left(2\sqrt{\alpha\sigma}\,\xi\right) \pm \sqrt{pq}\,\sec_A\left(2\sqrt{\alpha\sigma}\,\xi\right)\right]$$

$$Q_{14}(\xi) = -\sqrt{\frac{\alpha}{\sigma}}\left[\cot_A\left(2\sqrt{\alpha\sigma}\,\xi\right) \pm \sqrt{pq}\,\csc_A\left(2\sqrt{\alpha\sigma}\,\xi\right)\right]$$

$$Q_{15}(\xi) = \frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left[\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right) - \cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right]$$

Family 4: when $\alpha\sigma < 0$ and $\beta = 0$, then

$$Q_{16}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\,\tanh_A\left(\sqrt{-\alpha\sigma}\,\xi\right)$$

$$Q_{17}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\,\coth_A\left(\sqrt{-\alpha\sigma}\,\xi\right)$$

$$Q_{18}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\left[\tanh_A\left(2\sqrt{-\alpha\sigma}\,\xi\right) \pm i\sqrt{pq}\,\mathrm{sech}_A\left(2\sqrt{-\alpha\sigma}\,\xi\right)\right]$$

$$Q_{19}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\left[\coth_A\left(2\sqrt{-\alpha\sigma}\,\xi\right) \pm \sqrt{pq}\,\mathrm{csch}_A\left(2\sqrt{-\alpha\sigma}\,\xi\right)\right]$$

$$Q_{20}(\xi) = -\frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left[\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right) + \coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right]$$

Family 5: when $\beta = 0$ and $\sigma = \alpha$, then

$$Q_{21}(\xi) = \tan_A(\alpha\xi)$$

$$Q_{22}(\xi) = -\cot_A(\alpha\xi)$$

$$Q_{23}(\xi) = \tan_A(2\alpha\xi) \pm \sqrt{pq}\,\sec_A(2\alpha\xi)$$

$$Q_{24}(\xi) = -\cot_A(2\alpha\xi) \pm \sqrt{pq}\,\csc_A(2\alpha\xi)$$

$$Q_{25}(\xi) = \frac{1}{2}\left[\tan_A\left(\frac{\alpha}{2}\xi\right) - \cot_A\left(\frac{\alpha}{2}\xi\right)\right]$$

Family 6: when $\beta = 0$ and $\sigma = -\alpha$, then

$$Q_{26}(\xi) = -\tanh_A(\alpha\xi)$$

$$Q_{27}(\xi) = -\coth_A(\alpha\xi)$$

$$Q_{28}(\xi) = -\tanh_A(2\alpha\xi) \pm i\sqrt{pq}\,\mathrm{sech}_A(2\alpha\xi)$$

$$Q_{29}(\xi) = -\coth_A(2\alpha\xi) \pm \sqrt{pq}\,\mathrm{csch}_A(2\alpha\xi)$$

$$Q_{30}(\xi) = -\frac{1}{2}\left[\tanh_A\left(\frac{\alpha}{2}\xi\right) + \coth_A\left(\frac{\alpha}{2}\xi\right)\right]$$

Family 7: when $\beta^2 = 4\alpha\sigma$, then

$$Q_{31}(\xi) = \frac{-2\alpha(\beta\xi\ln(A) + 2)}{\beta^2\xi\ln(A)}$$

Family 8: when $\beta = \lambda$, $\alpha = m\lambda$ $(m \neq 0)$ and $\sigma = 0$, then

$$Q_{32}(\xi) = A^{\lambda\xi} - m$$

Family 9: when $\beta = \sigma = 0$, then

$$Q_{33}(\xi) = \alpha\xi\ln(A)$$

Family 10: when $\beta = \alpha = 0$, then

$$Q_{34}(\xi) = \frac{-1}{\sigma\xi\ln(A)}$$

Family 11: when $\alpha = 0$ and $\beta \neq 0$, then

$$Q_{35}(\xi) = -\frac{p\beta}{\sigma\left[\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + p\right]}$$

$$Q_{36}(\xi) = -\frac{\beta\left[\sinh_A(\beta\xi) + \cosh_A(\beta\xi)\right]}{\sigma\left[\sinh_A(\beta\xi) + \cosh_A(\beta\xi) + q\right]}$$

Family 12: when $\beta = \delta$, $\sigma = m\delta$ $(m \neq 0)$ and $\alpha = 0$, then

$$Q_{37}(\xi) = \frac{pA^{\delta\xi}}{q - mpA^{\delta\xi}}$$

In the above equations the generalized hyperbolic and triangular functions are defined as [22, 23]

$$\sinh_A(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{2}$$

$$\cosh_A(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{2}$$

$$\tanh_A(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}}$$

$$\coth_A(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}}$$

$$\mathrm{sech}_A(\xi) = \frac{2}{pA^{\xi} + qA^{-\xi}}$$

$$\mathrm{csch}_A(\xi) = \frac{2}{pA^{\xi} - qA^{-\xi}}$$

$$\sin_A(\xi) = \frac{pA^{i\xi} - qA^{-i\xi}}{2i}$$

$$\cos_A(\xi) = \frac{pA^{i\xi} + qA^{-i\xi}}{2}$$

$$\tan_A(\xi) = -i\,\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}}$$

$$\cot_A(\xi) = i\,\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}}$$

$$\sec_A(\xi) = \frac{2}{pA^{i\xi} + qA^{-i\xi}}$$

$$\csc_A(\xi) = \frac{2i}{pA^{i\xi} - qA^{-i\xi}}$$

where ξ is an independent variable, p and q are constants greater than zero and called deformation parameters.

Step 3. Determine the positive integer n in Eq. (4). This, usually, can be accomplished by balancing the linear term of highest order with the highest order nonlinear term (3), obtained in Step 2.

Step 4. Substitute Eq. (4) along with its required derivatives into Eq. (3) and compare the coefficients of powers of $Q(\xi)$ in resultant equation for obtaining the set of algebraic equations.

Step 5. By solving the overdetermined system of nonlinear algebraic equations by use of symbolic computation system Maple, we can get these unknowns $b_0, b_1, ..., b_n, \theta$.

3. Solutions to the unstable nonlinear Schrödinger equation

According to the method described in Section 2, using the travelling wave transformation [24–26]

$$u(x, t) = U(\xi)\exp(i\mu), \quad \xi = kx + \omega t, \quad \mu = \rho x + vt \tag{6}$$

we reduce Eq. (1) to the following second-order ordinary differential equation

$$k^2U'' - (\rho^2 + v + 2\gamma)U - 2\lambda U^3 = 0, \quad \omega = -2\rho k \tag{7}$$

Now, by balancing the highest order derivative term and the highest order nonlinear term in (7), we find $m = 1$. So, Eq. (1) has a formal solution of the form

$$u(\xi) = b_0 + b_1Q(\xi) \tag{8}$$

By substituting (8) into (7) and collecting all terms with the same order of $Q(\xi)$ together, the left-hand side of (7) is converted into polynomial in $Q(\xi)$. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for b_0, b_1 and v as follows:

$$Q^0(\xi):\quad k^2\ln^2(A)\alpha\beta b_1 - \rho^2 b_0 - vb_0 - 2\lambda b_0^3 - 2\gamma b_0 = 0$$

$$Q^1(\xi):\quad b_1\left[2k^2\ln^2(A)\alpha\sigma + k^2(Ln^2A)\beta^2 - \rho^2 - v - 2\gamma - 6\lambda b_0^2\right] = 0$$

$$Q^2(\xi):\quad 3k^2\ln^2(A)\beta\sigma b_1 - 6\lambda b_0 b_1^2 = 0$$

$$Q^3(\xi):\quad 2k^2\ln^2(A)\sigma^2 b_1 - 2\lambda b_1^3 = 0$$

Solving the above system of equations for b_0, b_1 and v, we obtain the following values:

$$b_0 = \pm\frac{1}{2}\frac{k\ln(A)\beta}{\sqrt{\lambda}} \tag{9a}$$

$$b_1 = \pm\frac{k\ln(A)\sigma}{\sqrt{\lambda}} \tag{9b}$$

$$v = \frac{k^2}{2}\ln^2(A)(4\alpha\sigma - \beta^2) - \rho^2 - 2\gamma \tag{9c}$$

From (9) and (6) and (8), we find the solutions of Eq. (1), as follows.
When $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then

$$u(x, t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{-\Delta}{\lambda}}\tan_A\left[\frac{\sqrt{-\Delta}}{2}k(x-2\rho t)\right] \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x, t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{-\Delta}{\lambda}}\cot_A\left[\frac{\sqrt{-\Delta}}{2}k(x-2\rho t)\right] \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x, t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{-\Delta}{\lambda}}\left\{\tan_A\left[\sqrt{-\Delta}\,k(x-2\rho t)\right] \pm \sqrt{pq}\sec_A\left[\sqrt{-\Delta}\,k(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x, t) = \mp\frac{k\ln(A)}{2}\sqrt{\frac{-\Delta}{\lambda}}\left\{\cot_A\left[\sqrt{-\Delta}\,k(x-2\rho t)\right] \pm \sqrt{pq}\csc_A\left[\sqrt{-\Delta}\,k(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x, t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{-\Delta}{\lambda}}\left\{\tan_A\left[\frac{\sqrt{-\Delta}}{4}k(x-2\rho t)\right] - \cot_A\left[\frac{\sqrt{-\Delta\sigma}}{4}k(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

where $\Delta = \beta^2 - 4\alpha\sigma$.
When $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then

$$u(x, t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{\Delta}{\lambda}}\tanh_A\left[\frac{\sqrt{\Delta}}{2}k(x-2\rho t)\right] \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{\Delta}{\lambda}}\coth_A\left[\frac{\sqrt{\Delta}}{2}k(x-2\rho t)\right] \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp\frac{k\ln(A)}{2}\sqrt{\frac{\Delta}{\lambda}}\left\{\tanh_A\left[\sqrt{\Delta}\,k(x-2\rho t)\right] \pm i\sqrt{pq}\,\mathrm{sech}_A\left[\sqrt{\Delta}\,k(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp\frac{k\ln(A)}{2}\sqrt{\frac{\Delta}{\lambda}}\left\{\coth_A\left[\sqrt{\Delta}\,k(x-2\rho t)\right] \pm \sqrt{pq}\,\mathrm{csch}_A\left[\sqrt{\Delta}\,k(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm\frac{k\ln(A)}{4}\sqrt{\frac{\Delta}{\lambda}}\left\{\tanh_A\left[\frac{\sqrt{\Delta}}{4}k(x-2\rho t)\right] + \coth_A\left[\frac{\sqrt{\Delta\sigma}}{4}k(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\Delta + \rho^2 + 2\gamma\right)t\right]\right\}$$

When $\alpha\sigma > 0$ and $\beta = 0$, then

$$u(x,t) = \pm k\ln(A)\sqrt{\frac{\sigma\alpha}{\lambda}}\tan_A\left[\sqrt{\alpha\sigma}\,k(x-2\rho t)\right] \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\sqrt{\frac{\sigma\alpha}{\lambda}}\cot_A\left[\sqrt{\alpha\sigma}\,k(x-2\rho t)\right] \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\sqrt{\frac{\sigma\alpha}{\lambda}}\left\{\tan_A\left[2k\sqrt{\alpha\sigma}\,(x-2\rho t)\right] \pm \sqrt{pq}\,\sec_A\left[2k\sqrt{\alpha\sigma}\,(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp k\ln(A)\sqrt{\frac{\sigma\alpha}{\lambda}}\left\{\cot_A\left[2k\sqrt{\alpha\sigma}\,(x-2\rho t)\right] \pm \sqrt{pq}\,\csc_A\left[2k\sqrt{\alpha\sigma}\,(x-2\rho t)\right]\right\}$$
$$\times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm\frac{k\ln(A)}{2}\sqrt{\frac{\sigma\alpha}{\lambda}}\left\{\tan_A\left[\frac{\sqrt{\alpha\sigma}}{2}k(x-2\rho t)\right] - \cot_A\left[\frac{\sqrt{\alpha\sigma}}{2}k(x-2\rho t)\right]\right\}$$
$$\times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

When $\alpha\sigma < 0$ and $\beta = 0$, then

$$u(x,t) = \pm k\ln(A)\sqrt{\frac{-\sigma\alpha}{\lambda}}\,\tanh_A\left[\sqrt{-\alpha\sigma}\,k(x-2\rho t)\right]$$
$$\times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\sqrt{\frac{-\sigma\alpha}{\lambda}}\,\coth_A\left[\sqrt{-\alpha\sigma}\,k(x-2\rho t)\right]$$
$$\times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\sqrt{\frac{-\sigma\alpha}{\lambda}}$$
$$\times\left\{\tanh_A\left[2k\sqrt{-\alpha\sigma}\,(x-2\rho t)\right] \pm i\sqrt{pq}\,\mathrm{sech}_A\left[2k\sqrt{-\alpha\sigma}\,(x-2\rho t)\right]\right\}$$
$$\times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp k\ln(A)\sqrt{\frac{-\sigma\alpha}{\lambda}}$$
$$\times\left\{\coth_A\left[2k\sqrt{-\alpha\sigma}\,(x-2\rho t)\right] \pm \sqrt{pq}\,\mathrm{csch}_A\left[2k\sqrt{-\alpha\sigma}\,(x-2\rho t)\right]\right\}$$
$$\times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha\sigma - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp\frac{k\ln(A)}{2}\sqrt{\frac{-\sigma\alpha}{\lambda}} \times\left\{\tanh_A\left[\frac{\sqrt{-\alpha\sigma}}{2}k(x-2\rho t)\right]+\coth_A\left[\frac{\sqrt{-\alpha\sigma}}{2}k(x-2\rho t)\right]\right\} \times\exp\left\{i\left[\rho x+\left(2k^2\ln^2(A)\alpha\sigma-\rho^2-2\gamma\right)t\right]\right\}$$

When $\beta = 0$ and $\sigma = \alpha$, then

$$u(x,t) = \pm k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\tan_A\left[k\alpha(x-2\rho t)\right] \times\exp\left\{i\left[\rho x+\left(2k^2\ln^2(A)\alpha^2-\rho^2-2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\cot_A\left[k\alpha(x-2\rho t)\right] \times\exp\left\{i\left[\rho x+\left(2k^2\ln^2(A)\alpha^2-\rho^2-2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\left\{\tan_A\left[2k\alpha(x-2\rho t)\right]\pm\sqrt{pq}\,\sec_A\left[2k\alpha(x-2\rho t)\right]\right\} \times\exp\left\{i\left[\rho x+\left(2k^2\ln^2(A)\alpha^2-\rho^2-2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\left\{\cot_A\left[2k\alpha(x-2\rho t)\right]\pm\sqrt{pq}\,\csc_A\left[2k\alpha(x-2\rho t)\right]\right\} \times\exp\left\{i\left[\rho x+\left(2k^2\ln^2(A)\alpha^2-\rho^2-2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm\frac{k\ln(A)}{2}\frac{\alpha}{\sqrt{\lambda}}\left\{\tan_A\left[\frac{k\alpha}{2}(x-2\rho t)\right]-\cot_A\left[\frac{k\alpha}{2}(x-2\rho t)\right]\right\} \times\exp\left\{i\left[\rho x+\left(2k^2\ln^2(A)\alpha^2-\rho^2-2\gamma\right)t\right]\right\}$$

When $\beta = 0$ and $\sigma = -\alpha$, then

$$u(x,t) = \pm k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\tanh_A\Big[k\alpha(x-2\rho t)\Big] \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha^2 - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\coth_A\Big[k\alpha(x-2\rho t)\Big] \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha^2 - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\left\{\tanh_A\Big[2k\alpha(x-2\rho t)\Big] \pm i\sqrt{pq}\,\mathrm{sech}_A\Big[2k\alpha(x-2\rho t)\Big]\right\} \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha^2 - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp k\ln(A)\frac{\alpha}{\sqrt{\lambda}}\left\{\coth_A\Big[2k\alpha(x-2\rho t)\Big] \pm \sqrt{pq}\,\mathrm{csch}_A\Big[2k\alpha(x-2\rho t)\Big]\right\} \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha^2 - \rho^2 - 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \mp\frac{k\ln(A)}{2}\frac{\alpha}{\sqrt{\lambda}}\left\{\tanh_A\left[\frac{k\alpha}{2}(x-2\rho t)\right] + \coth_A\left[\frac{k\alpha}{2}(x-2\rho t)\right]\right\} \times \exp\left\{i\left[\rho x + \left(2k^2\ln^2(A)\alpha^2 - \rho^2 - 2\gamma\right)t\right]\right\}$$

When $\beta^2 = 4\alpha\sigma$, then

$$u(x,t) = \pm\frac{k\ln(A)}{2\sqrt{\lambda}}\left[\beta + \frac{\beta k(x-2\rho t)\ln(A) + 2}{k(x-2\rho t)\ln(A)}\right]\exp\left[i\left(\rho x - (\rho^2 + 2\gamma)t\right)\right]$$

When $\beta = \alpha = 0$, then

$$u(x,t) = \pm\frac{1}{\sqrt{\lambda}\,(x-2\rho t)}\exp\left[i\left(\rho x - (\rho^2 + 2\gamma)t\right)\right]$$

When $\alpha = 0$ and $\beta \neq 0$, then

$$u(x,t) = \pm\frac{k\ln(A)\beta}{\sqrt{\lambda}}\left\{\frac{1}{2} - \frac{p}{\cosh_A\left[\beta k(x-2\rho t)\right] - \sinh_A\left[\beta k(x-2\rho t) + p\right]}\right\}$$
$$\times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\beta^2 + \rho^2 + 2\gamma\right)t\right]\right\}$$

$$u(x,t) = \pm\frac{k\ln(A)\beta}{\sqrt{\lambda}}\left\{\frac{1}{2} - \frac{\sinh_A\left[\beta k(x-2\rho t)\right] + \cosh_A\left[\beta k(x-2\rho t)\right]}{\sinh_A\left[\beta k(x-2\rho t)\right] + \cosh_A\left[\beta k(x-2\rho t)\right] + q}\right\}$$
$$\times \exp\left\{i\left[\rho x - \left(\frac{k^2}{2}\ln^2(A)\beta^2 + \rho^2 + 2\gamma\right)t\right]\right\}$$

Family 12: when $\beta = \delta$, $\sigma = m\delta$ $(m \neq 0)$ and $\alpha = 0$, then

$$u(x,t) = \pm\frac{k\ln(A)\delta}{\sqrt{\lambda}}\left\{\frac{1}{2} + m\left(\frac{pA^{k\delta(x-2\rho t)}}{q - mpA^{k\delta(x-2\rho t)}}\right)\right.$$
$$\left.\times \exp\left\{i\left[\rho x + \left(2\ln^2(A)k^2\delta^2 - \rho^2 - 2\gamma\right)t\right]\right\}\right\}$$

Remark: As far as we know, for the first time we describe and introduce the new extended direct algebraic method which is a new method for solving nonlinear partial differential equations. Thus, all the solutions of the unstable Schrödinger equation are new, which cannot be found in literature to our knowledge.

4. Conclusions

In this paper, we have succeeded to introduce, apply and describe the new extended direct algebraic method for solving unstable Schrödinger equation. A classical traveling wave transform was used to reduce the unstable Schrödinger equation to an ODE. The homogeneous balance procedure was implemented to determine the degree of the power series of the predicted solution. Then, the new extended direct algebraic method was applied and classical polynomial equation approach led to a system of equations. The solution of this system described the relations between the parameters used in the transform and the other parameters.

Thus, plenty of solutions in the traveling wave form have been constructed explicitly. The obtained solutions are of the forms of generalized hyperbolic, generalized trigonometric, exponential and rational functions. The method is the generalization of various techniques used in the related literature and can be used to the other nonlinear equations.

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