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ON SYSTEM RELIABILITY OF INCREASING MULTI-STATE LINEAR *k*-WITHIN-(*m,s*)-OF-(*m,n*):F LATTICE SYSTEM

ZWIĘKSZANIE NIEZAWODNOŚCI WIELOSTANOWYCH SYSTEMÓW LINIOWYCH TYPU *k*-W- (*m,s*) -Z- (*m,n*):F O STRUKTURZE KRATOWEJ

A "multi-state linear k-within-(m,s)-of-(m,n):F lattice system" (MS L(k,m,s,n:F)) comprises of m×n components, which are ordered in m rows and n columns. The state of system and components may be one of the following states: 0, 1, 2, …, H. The state of MS L(k,m,s,n:F) is less than j whenever there is at least one sub-matrix of the size m×s which contains k_l or more components that are in state less than l for all j ≤ l ≤ H. This system is a model for many applications, for example, tele communication, radar detection, oil pipeline, mobile communications, inspection procedures and series of microwave towers systems. In this paper, we propose new bounds of increasing MS L(k,m,s,n:F) reliability using second and third orders of Boole-Bonferroni bounds with i.i.d components. The new bounds are examined by previously published numerical examples for some special cases of increasing MS L(k,m,s,n:F). Also, illustration examples of modelling the system and numerical examples of new bounds are presented. Further, comparisons between the results of second and third orders of Boole-Bonferroni bounds are given.

Keywords: *network reliability, reliability engineering, structural reliability, system failure modelling, reliability optimization, probabilistic methods.*

"Wielostanowy system liniowy k-w- (m, s) -z- (m, n):F o strukturze kratowej" (MS L(k, m, s, n:F)) składa się z m × n elementów, uporządkowanych w m wierszach i n kolumnach. Stan systemu i elementów może być jednym z następujących stanów: 0, 1, 2, ..., H. Stan MS L (k, m, s, n: F) jest mniejszy niż j, gdy istnieje co najmniej jedna pod-matryca o rozmiarze m × s, która zawiera kl lub więcej elementów, które znajdują się w stanie mniejszym niż l dla wszystkich j ≤ l ≤ H. System ten stanowi model dla wielu zastosowań, na przykład w telekomunikacji, detekcji radarowej, rurociągach naftowych, komunikacji mobilnej, procedurach przeglądu oraz systemach wież radiolinii. W niniejszym artykule proponujemy nowe granice zwiększania niezawodności MS L (k, m, s, n: F) z wykorzystaniem drugiego i trzeciego stopnia nierówności Boole'a–Bonferroniego z niezależnymi elementami o jednakowym rozkładzie. Nowe granice omówiono na podstawie poprzednio publikowanych przykładów numerycznych dla niektórych szczególnych przypadków zwiększania MS L (k, m, s, n: F). Przedstawiono także przykłady ilustrujące modelowanie systemu oraz numeryczne przykłady nowych granic. Ponadto porównano wyniki uzyskane dla drugiego i trzeciego stopnia nierówności Boole'a–Bonferroniego.

Słowa kluczowe: *niezawodność sieci, inżynieria niezawodności, niezawodność konstrukcyjna, modelowanie uszkodzeń systemu, optymalizacja niezawodności, metody probabilistyczne.*

Notations

m,s,n,k,p_j	system parameters.
N	$= n - s + 1$.
H	highest state for the system and components.
k_j	minimum number of components that must be in state less than j in the sub matrix of the size $m\times s$, $j = 1, 2, \ldots, H$.
k_j^G	minimum number of components that must be in state greater than or equal j in the sub matrix of the size $m\times s$, $j = 1, 2, \ldots, H$; $k_j^G = (m\times s)-k_j+1$.
k	a vector of k_j-s.
k_j^G	a vector of k_j^G-s.
δ_j	number of components that are in state less than j inside the sub-matrix of the size $m\times s$.
$x_{i,j}$	the state of the component, which are located in the row i and the column j, $x_{i,j} \in \{0,1,\ldots,H\}$.
x	the states of all components, $x = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{pmatrix}$.
$\varphi(x)$	the structure function of the system, $\phi(x) \in \{0,1,2,\ldots,H\}$.
p_j	probability that state of the component is j, $\sum_{i=0}^{H} p_i = 1$.
P_j	probability that state of the component is greater than or equal j, $P_j = \sum_{i=j}^{H} p_i$.
Q_j	probability that state of the component is less than j, $Q_j = 1 - P_j$.
$A_{i,j}$	an event that at least k_l components in state less than l, $j \le l \le H$, of the sub-matrix with the size $m\times s$, that begin with the component (1, i) and end with the component (m, $i + s$ - 1).
μ	denote the random number of events among $A_{1,j}, A_{2,j}, \ldots, A_{N,j}$ which occur.

$S_{1,j}, S_{2,j}, S_{3,j}$: the binomial moments of μ.

$S_{1,j}$	$\sum_a \Pr(A_{a,j})$, for $1 \le a \le N$.
$S_{2,j}$	$\sum_{a,b} \Pr(A_{a,j}A_{b,j})$, for $1 \le a < b \le N$.
$S_{3,j}$	$\sum_{a,b,c} \Pr(A_{a,j}A_{b,j}A_{c,j})$, for $1 \le a < b < c \le N$.
R_j	probability that state of the system is greater than or equal j.
F_j	probability that state of the system is less than j, $F_j = 1 - R_j$.
$\lfloor z \rfloor$	the lower integer part for z.
E_j	maximum error of the estimation of R_j and F_j.
UB_j	the upper bound of R_j.
LB_j	the lower bound of R_j.
i.i.d	independent identically distributed.
MS	multi-state.
L(*k*,*m*,*s*,*n*:F)	linear *k*-within-(*m*,*s*)-of-(*m*,*n*):F lattice system
L(k^G,*m*,*s*,*n*:G)	linear *k*-within-(*m*,*s*)-of-(*m*,*n*):G lattice system

1. Introduction

A binary L(*k*,*m*,*s*,*n*:F) is a two dimensional grid. Its components have only the state 1 (operating) or state 0 (failed), and arranged in *m* rows and *n* columns. This system fails if at least one (*m*,*s*) sub-matrix of its components contains *k* or more failed components. Many papers studied its reliability, such as [13-16, 21, 23]. In the last few years, many systems generalized to MS systems, because the MS models give more limberness for modelling the equipment conditions. Such as, MS consecutive *k*-out-of-*n*:F system [9, 22, 24], MS *k*-out-of-*n*:F system [1, 10, 20], MS consecutive *k*-out-of-*r*-from-*n*: F system [8, 19] and MS L(*k*,*m*,*s*,*n*:F) [7]. In this paper, we study MS L(*k*,*m*,*s*,*n*:F). This system is a model for many applications. The system definition and illustration examples of modelling the system are given in section 2. The Boole-Bonferroni bounds are generalized in section 3, that will used for evaluation the proposed bounds. In section 4, the proposed bounds and an illustration example are given. The numerical results are presented in section 5.

2. The MS L(k,m,s,n:F)

The MS L(*k*,*m*,*s*,*n*:F) contains $m \times n$ components, that are ordered as a matrix of the degree $m \times n$. The possible states of MS L(*k*,*m*,*s*,*n*:F) and its components are: 0, 1, …,*H*. The state of MS L(*k*,*m*,*s*,*n*:F) is less than *j* whenever there is at least one sub-matrix of the size $m \times s$ which contains k_l or more components that are in state less than *l* for all $j \le l \le H$. In other words, $\phi(x) < j$ if at least one sub-matrix of the size $m \times s$ is in state less than *j*. The state of a sub-matrix of the size $m \times s$ is less than *j* if all the following inequalities are satisfied:

$$\delta_j \ge k_j,$$

$$\delta_{j+1} \ge k_{j+1},$$

$$\delta_{j+2} \ge k_{j+2},$$

$$\vdots$$

$$\delta_H \ge k_H.$$

The values of k vector, $k_1, k_2, \ldots, k_H$, categorize the MS L(*k*,*m*,*s*,*n*:F) to three cases:

Case1: When $k_1 \ge k_2 \ge \ldots \ge k_H$, the system is called a decreasing MS L(*k*,*m*,*s*,*n*:F). The exact reliability of decreasing MS L(*k*,*m*,*s*,*n*:F) evaluated in Ref. [7].

Case2: When $k_1 \le k_2 \le \ldots \le k_H$, the system is called an increasing MS L(*k*,*m*,*s*,*n*:F). In this case, that is more difficulty, new lower and upper bounds are proposed.

Case3: When $k_1 = k_2 = \ldots = k_H$, the system is called a constant MS L(*k*,*m*,*s*,*n*:F). This system is a special case of the increasing MS L(*k*,*m*,*s*,*n*:F) and decreasing MS L(*k*,*m*,*s*,*n*:F).

As with the binary system, the MS L(*k*,*m*,*s*,*n*:F) and the MS L(k^G,*m*,*s*,*n*:G) are considered as mirror images of each other. Further, the decreasing MS L(*k*,*m*,*s*,*n*:F) is an increasing MS L(k^G,*m*,*s*,*n*:G). The following examples illustrate this system.

Example 1:

A decreasing MS linear (k_1 = 4, k_2 = 3, k_3 = 2)-within-(2,2)-of-(2,4):F lattice system, which is an increasing MS linear ($k_1^G = 1, k_2^G = 2, k_3^G = 3$)-within-(2,2)-of-(2,4):G lattice system, consists of 8 components, that arranged in 2 rows and 4 columns. This system contains 3 sub-matrices of the degree 2×2. The state of any one of them is:

- less than 1, if $\delta_1 \ge 4$, $\delta_2 \ge 3$ and $\delta_3 \ge 2$,
- less than 2, if $\delta_2 \ge 3$ and $\delta_3 \ge 2$,
- less than 3, if $\delta_3 \ge 2$.

For state 1:

$\phi(x) < 1$, if at least one sub-matrix of the degree 2×2 is in state less than 1. For example, when $x = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix}$:

- The state of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is less than 1, such that $\delta_1 = \delta_2 = \delta_3 = 4$.
- The state of $\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}$ is less than 3, such that $\delta_1 = \delta_2 = 2$ and $\delta_3 = 3$.
- The state of $\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$ is less than 3, such that $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_3 = 2$.

Then $\phi\begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix} < 1$. Similarly, $\phi\begin{pmatrix} 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix} < 1$, $\phi\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} < 1$, $\phi\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} < 1$, etc.

For state 2:

$\phi(x) < 2$, if at least one sub-matrix of the degree 2×2 is in state less than 2. For example, when $x = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}$:

• The state of $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ is less than 3, such that $\delta_1 = 1$, $\delta_2 = 2$ and $\delta_3 = 3$.

• The state of $\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$ is less than 2, such that $\delta_1 = 0$, $\delta_2 = \delta_3 = 3$.

• The state of $\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$ is less than 2, such that $\delta_1 = 1$, $\delta_2 = \delta_3 = 3$.

Then $\phi\begin{pmatrix} 0 & 1 & 1 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix} < 2$. Similarly, $\phi\begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix} < 2$, $\phi\begin{pmatrix} 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} < 2$, $\phi\begin{pmatrix} 3 & 3 & 3 & 1 \\ 3 & 3 & 1 & 1 \end{pmatrix} < 2$, etc.

For state 3:

$\phi(x) < 3$, if at least one sub-matrix of the degree 2×2 is in state less than 3. For example, when $x = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix}$:

• The state of $\begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}$ is 3, such that $\delta_1 = 0$, $\delta_2 = \delta_3 = 1$.

• The state of $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$ is less than 3, such that $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_3 = 2$.

• The state of $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ is less than 3, such that $\delta_1 = 1$, $\delta_2 = 2$ and $\delta_3 = 4$.

Then $\phi\begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix} < 3$. Similarly, $\phi\begin{pmatrix} 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 0 \end{pmatrix} < 3$, $\phi\begin{pmatrix} 3 & 2 & 3 & 3 \\ 2 & 1 & 3 & 3 \end{pmatrix} < 3$, $\phi\begin{pmatrix} 3 & 2 & 3 & 3 \\ 2 & 1 & 2 & 0 \end{pmatrix} < 3$, etc.

Example 2:

An increasing MS linear (k_1=2, k_2=3, k_3=4)-within-(2,2)-of-(2,4):F lattice system, that is a decreasing MS linear ($k_1^G = 3, k_2^G = 2, k_3^G = 1$)-within-(2,2)-of-(2,4):G lattice system, consists of 8 components, that arranged in 2 rows and 4 columns. This system contains 3 sub-matrices of the degree 2×2. The state of any one of them is:

• less than 1, if $\delta_1 \geq 2$, $\delta_2 \geq 3$ and $\delta_3 \geq 4$,

• less than 2, if $\delta_2 \geq 3$ and $\delta_3 \geq 4$,

• less than 3, if $\delta_3 \geq 4$.

For state 1:

$\phi(x) < 1$, if at least one sub-matrix of the degree 2×2 is in state less than 1. For example, when $x = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}$:

• The state of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is less than 1, such that $\delta_1 = 2$ and $\delta_2 = \delta_3 = 4$.

• The state of $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ is less than 2, such that $\delta_1 = 1$, $\delta_2 = 3$ and $\delta_3 = 4$.

• The state of $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ is less than 3, such that $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_3 = 4$.

Then $\phi\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix} < 1$. Similarly, $\phi\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} < 1$, $\phi\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} < 1$, $\phi\begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} < 1$, etc.

For state 2:

$\phi(x) < 2$, if at least one sub-matrix of the degree 2×2 is in state less than 2. For example, when $x = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 \end{pmatrix}$:

• The state of $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ is less than 2, such that $\delta_1 = 1$, $\delta_2 = 3$ and $\delta_3 = 4$.

• The state of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is less than 2, such that $\delta_1 = 0$ and $\delta_2 = \delta_3 = 4$.

• The state of $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ is 3, such that $\delta_1 = 0$ and $\delta_2 = \delta_3 = 3$.

Then $\phi\begin{pmatrix} 0 & 1 & 1 & 3 \\ 2 & 1 & 1 & 1 \end{pmatrix} < 2$. Similarly, $\phi\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 2 & 0 & 0 \end{pmatrix} < 2$, $\phi\begin{pmatrix} 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix} < 2$, $\phi\begin{pmatrix} 3 & 3 & 1 & 1 \\ 3 & 3 & 1 & 1 \end{pmatrix} < 2$, etc.

For state 3:

$\phi(x) < 3$, if at least one sub-matrix of the degree 2×2 is in state less than 3. For example, when $x = \begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 2 & 0 \end{pmatrix}$:

• The state of $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ is 3, such that $\delta_1 = 0$, $\delta_2 = 1$ and $\delta_3 = 3$.

• The state of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is less than 3, such that $\delta_1 = 0$, $\delta_2 = 2$ and $\delta_3 = 4$.

• The state of $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$ is 3, such that $\delta_1 = 1$, $\delta_2 = 2$ and $\delta_3 = 3$.

Then $\phi\begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 2 & 0 \end{pmatrix} < 3$. Similarly, $\phi\begin{pmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 2 & 0 \end{pmatrix} < 3$, $\phi\begin{pmatrix} 2 & 1 & 2 & 0 \\ 2 & 3 & 2 & 1 \end{pmatrix} < 3$, $\phi\begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 2 & 2 \end{pmatrix} < 3$, etc.

Example 3: *A Surveillance Cameras System*

Given, a surveillance cameras system consists of 20 cameras that arranged in 4 rows and 5 columns. This system has 4 different surveillance levels:

- Good surveillance (state 3).
- Medium surveillance (state 2).
- Low surveillance (state 1).
- Non surveillance (state 0).

Each camera also has 4 different surveillance levels:

- Good surveillance, in the first time (state 3).
- Medium surveillance, after some time (state 2).
- Low surveillance, after more time (state 1).
- Non surveillance, the camera not works (state 0).

Then:

- The system state is less than 1, if at least one sub-matrix of the degree 4×3 contains at least 6 components in state less than 1.
- The system state is less than 2, if at least one sub-matrix of the degree 4×3 contains at least 4 components in state less than 2.
- The system state is less than 3, if at least one sub-matrix of the degree 4×3 contains at least 2 components in state less than 3.

We can represent such a system by a decreasing MS linear (6,4,2)-within-(4,3)-of-(4,5):F lattice system (or an increasing MS linear ($k_1^G = 7, k_2^G = 9, k_3^G = 11$)-within-(4,3)-of-(4,5):G lattice system).

Example 4: *A Radar Detection System*

Given, a radar detection system consists of 25 radar stations that arranged in 5 rows and 5 columns. This system has 4 different surveillance levels:

- Good detection (state 3).
- Medium detection (state 2).
- Low detection (state 1).
- Non detection (state 0).

Each station also has 4 different detection levels:

- Good detection, in the first time (state 3).
- Medium detection, after some time (state 2).
- Low detection, after more time (state 1).
- Non detection, the station not works (state 0).

Then:

- The system state is less than 1, if at least one sub-matrix of the degree 5×3 contains at least 3 components in state less than 1, at least 5 components in state less than 2 and at least 7 components in state less than 3.
- The system state is less than 2, if at least one sub-matrix of the degree 5×3 contains at least 5 components in state less than 2 and at least 7 components in state less than 3.
- The system state is less than 3, if at least one sub-matrix of the degree 5×3 contains at least 7 components in state less than 3.

We can represent such a system by an increasing MS linear (3,5,7)-within-(5,3)-of-(5,5):F lattice system (or a decreasing MS linear ($k_1^G = 13, k_2^G = 11, k_3^G = 9$)-within-(5,3)-of-(5,5):G lattice system).

3. Generalization of Boole-Bonferroni Bounds

The technique of Boole-Bonferroni bounds was derived by Prékopa and Boros [2, 17], and improved by many papers such as [2-5, 11, 12, 17]. This technique depends on the solution of the linear programming problem according to the definition of the binomial moments.

Let μ denote the random number of the events among $A_{1,j}, A_{2,j}, \dots, A_{N,j}$ which occur. Then:

$$S_{i,j} = E\left[\binom{\mu}{i}\right] = \sum_{l=1}^{N} \binom{l}{i} b_l, \qquad i = 1,2,\dots,N \quad (1)$$

where $b_l = \Pr(\mu = l)$ and $\binom{l}{i} = 0$, if $i > l$.

The proof of the definition of the expected value in formulae (1) can be found by Prékopa [18]. The value $S_{i,j}$ is called the ith binomial moment of μ.

If we take $b_1, b_2 \dots, b_N$ as variables and compute $S_{1,j}, S_{2,j}, \dots, S_{V,j}$; $V \prec N$, then we have two linear programming problems as follow:

$$\text{Minimize } \{b_1 + b_2 + \dots + b_V + \dots + b_N\} \quad (2)$$

Subject to:

$$\begin{aligned} b_1 + \binom{2}{1} b_2 + \dots + \binom{V}{1} b_V + \dots + \binom{N}{1} b_N &= S_{1,j} \\ b_2 + \dots + \binom{V}{2} b_V + \dots + \binom{N}{2} b_N &= S_{2,j} \\ \ddots \ddots \vdots \\ b_V + \dots + \binom{N}{V} b_N &= S_{V,j} \end{aligned}$$

$$b_1 \ge 0, b_2 \ge 0, \dots, b_V \ge 0, \dots, b_N \ge 0$$

$$\text{Maximize } \{b_1 + b_2 + \dots + b_V + \dots + b_N\} \quad (3)$$

Subject to:

$$\begin{aligned} b_1 + \binom{2}{1} b_2 + \dots + \binom{V}{1} b_V + \dots + \binom{N}{1} b_N &= S_{1,j} \\ b_2 + \dots + \binom{V}{2} b_V + \dots + \binom{N}{2} b_N &= S_{2,j} \\ \ddots \ddots \vdots \\ b_V + \dots + \binom{N}{V} b_N &= S_{V,j} \end{aligned}$$

$$b_1 \ge 0, b_2 \ge 0, \dots, b_V \ge 0, \dots, b_N \ge 0$$

The solutions of these problems give us the best possible lower and upper bounds respectively on the value of

$$\Pr(\mu \ge 1) = \Pr(A_{1,j} + \dots + A_{1,N}) = F_j \quad (4)$$

These bounds are called Boole-Bonferroni bounds. In the following, we give the known explicit solutions of the linear programming problems for $V = 2$ (the second order) and $V = 3$ (the third order).

3.1. The Second Order of Boole-Bonferroni Bounds:

By putting V=2 in the aforesaid linear programming problems and calculation $S_{1,j}$ and $S_{2,j}$, j = 1, 2, 3,…, H, then the lower bound of F_j is:

$$F_j \ge \frac{2}{u_j + 1} S_{1,j} - \frac{2}{u_j(u_j + 1)} S_{2,j}. \quad (5)$$

Where:

$$u_j = 1 + \left\lfloor \frac{2S_{2,j}}{S_{1,j}} \right\rfloor \quad (6)$$

And the upper bound of F_j is:

$$F_j \le S_{1,j} - \frac{2}{N} S_{2,j}. \quad (7)$$

3.2. The Third Order of Boole-Bonferroni Bounds:

By putting V=3 in the aforesaid linear programming problems and calculation $S_{1,j}$, $S_{2,j}$ and $S_{3,j}$, j = 1, 2, 3,…, H, then the lower bound of F_j is:

$$F_j \ge \frac{1}{N(\lambda_j+1)}\left[(\lambda_j+2N-1)S_{1,j} - \frac{2(2\lambda_j+N-2)}{\lambda_j}S_{2,j} + \frac{6}{\lambda_j}S_{3,j}\right] \quad (8)$$

Where:

$$\lambda_j = 1 + \left\lfloor \frac{-6S_{3,j}+2(N-2)S_{2,j}}{-2S_{2,j}+(N-1)S_{1,j}} \right\rfloor \quad (9)$$

And the upper bound of F_j is:

$$F_j \le Min\left\{1, S_{1,j} - \frac{2}{\omega_j(\omega_j+1)}\left[(2\omega_j-1)S_{2,j} - 3S_{3,j}\right]\right\} \quad (10)$$

Where:

$$\omega_j = 2 + \left\lfloor \frac{3S_{3,j}}{S_{2,j}} \right\rfloor \quad (11)$$

4. System Reliability of the Increasing MS L(*k*,*m*,*s*,*n*:F)

From the definition of $A_{i,j}$, $i = 1,2,...,N$, the increasing MS L(k,m,s,n:F) is in state less than j, if at least one event $A_{i,j}$, $i = 1,2,...,N$, occurred. Then:

$$F_j = \Pr\left\{\bigcup_{i\in\Omega} A_{i,j}\right\} \text{ for all } \Omega = \{1,2,...,N\} \quad (12)$$

Calculation F_j in equation (12) is very difficult, so we will propose an approximation for lower and upper bounds of increasing MS L(k,m,s,n:F) using Boole-Bonferroni bounds. Calculation these bounds required the knowledge of $S_{1,j}$, $S_{2,j}$ and $S_{3,j}$, that will be suggested in the following sections. Further, we can have the lower bounds and upper bounds of R_j as follows:

LBj = 1- (the upper bounds of F_j), (13)

UBj = 1- (the lower bound of F_j). (14)

Estimation R_j by one value can be given by the following formula:

$$\hat{R}_j = \frac{LB_j + UB_j}{2}. \quad (15)$$

The maximum error is:

$$E_j = UB_j - \hat{R}_j = \hat{R}_j - LB_j. \quad (16)$$

4. 1. Calculation the Binomial Moment $S_{1,j}$

The binomial moment $S_{1,j}$ can be given by:

$$\begin{aligned} S_{1,j} &= \Pr(A_{1,j}) + \Pr(A_{2,j}) + ... + \Pr(A_{N,j}) \\ &= N \times \Pr(A_{1,j}) \end{aligned} \quad (17)$$

Where:

$$\Pr(A_{1,j}) = \sum_{y=k_j}^{m\times s} \binom{m\times s}{y} Q_j^y \beta_j(m\times s, y), \quad (18)$$

$$\beta_j(m\times s, y) = \prod_{e=0}^{M-j-1} \sum_{i_e=0}^{\min(m\times s-y-I_e,\, m\times s-k_{M-e}-I_e)} \binom{m\times s-y-I_e}{i_e} \times p_{M-e}^{i_e} . p_j^{m\times s-y-I_{M-j}}, \quad j < H \quad (19)$$

$$\beta_M(m\times s, y) = p_M^{m\times s-y}, \quad (20)$$

$$I_0 = 0,\ I_a = I_a(i_0, i_1, ..., i_{a-1}) = \sum_{b=0}^{a-1} i_b, \text{ for } a = 1, 2,…, H - j. \quad (21)$$

4. 2. Calculation the Binomial Moment $S_{2,j}$

The binomial moment $S_{2,j}$ can be given by:

$$\begin{aligned} S_{2,j} &= \sum\nolimits_{a,b} \Pr(A_{a,j}A_{b,j}),\ 1 \le a < b \le N. \\ &= \sum_{a=1}^{N-1}\sum_{b=a+1}^{N} \Pr(A_{a,j}A_{b,j}) \\ &= \sum_{t=2}^{N} (N-t+1)\times \Pr(A_{1,j}A_{t,j}) \end{aligned} \quad (22)$$

The $\Pr(A_{a,j}A_{b,j})$, $1 \le a < b \le N$, can be calculated through the following two cases:

Case 1: If b-a > s-1, then:

$$\Pr(A_{a,j}A_{b,j}) = \left(\Pr(A_{a,j})\right)^2. \quad (23)$$

Case 2: If b-a ≤ s-1, then:

$$\Pr(A_{a,j}A_{b,j}) = \sum_{y_1=t_1}^{m_1}\binom{m_1}{y_1} Q_j^{y_1}.\Psi_j(m_1, y_1).\left[\sum_{y_2=t_2}^{m_2}\binom{m_2}{y_2} Q_j^{y_2}.\Psi'_j(m_2, y_2, I_{M-j})\right]^2, \quad (24)$$

$$\Psi_j(m_1,y_1)=\prod_{e=0}^{M-j-1}\sum_{i_e=0}^{m_3}\binom{m_1-y_1-I_e}{i_e}p_{M-e}^{i_e}.p_j^{m_1-y_1-I_{M-j}}\text{, j<M,} \quad (25)$$

$$\Psi_M(m_1,y_1)=p_M^{m_1-y_1}, \quad (26)$$

$$\Psi'_j(m_2,y_2,I_{M-j})=\prod_{g=0}^{M-j-1}\sum_{i'_g=0}^{m_4}\binom{m_2-y_2-I'_g}{i'_g}p_{M-g}^{i'_g}.p_j^{m_2-y_2-I'_{M-j}}, j<M, \quad (27)$$

$$\Psi'_M(m_2,y_2,I_0)=p_M^{m_2-y_2}, \quad (28)$$

$$I'_0=0\,,\; I'_a=I'_a(i'_0,i'_1,\ldots,i'_{a-1})=\sum_{b=0}^{a-1}i'_b \text{ , for } a=1,2,\ldots,M\text{-}j, \quad (29)$$

$t_1=\max(0,k_j+m(a-b))$, $\quad m_1=m(s-b+a)$,

$t_2=\max(0,k_j-y_1)$, $\quad m_2=m(b-a)$,

$m_3=\min(m_1-y_1-I_e, m\times s-k_{M-e}-I_e)$,

$m_4=\min(m_2-y_2-I'_g, m\times s-k_{M-g}-I_{g+1}-I'_g,\ldots,m\times s-k_{j+1}-I_{M-j}-I'_g)$.

4. 3. Calculation the Binomial Moment $S_{3,j}$

The binomial moment $S_{3,j}$ can be given by:

$S_{3,j}=\sum_{a,b,c}\Pr(A_{a,j}A_{b,j}A_{c,j})$, for all $1\le a<b<c\le N$.

$$=\sum_{a=1}^{N-2}\sum_{b=a+1}^{N-1}\sum_{c=b+1}^{N}\Pr(A_{a,j}A_{b,j}A_{c,j}) \quad (30)$$

The $\Pr(A_{a,j}A_{b,j}A_{c,j})$, $1\le a<b<c\le N$, can be calculated through the following five cases:

Case 1: *c-a* ≤ *s*-1

In this case, all the events $A_{a,j}$, $A_{b,j}$ and $A_{c,j}$ have common components. The number of common components between the events $A_{a,j}$, $A_{b,j}$ and $A_{c,j}$ is $m\times(s+a-c)$ components. Then:

1- When *j* < *M*,

$$\Pr(A_{a,j}A_{b,j}A_{c,j})=\prod_{e=1}^{5}\sum_{x_e=t_e}^{m_e}\binom{m_e}{x_e}Q_j^{x_e}.\left[\prod_{L=0}^{M-j-1}\sum_{d_{e,L}=0}^{g_{e,L}}\binom{m_e-x_e-D_{e,L}}{d_{e,L}}p_{M-L}^{d_{e,L}}\right].p_j^{m_e-x_e-D_{e,M-j}} \quad (31)$$

2- When *j* = *M*,

$$\Pr(A_{a,j}\,A_{b,j}A_{c,j})=\prod_{e=1}^{5}\sum_{x_e=t_e}^{m_e}\binom{m_e}{x_e}Q_j^{x_e}.p_j^{m_e-x_e}, \quad (32)$$

where:

$t_1=\max(0,\ k_j+m(a-c))$, $\quad m_1=m(s+a-c)$,

$t_2=\max(0,\ k_j-x_1+m(a-b))$, $\quad m_2=m_5=\mathrm{m}(c-b)$

$t_3=t_4=\max(0,\ k_j-x_1-x_2)$, $\quad m_3=m_4=m(b-a)$

$t_5=\max(0,k_j-x_1-x_3)$

$d_{e,M-j}=m_e-x_e-D_{e,M-j}$, $\quad D_{i,L}=\sum_{y=0}^{L-1}d_{i,y}$, $\quad i=1, 2, \ldots, 5$

$$g_{1,L}=\mathrm{Min}\left(m_1-x_1-D_{1,L},\ m\times s-k_{M-L}-D_{1,L}\right)$$

$$g_{2,L}=\mathrm{Min}\left(m_2-x_2-D_{2,L},\ m\times s-k_{M-L}-D_{2,L}-D_{1,L+1},\ldots,m\times s-k_j-D_{2,L}-D_{1,M-j+1}\right)$$

$$g_{3,L}=\mathrm{Min}\left(m_3-x_3-D_{3,L},m\times s-k_{M-L}-D_{3,L}-D_{1,L+1}-D_{2,L+1},\ldots,m\times s-k_j-D_{3,L}-D_{1,M-j+1}-D_{2,M-j+1}\right)$$

$$g_{4,L}=\mathrm{Min}\left(m_4-x_4-D_{4,L},\ m\times s-k_{M-L}-D_{4,L}-D_{1,L+1}-D_{2,L+1},\ldots,m\times s-k_j-D_{4,L}-D_{1,M-j+1}-D_{2,M-j+1}\right)$$

$$g_{5,L}=\mathrm{Min}\left(m_5-x_5-D_{5,L},\ m\times s-k_{M-L}-D_{5,L}-D_{1,L+1}-D_{3,L+1},\ldots,m\times s-k_j-D_{5,L}-D_{1,M-j+1}-D_{3,M-j+1}\right)$$

Case 2: *c-a* > *s*-1, *b-a* ≤ *s*-1, *c-b* ≤ *s*-1

In this case the events $A_{a,j}$, $A_{b,j}$ have common components, and so the events $A_{b,j}$, $A_{c,j}$. The number of common components between the events $A_{a,j}$, $A_{b,j}$ is $m\times(s+a-b)$ components and between $A_{b,j}$, $A_{c,j}$ is $m\times(s+b-c)$ components. But there are no any common components between the events $A_{a,j}$, $A_{c,j}$.

We can use the formulas (31) and (32) to calculate the $\Pr(A_{a,j}\ A_{b,j}A_{c,j})$ in this case, but with the following data:

$t_1=\max(0,\ k_j+m(c-a-2\mathrm{s}))$, $\quad m_1=m(c-a-s)$

$t_2=\max(0,\ k_j-x_1+m(c-b-s))$, $\quad m_2=m(a-b+s)$

$t_3=\max(0,\ k_j-x_1-x_2)$, $\quad m_3=m(b-c+s)$

$t_4=\max(0,\ k_j-x_2)$, $\quad m_4=m(b-a)$

$t_5=\max(0,\ k_j-x_3)$, $\quad m_5=m(c-b)$

$d_{e,M-j}=m_e-x_e-D_{e,M-j}$

$D_{i,L}=\sum_{y=0}^{L-1}d_{i,y}$, $\quad i=1, 2, \ldots, 5$

$$g_{1,L}=\mathrm{Min}\left(m_1-x_1-D_{1,L},\ m\times s-k_{M-L}-D_{1,L}\right)$$

$$g_{2,L}=\mathrm{Min}\left(m_2-x_2-D_{2,L},m\times s-k_{M-L}-D_{2,L}-D_{1,L+1},\ldots,m\times s-k_j-D_{2,L}-D_{1,M-j+1}\right)$$

$$g_{3,L}=\mathrm{Min}\left(m_3-x_3-D_{3,L},m\times s-k_{M-L}-D_{3,L}-D_{1,L+1}-D_{2,L+1},\ldots,m\times s-k_j-D_{3,L}-D_{1,M-j+1}-D_{2,M-j+1}\right)$$

$$g_{4,L}=\mathrm{Min}\left(m_4-x_4-D_{4,L},\ m\times s-k_{M-L}-D_{4,L}-D_{2,L+1},\ldots,m\times s-k_j-D_{4,L}-D_{2,M-j+1}\right)$$

$$g_{5,L}=\mathrm{Min}\left(m_5-x_5-D_{5,L},\ m\times s-k_{M-L}-D_{5,L}-D_{3,L+1},\ldots,m\times s-k_j-D_{5,L}-D_{3,M-j+1}\right)$$

Case 3: *c-a* > *s*-1, *b-a* ≤ *s*-1, *c-b* > *s*-1

In this case the events $A_{a,j}$, $A_{b,j}$ have common components. But the two events $A_{b,j}$, $A_{c,j}$ and so the two events $A_{a,j}$, $A_{c,j}$ are disjoint. The number of common components between the events $A_{a,j}$ and $A_{b,j}$

is $m\times(s + a - b)$ components. So, we can find the $\Pr(A_{a,j}\ A_{b,j}A_{c,j})$ by the following formula:

$$\Pr(A_{a,j}\ A_{b,j}A_{c,j}) = \Pr(A_{a,j}\ A_{b,j})\times\Pr(A_{c,j}) \tag{33}$$

Such that, the $\Pr(A_{a,j}\ A_{b,j})$ can be obtained by formulas (23) , (24) and the $\Pr(A_{c,j})$ can be obtained by formula (18).

Case 4: ***c-a > s*-1, *b-a > s*-1, *c-b ≤ s*-1**

In this case the two events $A_{a,j}$, $A_{b,j}$ and so the two events $A_{a,j}$, $A_{c,j}$ are disjoint. The events $A_{b,j}$ and $A_{c,j}$ have common components. The number of common components between the events $A_{b,j}$ and $A_{c,j}$ is $m\times(s + b - c)$ components. So, we can find the $\Pr(A_{a,j}\ A_{b,j}A_{c,j})$ by the following formula:

$$\Pr(A_{a,j}\ A_{b,j}A_{c,j}) = \Pr(A_{a,j}\ )\times\Pr(A_{b,j}A_{c,j}) \tag{34}$$

Such that, the $\Pr(A_{b,j}\ A_{c,j})$ can be obtained by formulas (23), (24) and the $\Pr(A_{a,j})$ can be obtained by formula (18).

Case 5: *c-a > s*-1, *b-a > s*-1, *c-b ≤ s*-1

In this case, all the events $A_{a,j}$, $A_{b,j}$, $A_{c,j}$ are disjoint.

So, we can find the $\Pr(A_{a,j}\ A_{b,j}A_{c,j})$ by the following formula:

$$\Pr(A_{a,j}\ A_{b,j}A_{c,j}) = \Pr(A_{a,j}\ )\times\Pr(A_{b,j})\times\Pr(A_{c,j}) \tag{35}$$

Such that, the $\Pr(A_{a,j}\ ),\Pr(A_{b,j}),\Pr(A_{c,j})$ can be obtained by formula (18).

Example 5.

Consider an increasing MS L(*k*,*m*,*s*,*n*:F) with the following data: n = 5, m = 2, s = 2, H = 3, the k vector is (k_1,k_2,k_3) = (1, 2, 3), and the state distribution of components is (p_0,p_1,p_2,p_3) = (0.1, 0.2, 0.4, 0.3). So that, (Q_1,Q_2,Q_3) = (0.1, 0.3, 0.7). In the following, we illustrate the calculations of $S_{1,j},S_{2,j},S_{3,j}$. The results of this example are listed in table 4.

At state 3:

$$\Pr(A_{1,3}) = \sum_{y=3}^{4}\binom{4}{y}(0.7)^{y}\beta_3(4,y) = 0.6517\ ,$$

$$\Pr(A_{1,3}A_{2,3}) = \sum_{y_1=1}^{2}\binom{2}{y_1}(0.7)^{y_1}.(0.3)^{2-y_1}.\left[\sum_{y_2=3-y_1}^{2}\binom{2}{y_2}(0.7)^{y_2}.(0.3)^{2-y_2}\right]^2 = 0.506611$$

$$\Pr(A_{1,3}A_{3,3}) = \Pr(A_{1,3}A_{4,3}) = \left[\Pr(A_{1,3})\right]^2 = 0.4247129$$

$$\Pr(A_{1,3}A_{2,3}A_{3,3}) = \sum_{x_1=0}^{0}\sum_{x_2=1}^{2}\sum_{x_3=3-x_2}^{2}\sum_{x_4=3-x_2}^{2}\sum_{x_5=3-x_3}^{2}\binom{0}{0}\binom{2}{x_2}\binom{2}{x_3}\binom{2}{x_4}\binom{2}{x_5}$$
$$\times(0.7)^{x_2+x_3+x_4+x_5}(0.3)^{8-(x_2+x_3+x_4+x_5)} = 0.3823593$$

$$\Pr(A_{1,3}A_{2,3}A_{4,3}) = \Pr(A_{1,3}A_{2,3})\times\Pr(A_{4,3}) = \Pr(A_{1,3}A_{2,3})\times\Pr(A_{1,3}) = 0.506611\times 0.6517$$
$$= 0.3301584$$

$$S_{1,3} = 4\times 0.6517 = 2.6068$$

$$S_{2,3} = 3\times\Pr(A_{1,3}A_{2,3}) + 2\times\Pr(A_{1,3}A_{3,3}) + \Pr(A_{1,3}A_{4,3})$$
$$= 3\times 0.506611\ +\ 2\times 0.4247129\ +\ 0.4247129 = 2.7939717$$

$$S_{3,3} = \Pr(A_{1,3}A_{2,3}A_{3,3}) + \Pr(A_{1,3}A_{2,3}A_{4,3}) + \Pr(A_{1,3}A_{3,3}A_{4,3}) + \Pr(A_{2,3}A_{3,3}A_{4,3})$$
$$= 2\times\Pr(A_{1,3}A_{2,3}A_{3,3}) + 2\times\Pr(A_{1,3}A_{2,3}A_{4,3})$$
$$= 2\times 0.3823593 + 2\times 0.3301584 = 1.4250354$$

$u_3 = 3$, $\lambda_3 = 2$, $\omega_3 = 3$

At state 2:

$$\Pr(A_{1,2}) = \sum_{y=2}^{4}\binom{4}{y}(0.3)^{y}\beta_2(4,y) = 0.2997$$
,

$$\Pr(A_{1,2}A_{2,2}) = \sum_{y_1=0}^{2}\binom{2}{y_1}(0.3)^{y_1}.\beta_2(2,y_1).\left[\sum_{y_2=2-y_1}^{2}\binom{2}{y_2}(0.3)^{y_2}.\beta_2'(2,y_2,I_1)\right]^2$$
$$= 0.159795$$

$$\Pr(A_{1,2}A_{3,2}) = \Pr(A_{1,2}A_{4,2}) = \left[\Pr(A_{1,2})\right]^2 = \left[0.2997\right]^2 = 0.0898201$$

$$\Pr(A_{1,2}A_{2,2}A_{3,2}) = \sum_{x_1=0}^{0}\binom{0}{0}(0.3)^0\sum_{d_{1,0}=0}^{0}\binom{0}{0}(0.3)^0(0.4)^0$$
$$\times\sum_{x_2=0}^{2}\binom{2}{x_2}(0.3)^{x_2}\sum_{d_{2,0}=0}^{g_{2,0}}\binom{2-x_2-D_{2,0}}{d_{2,0}}(0.3)^{d_{2,0}}(0.4)^{2-x_2-D_{2,1}}$$
$$\times\sum_{x_3=2-x_2}^{2}\binom{2}{x_3}(0.3)^{x_3}\sum_{d_{3,0}=0}^{g_{3,0}}\binom{2-x_3-D_{3,0}}{d_{3,0}}(0.3)^{d_{3,0}}(0.4)^{2-x_3-D_{3,1}}$$
$$\times\sum_{x_4=2-x_2}^{2}\binom{2}{x_4}(0.3)^{x_4}\sum_{d_{4,0}=0}^{g_{4,0}}\binom{2-x_4-D_{4,0}}{d_{4,0}}(0.3)^{d_{4,0}}(0.4)^{2-x_4-D_{4,1}}$$
$$\times\sum_{x_5=2-x_3}^{2}\binom{2}{x_5}(0.3)^{x_5}\sum_{d_{5,0}=0}^{g_{5,0}}\binom{2-x_5-D_{5,0}}{d_{5,0}}(0.3)^{d_{5,0}}(0.4)^{2-x_5-D_{5,1}}$$
$$= 0.0719061$$

$$\Pr(A_{1,2}A_{2,2}A_{4,2}) = \Pr(A_{1,2}A_{2,2})\times\Pr(A_{4,2}) = \Pr(A_{1,2}A_{2,2})\times\Pr(A_{1,2}) = 0.159795\times 0.2997$$
$$= 0.0478906$$

$$S_{1,2} = 4\times 0.2997 = 1.1988$$

$$S_{2,2} = 3\times\Pr(A_{1,2}A_{2,2}) + 2\times\Pr(A_{1,2}A_{3,2}) + \Pr(A_{1,2}A_{4,2})$$
$$= 3\times\ 0.159795\ +\ 2\times 0.0898201\ +\ 0.0898201 = 0.7488453$$

$$S_{3,2} = \Pr(A_{1,2}A_{2,2}A_{3,2}) + \Pr(A_{1,2}A_{2,2}A_{4,2}) + \Pr(A_{1,2}A_{3,2}A_{4,2}) + \Pr(A_{2,2}A_{3,2}A_{4,2})$$
$$= 2\times\Pr(A_{1,2}A_{2,2}A_{3,2}) + 2\times\Pr(A_{1,2}A_{2,2}A_{4,2})$$
$$= 2\times 0.0719061 + 2\times 0.0478906 = 0.2395934$$

$u_2 = 2$, $\lambda_2 = 1$, $\omega_2 = 2$

At state 1:

$$\Pr(A_{1,1}) = \sum_{y=1}^{4}\binom{4}{y}(0.1)^{y}\beta_1(4,y) = 0.1797\ ,$$

$$\Pr(A_{1,1}A_{2,1}) = \sum_{y_1=0}^{2}\binom{2}{y_1}(0.1)^{y_1}.\beta_1(2,y_1).\left[\sum_{y_2=\max(0,1-y_1)}^{2}\binom{2}{y_2}(0.1)^{y_2}.\beta_1'(2,y_2,I_2)\right]^2$$
$$= 0.078995$$

$$\Pr(A_{1,1}A_{3,1}) = \Pr(A_{1,1}A_{4,1}) = \left[\Pr(A_{1,1})\right]^2 = [0.1797]^2 = 0.0322921$$

$$\Pr(A_{1,1}A_{2,1}A_{3,1}) = \sum_{x_1=0}^{0}\binom{0}{0}(0.1)^0 \sum_{d_{1,0}=0}^{0}\binom{0}{0}(0.3)^0 \sum_{d_{1,1}=0}^{0}\binom{0}{0}(0.4)^0(0.2)^0$$
$$\times \sum_{x_2=0}^{2}\binom{2}{x_2}(0.1)^{x_2} \sum_{d_{2,0}=0}^{g_{2,0}}\binom{2-x_2-D_{2,0}}{d_{2,0}}(0.3)^{d_{2,0}} \sum_{d_{2,1}=0}^{g_{2,1}}\binom{2-x_2-D_{2,1}}{d_{2,1}}(0.4)^{d_{2,1}}(0.2)^2$$
$$\times \sum_{x_3=\max(0,1-x_2)}^{2}\binom{2}{x_3}(0.1)^{x_3} \sum_{d_{3,0}=0}^{g_{3,0}}\binom{2-x_3-D_{3,0}}{d_{3,0}}(0.3)^{d_{3,0}} \sum_{d_{3,1}=0}^{g_{3,1}}\binom{2-x_3-D_{3,1}}{d_{3,1}}(0.4)^{d_{3,1}}(0.2)^{2-x_3-D_{3,2}}$$
$$\times \sum_{x_4=\max(0,1-x_2)}^{2}\binom{2}{x_4}(0.1)^{x_4} \sum_{d_{4,0}=0}^{g_{4,0}}\binom{2-x_4-D_{4,0}}{d_{4,0}}(0.3)^{d_{4,0}} \sum_{d_{4,1}=0}^{g_{4,1}}\binom{2-x_4-D_{4,1}}{d_{4,1}}(0.4)^{d_{4,1}}(0.2)^{2-x_4-D_{4,2}}$$
$$\times \sum_{x_5=\max(0,1-x_3)}^{2}\binom{2}{x_5}(0.1)^{x_5} \sum_{d_{5,0}=0}^{g_{5,0}}\binom{2-x_5-D_{5,0}}{d_{5,0}}(0.3)^{d_{5,0}} \sum_{d_{5,1}=0}^{g_{5,1}}\binom{2-x_5-D_{5,1}}{d_{5,1}}(0.4)^{d_{5,1}}(0.2)^{2-x_5-D_{5,2}}$$
$$= 0.0234413$$

$$\Pr(A_{1,1}A_{2,1}A_{4,1}) = \Pr(A_{1,1}A_{2,1}) \times \Pr(A_{4,1}) = \Pr(A_{1,1}A_{2,1}) \times \Pr(A_{1,1}) = 0.078995 \times 0.1797$$
$$= 0.0141954$$

$$S_{1,1} = 4 \times 0.1797 = 0.7188$$

$$S_{2,1} = 3 \times \Pr(A_{1,1}A_{2,1}) + 2 \times \Pr(A_{1,1}A_{3,1}) + \Pr(A_{1,1}A_{4,1})$$
$$= 3 \times 0.078995 + 2 \times 0.0322921 + 0.0322921 = 0.3338613$$

$$S_{3,1} = \Pr(A_{1,1}A_{2,1}A_{3,1}) + \Pr(A_{1,1}A_{2,1}A_{4,1}) + \Pr(A_{1,1}A_{3,1}A_{4,1}) + \Pr(A_{2,1}A_{3,1}A_{4,1})$$
$$= 2 \times \Pr(A_{1,1}A_2,A_{3,1}) + 2 \times \Pr(A_{1,1}A_{2,1}A_{4,1})$$
$$= 2 \times 0.0234413 + 2 \times 0.0141954 = 0.0752734$$

$u_1 = 1$, $\lambda_1 = 1$, $\omega_1 = 2$

5. Numerical Results

The numerical calculations of increasing MS L(k,m,s,n:F) reliability are carried out using Visual Basic Program. The computer codes were written very carefully. The new bounds and computer codes examined by previously published numerical examples for some special cases of increasing MS L(k,m,s,n:F), as shown in tables 1-3.

- When m=1, the increasing MS L(k,1,s,n:F) becomes the increasing MS consecutive-k-out-of-s-from-n: F system. An example of increasing MS consecutive-k-out-of-s-from-n: F system in Ref [19] is examined by our bounds and given in table 1.
- When H=1 and m=1, the increasing MS L(k,1,s,n:F) becomes the binary consecutive-k-out-of-s-from-n: F system. An example of binary consecutive-k-out-of-s-from-n: F system in Ref [6] is examined by our bounds and given in table 2.
- When m=1 and s = n, the increasing MS L(k,1,n,n:F) becomes the increasing MS k-out-of-n:F system. An example of MS k-out-of-n:F system in Ref [10] is examined using formula (18) and given in table 3.

The bounds of the increasing MS L(k,m,s,n:F) reliability with H=3 and variant values of p_j, k_j, m, s, n are given in tables 4-7. These bounds are evaluated using second and third orders of Boole-Bonferroni bounds. The comparison between the results of second and third orders of Boole-Bonferroni bounds explained in tables 2-7 and figures 1-4. This comparison shows that the third order Boole-Bonferroni bounds are the best.

Table 1. $n = s = 4$, $m=1$, $H = 4$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$, $p_0=0.1$, $p_1=0.2$, $p_2=0.3$, $p_3=0.3$, $p_4=0.1$

State (j)	0	1	2	3	4
R_j	1	0.8669	0.7813	0.6112	0.3439

Table 2. n =15, m=1, s=10, H=4, k_1=4, k_2=6, k_3=7, k_4=9, p_0=0.1, p_1=0.2, p_2=0.3, p_3=0.3, p_4=0.1

Bounds	**S_1- S_2 based**	S_1- S_3 based	$\hat{R}_j$	E_j
LB_1 UB_1	0.9786406	0.9820220	0.9838254	0.0018034
	0.9884654	0.9856288		
LB_2 UB_2	0.8541696	0.8862806	0.8993991	0.0131184
	0.9270848	0.9125175		
LB_3 UB_3	0.1880035	0.3577432	0.42207593	0.0643328
	0.5517231	0.4864087		
LB_4 UB_4	0	0.0628391	0.1072802	0.0444411
	0.1957316	0.1517213		

Table 3. n =50, m=1, s=40, H=1, k_1=28, p_0=0.5, p_1=0.5

Bounds	**S_1- S_2 based**	S_1- S_3 based	$\hat{R}_j$	E_j
LB_1 UB_1	0.9560414	0.9696965	0.9764474	0.0067509
	0.9863147	0.9831983		

Table 4. $n=5, m=2, s=2, k_1=1, k_2=2, k_3=3, p_0=0.1, p_1=0.2, p_2=0.4, p_3=0.3$

Bounds	S_1- S_2 based	S_1- S_3 based	$\hat{R}_j$	E_j
LB_1	0.4481306	0.5397878	0.5491970	0.0094092
UB_1	0.6150613	0.5586062		
LB_2	0.1756226	0.3104519	0.3404011	0.0299492
UB_2	0.4504151	0.3703502		
LB_3	0	0.0089921	0.0473096	0.0383175
UB_3	0.1622619	0.0856270		

Table 5. $n=10, m=3, s=6, k_1=13, k_2=14, k_3=15, p_0=0.3, p_1=0.3, p_2=0.2, p_3=0.2$

Bounds	**S_1- S_2 based**	S_1- S_3 based	$\hat{R}_j$	E_j
LB_1	0.9988797	0.9990029	0.9990196	0.0000166
UB_1	0.9990861	0.9990362		
LB_2	0.7064799	0.7559996	0.7767656	0.0207660
UB_2	0.8306766	0.7975315		
LB_3	0.0000000	0.1815675	0.2344687	0.0529012
UB_3	0.3740597	0.2873698		

Table 6. $n=20, m=2, s=16, k_1=18, k_2=20, k_3=21, p_0=0.3, p_1=0.2, p_2=0.2, p_3=0.3$

Bounds	**S_1- S_2 based**	S_1- S_3 based	$\hat{R}_j$	E_j
LB_1	0.9944228	0.9951706	0.9955345	0.0003639
UB_1	0.9967637	0.9958983		
LB_2	0.7542582	0.7968765	0.8128056	0.0159291
UB_2	0.8527955	0.8287347		
LB_3	0.0047005	0.0967175	0.1186511	0.0219336
UB_3	0.1724146	0.1405848		

Table 7. $n=25, m=2, s=22, k_1=36, k_2=38, k_3=40, p_0=0.3, p_1=0.2, p_2=0.3, p_3=0.2$

Bounds	**S_1- S_2 based**	S_1- S_3 based	$\hat{R}_j$	E_j
LB_1	0.9969130	0.9973025	0.9973999	0.0000974
UB_1	0.9979420	0.9974973		
LB_2	0.7189426	0.7519136	0.7601564	0.0082428
UB_2	0.7928078	0.7683991		
LB_3	0.0214187	0.0456554	0.0503207	0.0046653
UB_3	0.0643166	0.0549860		

6. Conclusions.

In this paper, we proposed new lower and upper bounds for increasing MS L(k,m,s,n:F) reliability with i.i.d components using second and third orders Boole-Bonferroni bounds. The new bounds are examined by previously published numerical examples for some special cases of increasing MS L(k,m,s,n:F). The comparison between the results of second and third orders of Boole-Bonferroni bounds shows that the third order Boole-Bonferroni bounds are the best.

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