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THE VARIATIONAL ITERATION METHOD FOR A PENDULUM WITH A COMBINED TRANSLATIONAL AND ROTATIONAL SYSTEM

Muhammad AMIR*, Asifa ASHRAF **, Jamil Abbas HAIDER*

*Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan
**Department of Mathematics, University of Management and Technology, Lahore 54782, Pakistan

muhammadamir282959@gmail.com, asifaashraf90@gmail.com, jhaider@math.qau.edu.pk

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Abstract: The dynamic analysis of complex mechanical systems often requires the application of advanced mathematical techniques. In this study, we present a variation iteration-based solution for a pendulum system coupled with a rolling wheel, forming a combined translational and rotational system. Furthermore, the Lagrange multiplier is calculated using the Elzaki transform. The system under investigation consists of a pendulum attached to a wheel that rolls without slipping on a horizontal surface. The coupled motion of the pendulum and the rolling wheel creates a complex system with both translational and rotational degrees of freedom. To solve the governing equations of motion, we employ the variation iteration method, a powerful numerical technique that combines the advantages of both variational principles and iteration schemes. The Lagrange multiplier plays a crucial role in incorporating the constraints of the system into the equations of motion. In this study, we determine the Lagrange multiplier using the Elzaki transform, which provides an effective means to calculate Lagrange multipliers for constrained mechanical systems. The proposed solution technique is applied to analyse the dynamics of a pendulum with a rolling wheel system. The effects of various system parameters, such as the pendulum length, wheel radius and initial conditions, are investigated to understand their influence on the system dynamics. The results demonstrate the effectiveness of the variation iteration method combined with the Elzaki transform in capturing the complex behaviour of a combined translational and rotational system. The proposed approach serves as a valuable tool for analysing and understanding the dynamics of similar mechanical systems encountered in various engineering applications.

Key words: Elzaki transform, Variational iteration method, Combined translational and rotational system, Pendulum with rolling wheel

1. INTRODUCTION

A pendulum with a rolling wheel is a fascinating physical system that has numerous applications in engineering and physics. One of its most common applications is in the design of mechanical clocks, where the oscillation of the pendulum is used to regulate the motion of the clock's gears and maintain accurate timekeeping. The rolling wheel adds an extra dimension to the pendulum's motion, leading to new and interesting phenomena that can be studied and utilised in various applications. Another important application of a pendulum with a rolling wheel is in the study of dynamics and control systems. By analysing the system's motion and behaviour, engineers and scientists can gain valuable insights into how to control and stabilise complex mechanical systems. For example, this system can be used to design effective suspension systems for vehicles or to develop new control algorithms for robots and other automated machines [1-3]. Another interesting physical system is the combined translational and rotational system. This system is often encountered in many everyday devices, such as the wheels of a car or the rotor of a helicopter. By studying the motion of these systems, engineers can gain valuable insights into how to design more efficient and effective mechanical devices. Furthermore, the combined translational and rotational system has many applications in the field of robotics. By understanding how this system behaves and interacts with their environment, researchers can design more advanced and capable robots that can perform complex tasks and interact with the world in more natural ways.

F. Inada et al. [4] discussed the stability analysis of a two-degree-of-freedom system combined with a rotational and translational system. T. Koray Palazoğlu and Welat Miran [5] experimentally investigated the combined translational and rotational movement on radio frequency heating uniformity. The complex dynamics of a mechanical mechanism [6] can be solved both numerically and analytically by utilising the harmonic balance technique and RK fourth-order method. Two-dimensional numerical simulation [7] is used to investigate the mechanical activities on a moving invader into a granular material subject to a gravity field. The rotating pendulum system, three-degree-of-freedom auto-parametric system and spring–pendulum system have significant applications. The windmill and waterwheel aid in the development of energy harvesting. For many years, researchers have been striving to find a way to store the energy produced by heat and vibrations. The desire to power sensor networks and mobile devices without batteries that require external charging or maintenance is one of the driving forces behind the need for innovative energy-harvesting systems. Energy harvesting is also known as ambient power, or power harvesting is the process by which energy is derived from external sources such as solar electricity, wind, and thermal and kinetic energy; then stored for use by small, wireless autonomous devices such as wearable electronics devices, condition monitoring systems[8] and wireless networks of sensors [9]. Recently, pendulum energy harvesters have been used in ultra-low-frequency environments, including ocean waves, human motion and structural vibration.

Fig. 1. Classification of pendulum energy-harvesting technologies [10]

It is assumed that a rotational pendulum has a length and a lumped mass, and rotates at a constant speed. Wherever the pendulum's angular displacement in the vertical direction is, that is where the kinetic energy and potential energy exist. The Lagrange equation can be used to obtain the rotational pendulum equation. In many devices and structures, the vibrations that are excited generally have an improper and undesirable consequence. Currently, there are several various approaches to decrease or even stop the oscillating system's amplitude. A vibrating system coupled to an auxiliary mass using a spring is the most basic type of a dynamic vibration absorber. This kind is highly beneficial when the system becomes excited by a regularly changing force with a fixed frequency, particularly when the force's frequency is very close to the resonance frequency of the system under consideration [11]. The main system is coupled to an auto-parametric pendulum absorber as a secondary subsystem, which reduces the energy from the primary system [12]. Many constructed structures including clocks, percussion instruments, roller coasters and gravitational anomalies caused by earthquakes utilise pendulums from a practical perspective. Engineers are making an effort to comprehend the physics of the pendulum, centripetal force and inertia. A pendulum is used to improve the tempo of music. The oscillatory motion of a simple pendulum is one of the most investigated motions in applied physics and engineering. The motion of a pendulum plays a vital role in the historical background of physics and general topics in textbooks for undergraduate students. Thus, a simple pendulum is the most common mechanical example, and studying it eases the basics of classical mechanics. In many textbooks and engineering applications, the periodic motion is generated by reduced angle fluctuation using a simple pendulum [13]. The governing equation is non-linear if this restriction is not followed. However, this problem has an exact analytic solution within an integral formula [14]. Therefore, getting an exact, bounded approximated solution is helpful [15-16]. A simple pendulum has the most basic and comprehensive structure, which serves as the foundation for numerous complicated applications. Its significance in analysing the non-linear phenomena surrounding us is established in both engineering fields and fundamental areas such as physics and chemistry, and the lower dimensional compound system such as a swinging Atwood's machine [17], spring mass pendulum [18-19] and elastic pendulum [20-22] can demonstrate a wide variety of non-trivial phenomena including continuous processes and many types of resonance. A simple pendulum and another oscillating system, for example, a pendulum rotating with a different trajectory of its pivots, are components of different machine parts. Auto-parametric resonance is significant for this type of mechanical system. Hence, to determine the significance of a simple pendulum problem, the present work focused on what happens when a simple pendulum is attached to a lightweight spring.

Ordinary and partial differential equations are significant in many fields of science, including biology, mathematics, chemistry and applied physics. The physical importance of the condition that qualitatively governs dynamic behaviours and uses of practical physics include population growth, potential fields, electrical circuits and tree biology. Differential equations are generated from physical conditions. The solution to a linear differential equation is quite easy, but finding an analytical solution of a non-linear differential equation can be challenging in many cases. Consequently, approximation and numerical techniques are frequently used because the majority of differential equations do not have an exact closed form solution. There are many methods to solve non-linear problems; for instance, the perturbation method [23] is used to find the approximate solution of a parametric pendulum. V.S. Sorokin [24] analysed the motion of an inverted pendulum using the method of direct separation of motion (MDSM). A.I. Ismail [25] investigated the motion of a solid pendulum by using a large parameter method (LPM). M.M Khan [26] determined the solution of an undamped pendulum and a harmonic oscillator by using the variational iteration method (VIM). J Freundlich and D Soda [27] applied the multi-scale method to analyse the mechanical system with a spherical pendulum. The multi-harmonic energy balance method [28] is used to identify the energy-dependent pattern of the synchronisation of coupled pendulums. An energy-harvesting instrument [47] is used to control the kinematics of a spring–pendulum system.

The natural decomposition method [29] is utilised to determine the inviscid Burgers' equation. The variational iteration technique [30] is used to find the solution of integro-differential equation. The analytic solution of the Duffing–van der Pol oscillator (DVdP) is obtained by using the multiple timescales technique [31]. The applications of thermal oscillation in a heat shock are discussed [32]. Many of these exact solution techniques cannot handle all types of non-linear problems. Numerous computational techniques are employed to solve problems involving non-linear dynamics [33-34]. However, analytical methods are preferable to numerical methods because they allow for easier understanding of the basic physics of the problem. The variational iteration technique was introduced in 1998 [35] and has been successfully used to solve a wide range of non-linear problems [36]. The main objective of this approach is to construct a correction function by utilising a general Lagrange multiplier that is carefully selected because its correctional solution is significant than the initial trail function. Numerous results depending on the variational iteration technique are unable to demonstrate in many problems, because the integral of the correctional function is convolution type then modification of Elzaki transform can be applied. The Lagrange multiplier plays a vital role in the variational iteration technique, and variational theory is used for this purpose. Furthermore, to find out the Lagrange multiplier using the Elzaki variational method is much better than using the variational theory [37].

The current work focuses on analysing the motion of a pendulum attached to a rolling wheel by a lightweight spring and a combined translational and rotational system by using the variational iteration method and Elzaki transform. Understanding the dynamics of a pendulum with a rolling wheel is important in mechanical

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engineering, robotics, physics, etc. The system can be used to model a variety of real-world phenomena, such as the motion of a pendulum clock, the behaviour of a robotic arm or the dynamics of a vehicle with rolling wheels.

Overall, a pendulum with a rolling wheel is a fascinating example of a complex mechanical system that exhibits both translational motion and rotational motion, and it provides a rich area of research in various fields of science and engineering.

To perform the Elzaki transform based on the VIM, we follow the following four steps:

Apply the Elzaki transform to the non-linear differential equation, which involves converting the equation into a set of algebraic equations. This step can be accomplished using standard mathematical techniques.

Apply the inverse Elzaki transform to obtain an approximate solution to the differential equation.

Use the VIM to refine the approximate solution obtained in step 2. This involves iterating the solution until convergence is achieved.

Use the refined solution to obtain the final solution to the differential equation.

The combination of the Elzaki transform and the VIM provides an efficient and accurate method for solving a wide range of differential equations and has been applied in various fields of science and engineering.

The he article is organised as follows: Section 2 discusses the identification of the Lagrange multiplier by using the Elzaki transform. Section 3 outlines the suitability and correctness of the purposed method for a pendulum with a rolling wheel and a combined translational and rotational system. Section 4 ends with the concluding remarks.

2. IDENTIFICATION OF VARIATION ITERATION METHOD'S LAGRANGE MULTIPLIER BY USING THE ELZAKI TRANSFORM

The identification of variation iteration method's Lagrange multiplier is a crucial step in solving optimisation problems. One approach to finding this multiplier is using the Elzaki transform. The Elzaki transform is a mathematical technique that transforms the Lagrangian function into a new function that involves only optimisation variables. This transform eliminates the need for the Lagrange multiplier, simplifying the problem and making it easier to solve.

To use the Elzaki transform, the original Lagrangian function is first expressed as a function of optimisation variables and the Lagrange multiplier. Then, the Elzaki transform is applied to this function, which results in a new function that only involves optimisation variables. The derivative of this new function is set to zero to find the optimal values of the optimisation variables, which can then be used to solve the original optimisation problem.

The use of the Elzaki transform in the identification of the Lagrange multiplier in the variation iteration method has proven to be an effective and efficient approach. It has been successfully applied in various fields, including engineering, economics and physics, to solve a range of optimisation problems. The technique is especially useful for complex problems with multiple constraints and variables, where traditional methods may not be suitable.

Consider a general non-linear oscillatory system:

$$v''(t) + f(v) = 0 \tag{2.1}$$

Initial conditions for this system can be written as follows:

$$v(0), v'(0) = 0$$

Equation (1) can be rewritten in the following form:

$$v'' + \Omega^2 v' + g(v) = 0, \tag{2.2}$$

where

$$g(v) = f(v) - \Omega^2 v$$

According to the VIM [38], we developed the correctional functional for equation (2) in the following manner:

$$v_{n+1}(t) = v_n(t) + \int_0^t \lambda(t,\eta)[v_n''(\eta) + \Omega^2 v_n(\eta)\tilde{g}(v_n)]d\eta, \quad n = 0,1,2,.... \tag{2.3}$$

Here, λ represents the Lagrange multiplier, $\mathrm{v_n}$ denotes the approximate solution and $\tilde{\mathrm{g}}$ is the restricted variant. We then used another way of finding the Lagrange multiplier, which is a significant part of the variational technique. Generally, the Lagrange multiplier can be express as follows [39-40]:

$$\lambda = \lambda(t-\eta)$$

We applies the Elzaki transform [41] on both sides of equation (3), and using the properties of proposed transform, the correctional function can be written as follows:

$$E[v_{n+1}(t)] = E[v_n(t)] + E[\int_0^t \lambda(t-\eta)[v_n''(\eta) + \Omega^2 v_n(\eta) + \tilde{g}(v_n)]d\eta = E[v_n(t)] + E[\lambda(t) * (v_n''(\eta) + \Omega^2 v_n(\eta) + \tilde{g}(v_n))] = E[v_n(t)] + \frac{1}{w}E[\lambda(t)]E[(v_n''(\eta) + \Omega^2 v_n(\eta) + \tilde{g}(v_n))] = \tag{2.4}$$

$$\mathrm{E}[v_n(t)] + \frac{1}{w}\mathrm{E}[\lambda(t)][\left(\frac{1}{w^2} + \Omega^2\right)\mathrm{E}[v_n(t)] - sv_n(0) - v_n'(0) + \mathrm{E}[\tilde{g}(v_n)]] \tag{2.5}$$

By using the variation with respect to $\mathrm{v_n}$ for calculating the value λ in equation (2.5), the stationary condition can be express as follows:

$$E[\delta v_n] + \frac{1}{w}E[\lambda]\left(\frac{1}{w^2} + \omega^2\right)E[\delta v_n] = 0 \tag{2.6}$$

From equation (2.6), the following equation can be obtained:

$$E[\lambda(t)] = -\frac{w^3}{1+\Omega^2 w^2} \tag{2.7}$$

We assumed that

$$\frac{\delta}{\delta v_n}E[\tilde{g}(v_n)] = 0$$

We used the inverse Elzaki transform to determine the value of λ:

$$\lambda(t) = -\frac{1}{\Omega}\,sinwt\,, \tag{2.8}$$

which is similar to that used by Anjum and He [42].

By utilising equation (4), we obtained the following recursive formula:

$$E[v_{n+1}(t)] = E[v_n(t)] - \frac{1}{\Omega}E\left[\int_0^t sinw(t-\eta)[v_n''(\eta) + \Omega^2 v_n(\eta) + \tilde{g}(v_n)]\right]d\eta$$

or

$$E[v_{n+1}(t)] = E[v_n(t)] - \frac{1}{\Omega} E[sinwt][v_n''(\eta) + \Omega^2 v_n(\eta) + \tilde{g}(v_n)] \quad (2.9)$$

Using the Elzaki transform properties, we can find higher order solutions by using the aforementioned recursive formula.

3. PENDULUM WITH ROLLING WHEEL

A pendulum with a rolling wheel is a classic physics problem that involves the motion of a wheel attached to a pendulum. The system consists of a wheel that is free to rotate and a pendulum that is free to swing back and forth. The wheel is attached to the pendulum at its centre of mass, and the entire system is free to move in a vertical plane.

As the pendulum swings, the wheel rolls along the ground, creating a complex motion that can be challenging to analyse. The motion of the wheel is affected by both its rotational motion and linear motion, making it difficult to predict its behaviour.

Despite its complexity, the pendulum with a rolling wheel is an important model for understanding the principles of mechanics and is often used to teach physics students the relationship between rotational motion and linear motion. It also has practical applications in areas such as robotics, where similar systems are used to control the movement of robotic arms and other devices.

Overall, the pendulum with a rolling wheel is a fascinating example of the interplay between rotational motion and linear motion and provides a valuable tool for understanding the principles of mechanics.

Fig.1 shows the movement of a pendulum connected to a rotating wheel and restrained by a lightweight spring [43]. Cartesian coordinates are utilised for a better fit, with the x-axis being parallel to the horizon and the y-axis being vertically upwards.

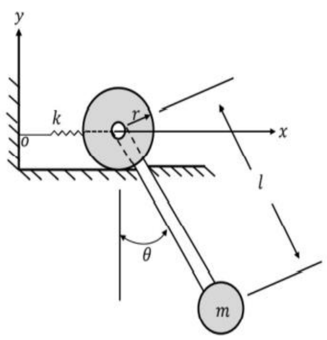

Fig. 2. Dynamic model

The given system's x-axis can be written as follows:

$$x = x_{wheel} + x_{pend} = r\theta + l_o sin\theta \quad (3.1)$$

The supposed system's y-axis can be written as follows:

$$y = -l_o cos\theta$$

As a result, the bob's location is specified as follows:

$$r = (r\theta + l_o sin\theta, -l_o cos\theta) \quad (3.2)$$

The kinetic energy of the system can be formulated as follows:

$$T = \frac{m}{2}(r^2 + l_o{}^2 + 2rl_o cos\theta)\dot{\theta}^2$$

The potential energy is given as follows:

$$V = \frac{1}{2}kx_{wheel}^2 - mgl_o cos\theta \quad (3.3)$$

The Lagrangian function of the system is given as follows:

$$L = \frac{m}{2}\left(r^2 + l_o{}^2 + 2rl_o cos\theta\right)\dot{\theta}^2 - \frac{1}{2}kr^2\theta^2 - mgl_o cos\theta$$

The organisation thus takes one degree of freedom.

Consequently, the motion equation becomes as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or

$$m\left(r^2 + l_o{}^2 + 2rl_o cos\theta\right)\ddot{\theta} - mr\dot{\theta}^2 sin\theta + kr^2\dot{\theta}^2 + mglsin\theta = 0 \quad (3.4)$$

Nayfeh [44] provided similar derivations of Eq. (3.4) in different cases for numerous mechanical problems, and El-Dib and Moatimid [45] provided derivations for a rigid rod that rocks across a circular surface. To simplify the governing equation of motion and minimise the scaling effects, the following non-dimensional parameters are modified.

Consider l, m and $\sqrt{\frac{\mathrm{l}}{\mathrm{g}}}$ to be the characteristic length, mass, and time respectively; as a result, Eq. (3.4) can be written as follows:

$$\left(r^2 + l_o{}^2 + 2rl_o cos\theta\right)\ddot{\theta} - r\dot{\theta}^2 sin\theta + kr^2\dot{\theta}^2 + sin\theta \quad (3.5)$$

For small values of θ, the Taylor expansion is expanded, assuming $sin\theta \cong \theta - \frac{\theta^3}{3!}$ and $cos\theta \cong 1 - \frac{\theta^2}{2!}$.

Then, Equation (3.5) is formulated as follows:

$$(1 - \alpha\theta^2)\ddot{\theta} + \omega^2\dot{\theta}^2 - \alpha\theta\dot{\theta}^2 + \frac{\alpha}{6}\theta^3\dot{\theta}^2 - \beta\theta^3 = 0 \quad (3.6)$$

where

$$= \frac{r}{(r+1)^2}, \omega^2 = \frac{kr^2+1}{(r+1)^2} \ and \ \beta = \frac{1}{6(r+1)^2}$$

Equation (3.6) can be rewritten as follows:

$$(1 - \alpha v^2)\ddot{v} + \Omega^2 v - \alpha v\dot{v}^2 + \frac{\alpha}{6}v^3\dot{v}^2 - \beta v^3 = 0 \quad (3.7)$$

Initial conditions are given as follows:

$$v(0) = 1, v'(0) = 0$$

The correctional functional is written as follows:

$$E[v_{n+1}(t)] = E[v_n(t)] - \frac{1}{\Omega}E[sin\Omega t]E[(1 - \alpha v_n{}^2)\ddot{v_n} + \Omega^2 v_n - \alpha v_n\dot{v_n}{}^2 + \frac{\alpha}{6}v_n{}^3\dot{v_n}{}^2 - \beta v_n{}^3] \quad (3.8)$$

Substituting $\mathrm{n} = 0$, we have

$$E[v_1(t)] = E[v_0(t)] - \frac{1}{\Omega}E[sin\Omega t]E[(1 - \alpha v_0{}^2)\ddot{v_0} + \Omega^2 v_0 - \alpha v_0\dot{v_0}{}^2 + \frac{\alpha}{6}v_0{}^3\dot{v_0}{}^2 - \beta v_0{}^3]$$

Assuming the initial solution is $\mathrm{v_0(t)} = \mathrm{Acos\Omega t}$, we have

$$E[v_1(t)] = E[v_0(t)] - \frac{1}{\omega}E[sin\Omega t]E[-A\omega^2 cos\Omega t - \alpha A^2 cos^2\Omega t(-a\Omega^2 cos\omega t) + \Omega^2 Acos\omega t - \alpha Acos\Omega t(-A\Omega sin\Omega t)^2 + \frac{\alpha}{6}A^3 cos^3\Omega t(-A\Omega sin\Omega t)^2 - \beta A^3 cos^3\Omega t = 0$$

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After some calculations, the previous expression becomes as follows:

$$E[v_1(t)] = E[Acos\Omega t] - \frac{1}{\Omega}\Bigg[\Big(-A\Omega^2 + \frac{3A^3\alpha\Omega^2}{4} + A\Omega^2 \alpha A^3\Omega^2 + \frac{3A^3\alpha\Omega^2}{4} + \frac{A^5\alpha\Omega^2}{8} - \frac{5A^5\alpha\Omega^2}{48} - \frac{3A^3\beta}{4}\Big) E[cos\Omega t]E[sin\Omega t] + \Big(\frac{A^3\alpha\Omega^2}{4} - \frac{A^3\alpha\Omega^2}{4} + \frac{A^5\alpha\Omega^2}{24} - \frac{5A^5\alpha\Omega^2}{6} - \frac{A^3\beta}{4} - \Big) E[cos3\Omega t]E[sin\Omega t] - \frac{A^5\alpha\Omega^2}{96}E[cos5\Omega t]E[sin\Omega t]\Bigg] \tag{3.9}$$

Taking the inverse Elzaki transform of equation (3.9), we get the first-order approximate results as follows:

$$-\frac{1}{\Omega}[\Big(-A\Omega^2 + \frac{3A^3\alpha\Omega^2}{4} + A\Omega^2 - \alpha A^3\Omega^2 + \frac{3A^3\alpha\Omega^2}{4} + \frac{A^5\alpha\Omega^2}{8} - \frac{5A^5\alpha\Omega^2}{48} - \frac{3A^3\beta}{4}\Big)$$

This results in the expression for the system's angular frequency.

$$\Omega = \sqrt{\frac{\frac{9}{7}\beta A^3}{\frac{3}{4}\alpha A^5 - \alpha A^2 - \frac{\alpha 3}{12}}}$$

4. COMBINED TRANSLATIONAL AND ROTATIONAL SYSTEM

A combined translational and rotational system is a physical system that involves both translational motion (motion in a straight line) and rotational motion (motion around an axis). This system can be found in a wide range of mechanical devices, from car engines to amusement park rides. In a combined system, both translational motion and rotational motion are interconnected and affect each other. For example, in a car engine, the pistons move in a straight line to create rotational motion in the crankshaft, which in turn drives the wheels to create translational motion of the vehicle. Analysing a combined translational and rotational system can be complex as the motion of each component is interdependent. This requires an understanding of the physics of both translational motion and rotational motion, as well as the principles of torque and energy transfer. Despite their complexity, a combined translational and rotational system is vital to many everyday devices and are crucial to the functioning of modern technology. Studying these systems can lead to advancements in engineering and physics and can help us better understand the world around us. The idealised description of a mechanical device is a homogenous, uniform wheel rolling smoothly over a horizontal surface. A combined translational and rotational system is shown geometrically [36,46] in Figure 2.

The device's frame is secured to the centre of the wheel by a linear spring and a force. At the top of the wheel, the force $F = cos\Omega t$ is used.

$$\tilde{v}'' + \frac{2k}{3m}\tilde{v} - \frac{2}{3m}cos^2\,\Omega t = 0 \tag{4.1}$$

Initial conditions of equation (4.1) are as follows:

$$\tilde{v}'(0) = A, \quad \tilde{v}(0) = 0 \tag{4.2}$$

To simplify this model, α and β are taken as follows:

$$\alpha = \frac{2k}{3m}, \beta = \frac{2}{3m}$$

Equation (4.2) can be rewritten as follows:

$$\tilde{v}'' + \alpha\tilde{v} - \beta\,cos\Omega t = 0 \tag{4.3}$$

Now, we rewrite the equation according to the solution of the variational iterative technique with the Elzaki transform:

$$\tilde{v}'' + \Omega^2\tilde{v} + g(\tilde{v}) = 0 \tag{4.4}$$

where

$$g(\tilde{v}) = (\alpha - \Omega^2)\tilde{v} - \beta\,\,cos^2\,\Omega t$$

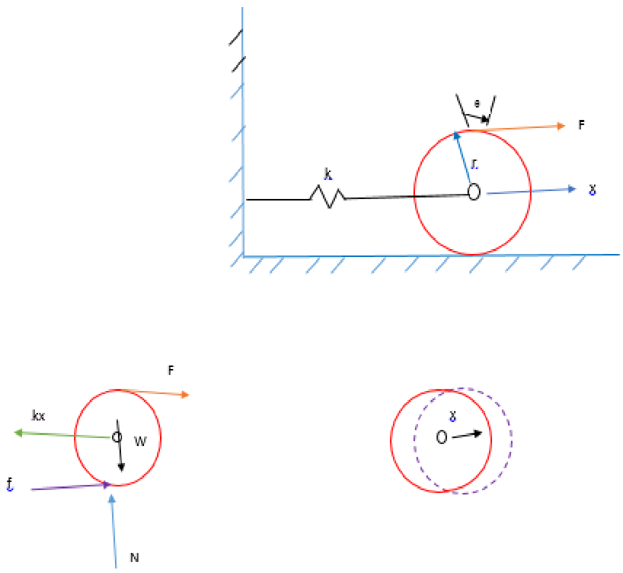

Fig. 3. Combined translational and rotational system

The iterative formula of the aforementioned model can be written as follows:

$$E[\tilde{v}_{n+1}(t)] = E[\tilde{v}_n(t)] - E[\int_0^t \frac{1}{\Omega}sin\Omega(t-\eta)[\tilde{v}_n''(\eta) + \tilde{\Omega}^2\tilde{v}_n(\xi) + \tilde{g}(\tilde{v}_n)]d\eta$$

$$E[\tilde{v}_{n+1}(t)] = E[\tilde{v}_n(t)] - \frac{1}{\Omega}E[sin\Omega t]E[\tilde{v}_n''(t) + \tilde{\Omega}^2\tilde{v}_n(t) + g(\tilde{v}_n)]$$

$$= E[\tilde{v}_n] - \frac{1}{\Omega}E[sin\Omega t]E[\tilde{v}_n'' + \alpha\tilde{v}_n(t) - \beta\,\,cos^2\,\Omega t] \tag{4.5}$$

We assume the initial solution as

$$\tilde{v}_0(t) = A\,cos\,\Omega t \tag{4.6}$$

By utilising equation (4.6) to find the solution of equation, we substitute $n = 0$:

$$E[\,\tilde{v}_1(t)] = E[A\,cos\,\Omega t] - \frac{1}{\Omega}E[sin\Omega t]E[-A\Omega^2\,cos\,\Omega t\, + \alpha A\,cos\,\Omega t - \beta\,cos\Omega t] \tag{4.7}$$

We apply the inverse Elzaki transform to equation (4.7) to obtain the first approximate solution:

$$\tilde{v}_1(t) = A\,cos\,\Omega t - \frac{1}{\Omega}(-4A\Omega^2 + \alpha A - 3\beta)\left(\frac{1}{2\Omega^2}sin\,\Omega t - \frac{1}{2\Omega}t\,cos\,\Omega t\right) \tag{4.8}$$

The second term expressed the secular term, so amplitude increases with time. To ensure convergence, we must eliminate the secular term to obtain the approximate solution:

$$(-4A\Omega^2 + \alpha A - 3\beta) = 0 \tag{4.9}$$

Simplify equation (4.9), the angular frequency is expressed as follows:

$$\Omega = \sqrt{\alpha - \frac{3\beta}{4A}}$$

$$\tilde{v}_1(t) = A\,cos(\sqrt{\alpha - \frac{3\beta}{4A}})t,$$

which is similar to that obtained by using the Laplace-basedv ariational iteration method [46].

5. CONCLUDING REMARKS

The objective of most researchers working on non-linear differential equations is to find analytical and numerical solutions. The current research emphasised on the basic pendulum problem when it was connected with a lightweight spring because it is significant in several fields. So, the purpose of this study was to analyse a pendulum with a spinning wheel, which is connected by a lightweight spring. The conserved equation generates a non-linear equation under some particular situations. Unfortunately, we failed to completely remove the secular terms when utilising the existing traditional techniques. As a result, the approximate solution that was obtained shows increased amplitude with time. A combination of variational iteration and Laplace transforms is used to find the solution of a translational and rotational system and a basic pendulum attached to a wheel and lightweight spring. A pendulum with a rolling wheel, also known as a rolling pendulum, is a system that consists of a pendulum attached to a wheel that rolls along a track. A combined translational and rotational system is a system in which an object moves both in a straight line and rotates about an axis simultaneously. This system has several interesting applications in physics and engineering listed as follows:

– Demonstrating conservation of energy: A rolling pendulum is an excellent demonstration of the conservation of energy. As the pendulum swings back and forth, the energy is transferred between potential energy (when the pendulum is at its highest point) and kinetic energy (when the pendulum is at its lowest point).
– Studying non-linear dynamics: The motion of a rolling pendulum is highly non-linear and chaotic. It is an interesting system for studying non-linear dynamics and chaos theory.
– Developing mechanical systems: A rolling pendulum can be used to develop mechanical systems that convert linear motion into rotational motion. This can be useful in designing various machines and devices.
– Measuring gravitational acceleration: The period of a rolling pendulum is dependent on the gravitational acceleration. Therefore, it can be used as a tool to measure gravitational acceleration accurately.
– Understanding motion and stability: A rolling pendulum is a great example of how motion and stability are interconnected. As the pendulum swings back and forth, the wheel rolls along the track, and the system's stability changes.
– Automotive engineering: Cars, trucks and other vehicles are examples of combined translational and rotational systems. The wheels of a vehicle move in a straight line while rotating about an axis, and the engine generates a rotational force to propel the vehicle forward.
– Robotics: Robotics is another field where combined translational and rotational systems are used. Robots with rotating arms can move in a straight line while rotating about an axis, making them useful in applications such as assembly lines and manufacturing.
– Mechanical engineering: Many mechanical systems, such as gears, pulleys and belts, involve combined translational and rotational motion. These systems are used in machines such as engines, motors and turbines.
– Sports equipment: Sports equipment such as baseballs, footballs and golf balls all involve combined translational and rotational motion. This motion affects the ball's trajectory and can be optimised for maximum distance or accuracy.
– Physics education: Combined translational and rotational systems can be used as teaching tools in physics classrooms. Simple examples such as rolling balls and wheels can be used to illustrate concepts such as angular momentum, torque and conservation of energy.

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Muhammad Amir: https://orcid.org/0009-0002-4871-4312

Jamil Abbas Haider: https://orcid.org/0000-0002-7008-8576

Asifa Ashraf: https://orcid.org/0009-0005-4786-7757