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CORRIGENDUM TO "ON DEFINABLE COMPLETENESS FOR ORDERED FIELDS" REPORTS ON MATHEMATICAL LOGIC, 54 (2019), 95–100

Short story: The ordered field we thought made Theorem 2.1 true in fact instead answers Question 1.1. The content of this corrigendum is all thanks to Emil Jerabek [2].

Whether Theorem 2.1 holds was asked on Mathoverflow [1] by an anonymous user in February 2021, then another researcher referred to our paper. Shortly thereafter, Emil Jerabek got involved and observed that the lemma we used for our main example was incorrect, and kindly informed the author [2]: The Hahn series field $F = \mathbb{R}((t^\Gamma))$ with real coefficients and with exponents in dyadic rationals Γ (like any other non-real closed Hahn series field, see further below) is not 0-definably complete: here an element $x \in (0,1)_F$ is a positive infinitesimal if and only if $(\forall y > 0)(\exists z)(xy < z^3 < y)$.

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The error was tracked by Jerabek as follows ([1, 2]). Our Lemma 2.1 fails. [The author apologizes to the journal RML and its readers for the confusion, and to Lou van den Dries for not more carefully checking the model he suggested to try (in private communication we referenced) and the argument sketch he gave.] The issue with the proof of Lemma 2.1 is that the dense but *non-Archimedean* group $\Gamma' = \mathbb{Q} \oplus \Gamma$ is not regularly dense: take, e.g., $a = (0, 1)$, $b = (1, 1)$, then there is no $u \in \Gamma'$ such that $3u \in (a, b)$. This is so because the order on $\mathbb{Q} \oplus \Gamma$ was defined such that the summand $\mathbb{Q}$ is convex, i.e., as the inverse lexicographic order (the convexity of $\mathbb{Q}$ was required later to construct the v^* valuation.) Hence the Robinson-Zakon criterion could not be used. The ordered groups Γ and Γ' are not elementarily equivalent: in the former there is no strictly positive element below which all positive elements are 3-divisible, while in the latter any positive element of the convex subgroup $\mathbb{Q}$, say $u = (1, 0)$, witnesses that.

In [1] Jerabek further shows that if an ordered field is 0-definably complete, then any parametrically definable gap would have to be regular. Therefore $D_\emptyset DCOF + D_p S_{rc} COF \equiv RCF$. Any 0-definably Dedekind complete ordered field that is not real closed would have to fail $D_p S_{rc} COF$, so would not be Scott complete. It cannot be an entire Hahn series field (with nonzero value group), since such fields are all Scott complete.

Addendum

In Question 1.1, we had asked whether $D_\emptyset S_{rc} COF \nvdash D_\emptyset DCOF$, and in our Fact 1.1 we had noted $D_p S_{rc} COF \nvdash D_p DCOF (\equiv RCF)$. The ordered field $F = \mathbb{R}((t^\Gamma))$, is Scott complete and now we know $F \not\models D_\emptyset DCOF$. Therefore $D_p S_{rc} COF \nvdash D_\emptyset DCOF$. This simultaneously answers Question 1.1 and strengthens Fact 1.1. The resolution of Question 1.1 is in a stronger sense (by Fact 1.2). The strengthening of Fact 1.1 is also believed to be proper since fortunately the statement that $D_\emptyset DCOF$ is strictly weaker than RCF still appears to hold, see the conclusive forcing argument by Dmytro Taranovsky in [1]. Granting that, the Jerabek decomposition above of RCF can be written with two incomparable theories as follows:

$RCF \equiv D_\emptyset D_{\mathrm{Irreg}} COF + D_p S_{rc} COF$;

$D_\emptyset D_{\mathrm{Irreg}} COF \nvdash D_p S_{rc} COF$, and $D_p S_{rc} COF \nvdash D_\emptyset D_{\mathrm{Irreg}} COF$.

The component $D_\emptyset D_{\mathrm{Irreg}} COF$ expresses that there are no 0-definable irregular gaps (it matches the parameter-free version of "non-valuational", a more standard terminology but often used when the structure is weakly o-minimal.)

Acknowledgement

The author is grateful to Emil Jerabek for finding the error and informing him, with all the above details.

References

[1] user107952, Is this theory the complete theory of the real ordered field?, MathOverflow, 2021, https://mathoverflow.net/q/383259 (accessed March 8, 2021).

[2] E. Jerabek, private communication, March 2021.

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