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Characteristic points and cycles in planar kinematics with application to the human gait

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Purpose: We present a novel method to process kinematical data typically coming from measurements of joints. This method will be illustrated through two examples. Methods: We adopt theoretical kinematics together with the principle of least action. We use motion and inverse motion for describing the whole experimental situation theoretically. Results: By using the principle of least action, the data contain information about inherent reference points, which we call characteristic points. These points are unique for direct and inverse motion. They may be viewed as centers of the fixed and moving reference systems. The respective actions of these characteristic points are analytically calculated. The sum of these actions defines the kinematical action. This sum is by design independent of the choice of reference system. The minimality of the kinematical action can be used again to select numerically one representative cycle in empirically given, approximately periodic motions. Finally, we illustrate the theoretical approach making use of two examples worked out, hinge movement and the sagittal component of the movement of a human leg during gait. Conclusions: This approach enables automatic cycle choices for evaluating large databases in order to compare and to distinguish empirically given movements. The procedure can be extended to three dimensional movements.

Key words: *automatic cycle choice, center of motion, planar kinematics, principle of least action*

1. Introduction

In medical in vivo diagnostic procedures kinematical data are measured to assess the individual movement structure of human extremities. An example is the human gait analysis. The data coming from different probands are not directly comparable. Here, we present a first step in developing a novel method to process such kinematical data. This method does not depend on any underlying mechanical linkage. In the following, we apply the method to data coming from cyclic planar motions. Using the data itself we characterise a point which can serve as the origin of a reference coordinate system. The long-term objective of this implementation is to distinguish reliably the measurements of quasi-planar movements coming from healthy or ill human subjects.

To illustrate how the data processing is to run, we consider cyclic motions of a given four-bar linkage [8] and look at coupler curves of some points of the moving plane (Fig. 1). The points draw a distinct pattern onto the plane of reference. In regions far outside, the rotation component dominates: the moving points run along wide closed paths with large velocities while inner points run along smaller circuits or loops with smaller velocities. Hence, in planar cyclic motions, a region exists where the points move along small closed paths with small velocities defining a kind of a *center of motion* where the "action" of the planar movement is minimal. The position of this "center" is characteristic for the movement. This center may not be confused with the geometrically defined instantaneous center of rotation (ICR, Fig. 1), which runs along the centrodes.

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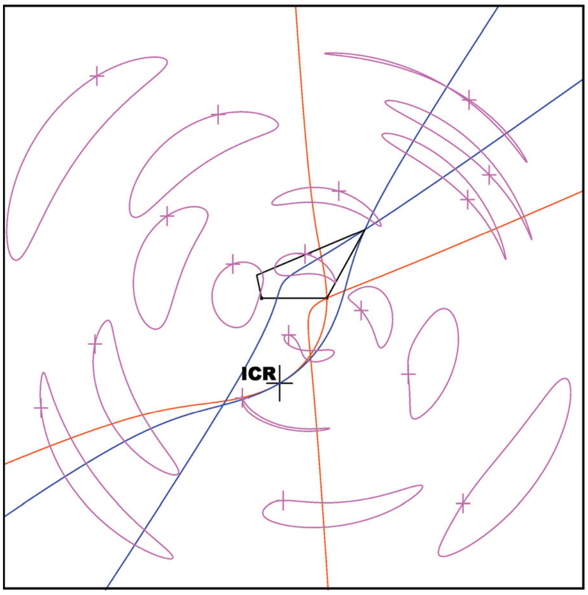

Fig. 1. A crank-rocker mechanism as an example for a periodic planar motion. Black: Actual position of the four-bar link (fixed support is horizontal). Black cross: Actual position of the instanteneous center of rotation (ICR). Red: Fixed centrode. The centrodes belong to the direct and inverse motion. Since they roll on each other, both can be used to reconstruct the motion. Blue: Actual position of the moving centrode. Magenta: Coupler curves. Magenta crosses: Actual positions of coupler points. In regions far outside the motion-generating four-bar linkage, the rotation component dominates the motion. The coupler points achieve large velocities and thus a large action, but near the support, the moving points move within a region of small action

We apply the well-known principle of least action coming from dynamics [14] to measured kinematical data (e.g., coupler curves) of one-parametric planar motions. Usually, this principle is used to derive equations of motion [10]. In processing measured kinematical data, however, it can be used to determine some characteristic properties of given quasi-periodic motions of human extremities. Our calculations provide expressions for the coordinates of mathematically stationary, characteristic points as a function of the considered time interval. We also use the minimality of the action to estimate the period and the initial time of the experimentally given quasi periodic data. By means of this special cycle, the coordinates of characteristic points are finally determined which represent characteristic coefficients for the motion. By two examples we show, how the theory is applied to experimental data.

The principle of least action is applied to one-parametric plane motions by explicitly calculating the minimal actions S_0 (relative to the fixed body, direct motion) and $\overline{S}_0$ (relative to the moving body, inverse motion). By design, these minimal actions neither depend on the space variables, nor on their reference systems. They are only functions of the common two time variables (initial and final time). If appropriate limits for the integrals are used, characteristic coefficients can be obtained for the episode in question. We do not discuss the dependence of the results on the parametrization, since the time is the appropriate one. If we adopt the natural parametrization (rotational angle between the fixed and the moving reference system), it can be shown that also a relation between the action and the characteristic direction of the motion exists [5]. This aspect is subject of a separate paper [6]. From now on we exclude the natural parametrization from our investigation.

We introduce the novel concept of *kinematical action* $S_K = S_0 + \overline{S}_0$ which is no longer dependent on the direct or inverse motion as reference. Through $S_K = S_0 + \overline{S}_0$, our main concept, we characterize the movement. By means of the desired minimality of $S_K = S_0 + \overline{S}_0$, the two remaining time parameters are determined which represent the integration limits, the initial and final time of the cycle considered. By calculating the average duration of a period and the initial time of the motion, we select one cycle of the nearby periodic motion in question, the *cycle of minimal action*. These integration limits are finally used to compute the numerical values of the characteristic coefficients.

We establish the applicability of our theoretical scheme to experimental data (Section 6.4) using human gait as an example of virtually periodic motion. The approximatively planar motions of femur and tibia relative to the hip joint were measured by the sagittal angles of the hip and the knee joint as functions of time. We adopt the double hinge as a mathematical model to describe main features of gait (Sections 6.1, 6.2). Using this approach, we calculate the characteristic, mathematically stationary points of direct and the inverse action. By calculating the average duration of a period and the initial time of a real measurement the cycle of minimal action is determined and characteristic points can be located in relation to the knee joint.

2. Methods: Planar kinematics and the principle of least action

Planar one-parametric motions can be described by using a rotation matrix $\mathbf{R}(t)$ and a displacement vector $\mathbf{d}(t)$. We adopt the more recent notation of Bottema and Roth [3] in preference to Blaschke and Müller [2].

The trajectory $\mathbf{x}(t)$ in the fixed system (the laboratory) of a point ξ belonging to the moving system (the moved rigid body) is given by

$$\mathbf{x}(t) = \mathbf{R}(t)\boldsymbol{\xi} + \mathbf{d}(t)\,. \tag{1}$$

It is a two-step transformation: First, the orientation is changed through $\mathbf{R}(t)$, then the origin is translated by $\mathbf{d}(t)$. The vector $\mathbf{x}(t)$ is a linear function of the position vector ξ. Equation (1) is the time-dependent transformation between fixed and moving system. We consider the vector ξ as time independent for direct motion.

With the coordinates

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\quad \mathbf{d}(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix},\quad \boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{2}$$

and

$$\mathbf{R}(t) = \begin{pmatrix} \cos(\alpha(t)) & -\sin(\alpha(t)) \\ \sin(\alpha(t)) & \cos(\alpha(t)) \end{pmatrix}. \tag{3}$$

Expressed in components, equation (1) reads

$$x(t) = \xi \cdot \cos\alpha(t) - \eta \cdot \sin\alpha(t) + v(t)\,, \tag{4}$$

$$y(t) = \xi \cdot \sin\alpha(t) + \eta \cdot \cos\alpha(t) + w(t)\,. \tag{5}$$

$\alpha(t)$ is the rotation angle between the fixed and the moving reference system. The functions $\alpha(t)$, $v(t)$ and $w(t)$ represent an one-parameter subgroup of the planar motion and reduce the three degrees of freedom to one. They describe the possible trajectories belonging to different ξ.

Now, we apply the formalism of the principle of least action to planar kinematics to remove the dependence on the chosen coordinate system of the moving frame.

The Lagrange function typically constitutes a suitable action. It is usually constructed by a linear combination of the squares of the first derivatives of the coordinates. Since we are dealing with kinematics in which forces (and therefore the potential energy) are not taken into account, the desired lagrangian has a simple form. We take the definition of the kinetic energy of a point with mass m at its trajectory $\mathbf{x}(t)$

$$E_{kin} = \frac{1}{2} m (\dot{\mathbf{x}})^2\,.$$

The derivative removes the dependence on the choice of the origin of the moving coordinate system and the square eliminates the dependence on its orientation: the kinetic energy does only depend on the chosen point of the moving frame. To this energy corresponds the action

$$S = \int_{t_1}^{t_2} E_{kin} dt\,,$$

called the characteristic function or energy functional. Setting $m = 1$, action is defined by

$$S = \frac{1}{2} \int_{t_1}^{t_2} (\dot{\mathbf{x}}^T\ \dot{\mathbf{x}}) dt\,. \tag{6}$$

The principle of least action requires: S = min. The condition

$$\delta S = 0 \tag{6a}$$

where δ denotes the variation, must be satisfied [11]. We call a point fulfilling this condition: stationary or characteristic point, respectively.

The action is a function of the coordinates ξ of the moving plane. Because of the quadratical dependence on the velocity the action S is a quadratic form in ξ. Hence, its minimum at point ξ_0 will be unique and characteristic of the moving frame.

Since the motion is empirically given the time stamps are discrete: $\mathbf{R}(t_i)$ =: $\mathbf{R}_i$ and $\mathbf{d}(t_i)$ =: $\mathbf{d}_i$. The integrals are reduced to sums, the differentials are differences.

2.1. Intermediary result: The point of minimal action

In order to obtain the unique point ξ_0 with minimal action, the actions of the points of the moving plane must be explicitly calculated for a planar one-parametric motion and expressed in terms of some characteristic integrals, which can be numerically calculated for an empirically given episode of a motion.

Taking the derivative of equation (1) we get the velocity

$$\mathbf{v}(t) := \dot{\mathbf{x}}(t) = \dot{\mathbf{R}}(t)\boldsymbol{\xi} + \dot{\mathbf{d}}(t)\,.$$

We insert $\mathbf{v}(t)$ into equation (6)

$$2S = \frac{1}{2} \int_{t_1}^{t_2} \mathbf{v}(t)^T \mathbf{v}(t) dt = \int_{t_1}^{t_2} (\boldsymbol{\xi}^T \dot{\mathbf{R}}^T \dot{\mathbf{R}} \boldsymbol{\xi} + 2\dot{\mathbf{d}}^T \dot{\mathbf{R}} \boldsymbol{\xi} + \dot{\mathbf{d}}^2) dt. \tag{7}$$

We made use of the fact that transposition leaves scalars unchanged, e.g., $\mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\mathbf{A}^T\mathbf{x}$ for arbitrary vectors **x** and **y** and an adequate matrix **A**[1].

It can be shown by direct calculation that

$$\dot{\mathbf{R}}(t)^T\dot{\mathbf{R}}(t) = \dot{\mathbf{R}}(t)\dot{\mathbf{R}}(t)^T = \mathbf{I}\dot{\alpha}(t)^2 \tag{8}$$

where **I** is the 2×2 unit matrix. Using equation (8), the action becomes

$$2S(\boldsymbol{\xi}) = \boldsymbol{\xi}^T\boldsymbol{\xi}\int\dot{\alpha}^2 dt + 2\left(\int\dot{\mathbf{d}}^T\dot{\mathbf{R}}dt\right)\boldsymbol{\xi} + \int\dot{\mathbf{d}}^2 dt\,. \tag{9}$$

Expressing the latter in components produces

$$\begin{aligned}2S(\xi,\eta) &= (\xi^2+\eta^2)\int\dot{\alpha}(t)^2 dt + \int(\dot{v}(t)^2+\dot{w}(t)^2)dt\\ &+\xi\int 2(-\dot{v}(t)\cdot\sin\alpha(t)+\dot{w}(t)\cdot\cos\alpha(t))\dot{\alpha}(t)dt\\ &+\eta\int 2(-\dot{v}(t)\cdot\cos\alpha(t)-\dot{w}(t)\cdot\sin\alpha(t))\dot{\alpha}(t)dt\\ &=:(\xi^2+\eta^2)I_R + I_T + \xi 2I_\xi + \eta 2I_\eta\,.\end{aligned} \tag{10}$$

The four characteristic coefficients of the action are defined as

$$I_R = \int\dot{\alpha}(t)^2 dt\,, \tag{11}$$

$$I_T = \int(\dot{v}(t)^2+\dot{w}(t)^2)dt\,, \tag{12}$$

$$I_\xi = \int(-\dot{v}(t)\cdot\sin\alpha(t)+\dot{w}(t)\cdot\cos\alpha(t))\dot{\alpha}(t)dt\,, \tag{13}$$

$$I_\eta = \int(-\dot{v}(t)\cdot\cos\alpha(t)-\dot{w}(t)\cdot\sin\alpha(t))\dot{\alpha}(t)dt\,. \tag{14}$$

These expressions exclude pure translations. For pure translations, equation (10) no longer depends on ξ, so that the whole moving plane will be a solution.

If we parametrize the action via the time (and not by the rotational angle α between the fixed and the moved plane), the coordinates of the searched characteristic point can be calculated by using the minimality of the action, as mentioned above

$$0 = \left.\frac{\partial(2S)}{\partial\boldsymbol{\xi}}\right|_{\boldsymbol{\xi}_0} = \boldsymbol{\xi}_0\int\dot{\alpha}^2 dt + \int\dot{\mathbf{R}}^T\dot{\mathbf{d}}dt\,. \tag{15}$$

Hence, we obtain

$$\boldsymbol{\xi}_0 = -\frac{\int_{t_1}^{t_2}\dot{\mathbf{R}}^T\dot{\mathbf{d}}dt}{\int_{t_1}^{t_2}\dot{\alpha}^2 dt}\,. \tag{16}$$

ξ_0 is the characteristic point of direct motion. For a hinge (see Section 5), this point and the hinge axis coincide. Note: in contrast to points ξ of the moving plane, the characteristic point ξ_0 is a function of the initial and final times t_1 and t_2 which represent the integration limits. Using equations (11), (13), (14) we get the components

$$\xi_0 = \frac{\int(\dot{v}(t)\cdot\sin\alpha(t)-\dot{w}(t)\cdot\cos\alpha(t))\dot{\alpha}(t)dt}{\int\dot{\alpha}(t)^2 dt} = -\frac{I_\xi}{I_R}, \tag{17}$$

$$\eta_0 = \frac{\int(\cos\alpha(t)\dot{v}(t)+\sin\alpha(t)\dot{w}(t))\dot{\alpha}(t)dt}{\int\dot{\alpha}(t)^2 dt} = -\frac{I_\eta}{I_R}. \tag{18}$$

This characteristic point ξ_0 removes the remaining dependence of the action on ξ simply by substituting ξ with ξ_0 in S: by inserting equation (16) into equation (9), $S_0 := S(\xi_0)$ we define the *minimal action*

$$\begin{aligned}2S_0 &= \frac{\left(\int\dot{\mathbf{R}}^T\dot{\mathbf{d}}dt\right)^T\int\dot{\mathbf{R}}^T\dot{\mathbf{d}}dt}{\left(\int\dot{\alpha}^2 dt\right)}\int\dot{\alpha}^2 dt\\ &+2\left(\int\dot{\mathbf{d}}^T\dot{\mathbf{R}}dt\right)^T\left(-\frac{\int\dot{\mathbf{R}}^T\dot{\mathbf{d}}dt}{\left(\int\dot{\alpha}^2 dt\right)}\right)+\int\dot{\mathbf{d}}^2 dt\\ &=\int\dot{\mathbf{d}}^2 dt - \frac{\left(\int\dot{\mathbf{R}}^T\dot{\mathbf{d}}dt\right)^2}{\int\dot{\alpha}^2 dt}.\end{aligned} \tag{19}$$

Expressed with the characteristic integral coefficients, equation (19) becomes

$$2S_0(t_1,t_2) = I_T(t_1,t_2) - \frac{I_\xi^2(t_1,t_2)+I_\eta^2(t_1,t_2)}{I_R(t_1,t_2)}. \tag{20}$$

Using equations (17) and (18), we re-express equation (20) as a function of the coordinates of its characteristic points

$$\begin{aligned}2S_0 &= 2S(\xi_0(t_1,t_2),t_1,t_2)\\ &= I_T(t_1,t_2) - (\xi_0^2(t_1,t_2)+\eta_0^2(t_1,t_2))I_R(t_1,t_2).\end{aligned} \tag{21}$$

[1] Note: This vectorial notation allows an easy generalization to three dimensions.

The previous equation shows that S_0 only depends on two time variables with its characteristic integral coefficients and ξ_0. Accordingly, the minimal action is only a function of the upper and lower limits of the integration: S_0 no longer possesses any kind of dependence on the reference system.

Note: $S_0(t_1, t_2) := S(\xi_0(t_1, t_2), t_1, t_2)$ is indeed a minimum. The Hessian

$$\frac{\partial^2 S}{\partial \xi^2} = \frac{1}{2}\mathbf{I}\int \dot{\alpha}^2 dt = \frac{1}{2}\mathbf{I}\cdot I_R$$

is always greater than zero for motions which are not pure translations.

2.2. Intermediary result: Minimal action for inverse motion

To derive the minimal action for inverse motion, we have to exchange the fixed with the moving reference system

$$\xi(t) = \mathbf{R}^T(t)\cdot(\mathbf{x}-\mathbf{d}(t)).$$

Point $\mathbf{x}$ is now time independent. Again, it is a two-step transformation with the translation of the origin coming first. Inverse motion is obtained through the substitutions

$$\mathbf{R} \mapsto \overline{\mathbf{R}} = \mathbf{R}^T$$

and

$$\mathbf{d} \mapsto \overline{\mathbf{d}} = -\mathbf{R}^T\cdot\mathbf{d}.$$

The respective unique characteristic point is achieved by applying the formulas of the previous section to the substitutions above. Taking the time derivative of the coordinates in the moving system $\xi(t)$ results in

$$\dot{\xi}(t) = \dot{\mathbf{R}}^T(t)\cdot(\mathbf{x}-\mathbf{d}(t)) - \mathbf{R}^T(t)\cdot\dot{\mathbf{d}}(t)$$

and inserting this into the corresponding action produces

$$2\overline{S} = \int \dot{\xi}^T\cdot\dot{\xi}dt = \int ((\mathbf{x}-\mathbf{d})^T\cdot\dot{\mathbf{R}} - \dot{\mathbf{d}}^T\cdot\mathbf{R}) \cdot(\dot{\mathbf{R}}^T\cdot(\mathbf{x}-\mathbf{d}) - \mathbf{R}^T\cdot\dot{\mathbf{d}})dt. \tag{22}$$

$\overline{S} = \frac{1}{2}\int \dot{\xi}^T\cdot\dot{\xi}dt$ is the *inverse action*.

We consider $\mathbf{I} = \mathbf{R}\,\mathbf{R}^T$ or $\mathbf{I} = \mathbf{R}^T\,\mathbf{R}$, whose derivative gives the antisymmetry of both matrices

$$\dot{\mathbf{R}}\,\mathbf{R}^T = \dot{\alpha}\,J = -\mathbf{R}\,\dot{\mathbf{R}}^T, \tag{23}$$

with the matrix

$$\mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Especially, it holds: $\mathbf{J}^2 = -\mathbf{I}$ and $\mathbf{J}^T\,\mathbf{J} = \mathbf{I}$. Using equation (23), we rewrite equation (22) as

$$2\overline{S} = \int (\dot{\alpha}^2(\mathbf{x}-\mathbf{d})^2 + 2\dot{\alpha}\,\dot{\mathbf{d}}^T\mathbf{J})\cdot(\mathbf{x}-\mathbf{d}) + \dot{\mathbf{d}}^T\dot{\mathbf{d}})dt. \tag{24}$$

Analogously to the case of direct motion we calculate the characteristic point of inverse motion

$$\mathbf{x_0} = \frac{\int \dot{\alpha}\mathbf{d}\,dt + J\int \dot{\alpha}\,\dot{\mathbf{d}}dt}{\int \dot{\alpha}^2 dt} =: \frac{1}{I_R}\begin{pmatrix} I_x \\ I_y \end{pmatrix}. \tag{25}$$

The components are

$$x_0 = \frac{\int v(t)\dot{\alpha}(t)^2 dt - \int \dot{w}(t)\dot{\alpha}(t)dt}{\int \dot{\alpha}(t)^2 dt},$$

$$y_0 = \frac{\int w(t)\dot{\alpha}(t)^2 dt + \int \dot{v}(t)\dot{\alpha}(t)dt}{\int \dot{\alpha}(t)^2 dt}.$$

For $\mathbf{x_0}$, the characteristic point of the inverse motion, we express its minimal inverse action using the characteristic integral coefficients. The minimal action $\overline{S}_0$ is

$$2\overline{S}_0 = I_T - I_R\mathbf{x_0}^T\mathbf{x_0} + \underbrace{\int \dot{\alpha}^2\mathbf{d}^T\mathbf{d}dt}_{=:I_D} - 2\underbrace{\int \dot{\alpha}^2\dot{\mathbf{d}}^T\mathbf{J}\cdot\mathbf{d}\,dt}_{=:I_I}, \tag{26}$$

where $\overline{S}_0$, given by $\mathbf{x}_0$, I_T, I_R and the new terms $I_D = \int \dot{\alpha}^2\,(v^2 + w^2)dt$, $I_I = \int \dot{\alpha}(\dot{w}v - \dot{v}w)dt$, no longer depends on space variables, but just on two time variables, the limits of integration.

3. Result: Kinematical action

The characteristic points do *not* depend on the origins of the reference systems. This independence implies the "motion invariance". Nonetheless, the minimal actions of direct and inverse motion are not equivalent because of different characteristic integral coefficients.

We introduce a novel concept: the kinematical action. It is independent of the chosen map between both reference systems. The functions $S_0 = S(\xi_0)$ and $\overline{S}_0 =$

$\bar{S}(x_0)$ only depend on the lower and upper limits (t_1, t_2) of integrations. We add both to obtain the kinematical action

$$S_K(t_1,t_2) := S_0(t_1,t_2) + \bar{S}_0(t_1,t_2).$$

It is, by design, invariant under reversal of direct and inverse motion. Since S_K is a sum of two integrals, for a fixed initial time t_A, the following equation applies

$$S_K(t_1,t_2) = S_K(t_A,t_2) - S_K(t_A,t_1),$$

so $S_K(t_A, t)$ has the properties of an antiderivative. Its total time derivative with respect to the upper integration limit is given by

$$\dot{S}_K(t_A,t) = \dot{S}_0(t_A,t) + \dot{\bar{S}}_0(t_A,t) = \frac{1}{2}\frac{I_R^2}{\dot{I}_R}(\dot{\xi}_0^2 + \dot{x}_0^2) \quad (27)$$

(see the Appendix). This is, by definition, the Lagrange function of the system under consideration.

The expression $S_K(t_1, t_2)$, the kinematical action, represents the core of our theory because by means of this quantity characteristic properties of a given motion are found independently of the reference system.

4. Further result: The cycle of least action

We would like to compare measurements with different durations. The measurements may begin at moment t_A and finish at moment t_E. Within this measuring interval $t_A - t_E$ we consider the period with start t and end $t + T$ as a shorter time interval which is taken from $t \geq t_A$ to $t + T \leq t_E$. For a given time t the kinematical action $S_K(t, t + T)$ can be calculated for the given period T. Varying the initial time $t \in (t_A, t_E - T)$ we get a distribution of the possible kinematical actions which are related to the numerous possible periods T. This distribution is characterised by the mean $\overline{S_K(T)}$ and standard deviation $sd_t S_K(t, t + T)$.

Both, the mean and the standard deviation, depend on the length of interval T. For T close to zero, the mean grows monotonically with increasing interval. The standard deviation $sd_t S_K(t, t + T)$, however, shows a distinctive minimum which gives the averaged period of the cyclic moving plane. From now on, this minimum T_0 can be considered as fixed.

Since the data of S_K are gained empirically, this minimization can only be implemented numerically. For this calculation process an appropriate initial time $t = t_0$ is searched for the averaged period T_0. The initial time t_0 is determined requiring that the action with T_0 inserted has the minimum

$$S_K(t_0, t_0 + T_0) = \min_{t_0}.$$

Using the initial time t_0 and the averaged period T_0, the *cycle of least action* is figured out.

5. First example: Hinge movement

Hinge movement is a pure rotation around the hinge axis, located from the origins by the two constant vectors, S in the fixed and σ in the moving system. It holds

$$\mathbf{x} - \mathbf{s} = \mathbf{R} \cdot (\boldsymbol{\xi} - \boldsymbol{\sigma}),$$

or,

$$\mathbf{x} = \mathbf{R} \cdot \boldsymbol{\xi} + \underbrace{\mathbf{s} - \mathbf{R} \cdot \boldsymbol{\sigma}}_{=:\mathbf{d}_s}. \quad (28)$$

Using equations (16) and (23), we obtain for the characteristic point of the direct motion

$$\boldsymbol{\xi}_0 = -\frac{\int \dot{\mathbf{R}}^T \cdot \dot{\mathbf{d}}_s dt}{\int \dot{\alpha}^2 dt} = \boldsymbol{\sigma}.$$

The analogous calculation for the inverse motion yields

$$\mathbf{x}_0 = \mathbf{s}.$$

As expected, the characteristic points coincide with the hinge axis, represented by the two vectors **s** and **σ**. Their velocity and action are zero, since they are constant.

6. Second example: The human leg as a double hinge

We consider a double hinge, called also dimeric link chain, as a most simple model of non-trivial, planar movement. This model is widely used in robotics [13], gait synthesis [1], and in biomechanics [9], [12]. For the sagittal component of movement of the human leg during walking, this conception was introduced by Braune and Fischer [4], see Fig. 2.

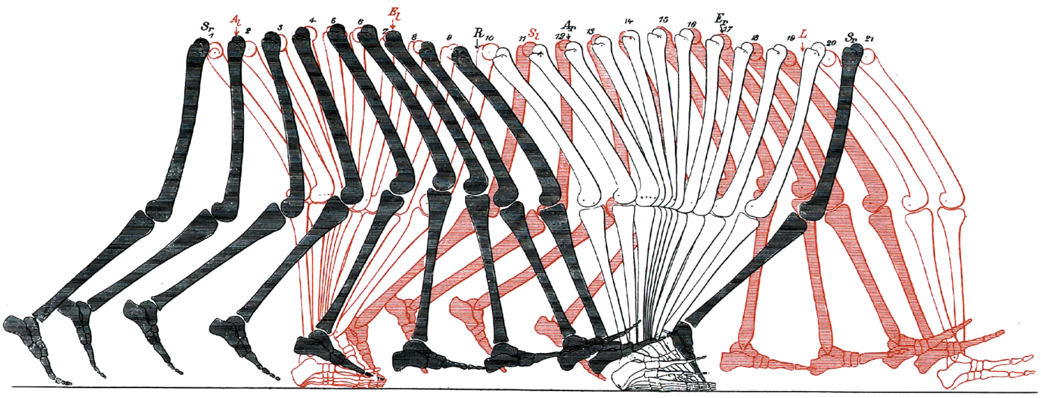

Fig. 2. The average bipedal walk of a human, from Braune and Fischer [4]

The motion of the human leg during gait is measured by using the spatial trajectories of three markers on the trochanter major, the knee center and the ankle. Considering only the respective sagittal components the leg is attributed to a double hinge (Fig. 3). The origin of the fixed system is the center of the hip joint whose x-axis points forwards. The origin of the moving system is the knee marker. The ξ-axis of the moving system connects the knee and the ankle marker. Accordingly, motion is reduced to the hip and the knee angle, as it is just common in gait analysis.

The mathematical procedure is: (a) we calculate the characteristic points analytically; (b) we insert these results into their respective minimal actions and add them to the kinematical action; (c) we calculate the cycle of least action and use it to finally determine the numerical values of the characteristic points.

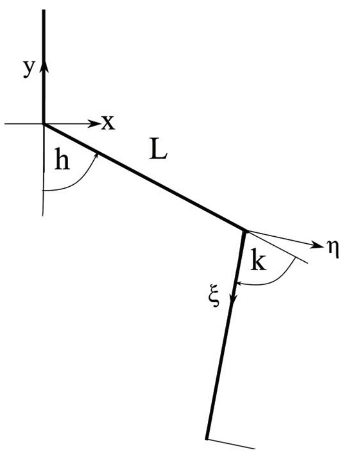

Fig. 3. The leg as a two-link chain. The axes of the fixed coordinate system (trunk) are x and y. The axes of the moving reference (tibia) are η and ξ. L is the length of the femur. The direction of the coordinate transformations is from above (trunk) to below (foot). Angles (h, k) are measured sagittally and relate to the inflection of the last segment. The sign of knee angle k is in accordance with the medical convention of a positive extension being represented mathematically as negative

6.1. Direct motion of a double hinge

A tibia-fixed point of the double hinge is described in Cartesian coordinates by the equations

$$x(\xi,\eta,t) = L\sin(h(t)) + \xi\sin(h(t)-k(t)) + \eta\cos(h(t)-k(t)), \quad (29)$$

$$y(\xi,\eta,t) = -L\cos(h(t)) - \xi\cos(h(t)-k(t)) + \eta\sin(h(t)-k(t)), \quad (30)$$

where $h - k \equiv \alpha$ is the resulting total angle.
By differentiating equations (29), (30), the velocities $\dot{x}(t)$ and $\dot{y}(t)$ are obtained through which the reduced action is calculated

$$2S = (\xi^2+\eta^2)\int(\dot{h}-\dot{k})^2 dt + L^2\int \dot{h}^2 dt + 2\xi L\int \cos(k)\dot{h}(\dot{h}-\dot{k})dt + 2\eta L\int \sin(k)\dot{h}(\dot{h}-\dot{k})dt.$$

The characteristic point and minimal action are

$$\xi_0 = -\frac{L\int(\cos k(t))\dot{h}(t)(\dot{h}(t)-\dot{k}(t))dt}{\int(\dot{h}(t)-\dot{k}(t))^2 dt},$$

$$\eta_0 = -\frac{L\int(\sin k(t))\dot{h}(t)(\dot{h}(t)-\dot{k}(t))dt}{\int(\dot{h}(t)-\dot{k}(t))^2 dt},$$

$$2S_0 = L^2\int \dot{h}^2 dt - (\xi_0^2+\eta_0^2)\int(\dot{h}-\dot{k})^2 dt. \quad (31)$$

6.2. Inverse motion

Inverting equations (29), (30) results in

$$\xi(x,y,t) = x\sin(h(t)-k(t))$$
$$- y\cos(h(t)-k(t)) - L\cos(k(t)),$$

$$\eta(x,y,t) = x\cos(h(t)-k(t))$$
$$+ k(t)) + y\sin(h(t)-k(t)) - L\sin(k(t)).$$

Next, the time derivative is taken and the squared results are added. We obtain the inverse action

$$2\overline{S} = (x^2+y^2)\int(\dot{h}-\dot{k})^2 dt + L^2\int \dot{k}^2 dt$$
$$+ 2xL\int(\sin h)\dot{k}(\dot{h}-\dot{k})dt$$
$$- 2yL\int(\cos h)\dot{k}(\dot{h}-\dot{k})dt.$$

The characteristic point and minimal action are

$$x_0 = -\frac{L\int(\sin h(t))(\dot{h}(t)-\dot{k}(t))\dot{k}(t)dt}{\int(\dot{h}(t)-\dot{k}(t))^2 dt}, \qquad (32)$$

$$y_0 = -\frac{L\int(\cos h(t))(\dot{h}(t)-\dot{k}(t))\dot{k}(t)dt}{\int(\dot{h}(t)-\dot{k}(t))^2 dt}, \qquad (33)$$

$$2\overline{S}_0 = L^2\int \dot{k}^2 dt - (x_0^2+y_0^2)\int(\dot{h}-\dot{k})^2 dt. \qquad (34)$$

6.3. Kinematical action of the double hinge

Using equations (31) and (34), the kinematical action of the leg motion is

$$2S_K = 2S_0 + 2\overline{S}_0 = L^2\int(\dot{h}^2+\dot{k}^2)dt$$
$$-(\xi_0^2+\eta_0^2+x_0^2+y_0^2)\int(\dot{h}-\dot{k})^2 dt.$$

We rescale the lengths of different individuals by setting the femur to unit length $L = 1$.

6.4. Application to experimental data

Our approach, combined with a statistical analysis regarding a small group of sound patients, results in a partial dependence of the characteristic points on the side of the leg used as shown in the following and illustrated by Fig. 8 and Table 1.

Table 1. The numerical values of the characteristic points: Ten volunteers times two legs (R, L), direct (ξ, η) and inverse motion (x, y)

Volunteer-Id	Leg	ξ	η	x	y
G0	R	–0.087	–0.075	0.200	–0.819
G1	R	–0.062	–0.097	0.339	–0.776
G2	R	–0.099	–0.068	0.159	–0.814
G3	R	0.005	–0.044	0.219	–0.909
G4	R	–0.063	–0.073	0.257	–0.825
G5	R	–0.051	–0.046	0.225	–0.869
G6	R	–0.021	–0.034	0.251	–0.891
G7	R	–0.043	–0.067	0.231	–0.843
G8	R	–0.033	–0.049	0.237	–0.867
G9	R	–0.024	–0.055	0.228	–0.884
G0	L	–0.041	–0.049	0.186	–0.867
G1	L	–0.014	–0.052	0.256	–0.893
G2	L	–0.082	–0.058	0.188	–0.833
G3	L	–0.011	–0.038	0.256	–0.896
G4	L	–0.045	–0.046	0.164	–0.878
G5	L	–0.084	–0.055	0.253	–0.833
G6	L	–0.004	–0.024	0.239	–0.910
G7	L	0.010	–0.038	0.217	–0.921
G8	L	–0.034	–0.050	0.154	–0.886
G9	L	–0.045	–0.044	0.158	–0.879

Four male and six female healthy subjects were advised to walk at a medium speed on plain ground (height; 162 cm to 190 cm (mean 175 cm); weight: 51 kg to 82 kg (mean 71 kg); age: 23 yrs to 68 yrs (mean 40 yrs). In total 20 leg measurements were analyzed. We started with the spatial trajectories of three markers near to the centers of the hip, knee and ankle joint and obtained the sagittal hip angle h_i and the sagittal knee angle k_i in the medical sign convention (Fig. 3). In Fig. 4, the measured hip and knee angles are shown as functions of the time frames.

Since the sample rate was sufficiently high (120 frames/sec), the derivatives were approximated by forward differences and the integrals by cumulative sums: thus the characteristic coefficients were given as numerical lists $I(t_A = 0, t_i)$. From these, we calculated the minimal actions $S_0(t_1, t_2)$ and $\overline{S}_0(t_1, t_2)$ as a function of two arguments using the differences $I(t_A = 1, t_2) - I(t_A = 1, t_1)$. The corresponding actions with $t_A = 1$ are shown in Fig. 5.

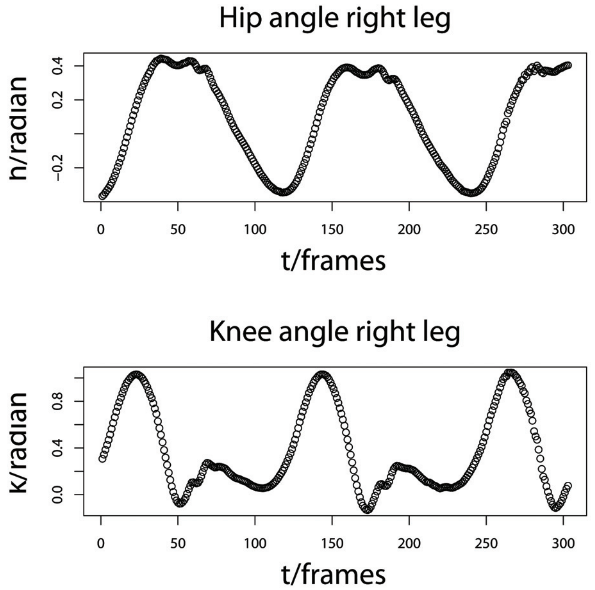

Fig. 4. The hip and the knee angle (in radians) of the right leg of a healthy human subject in the time frame

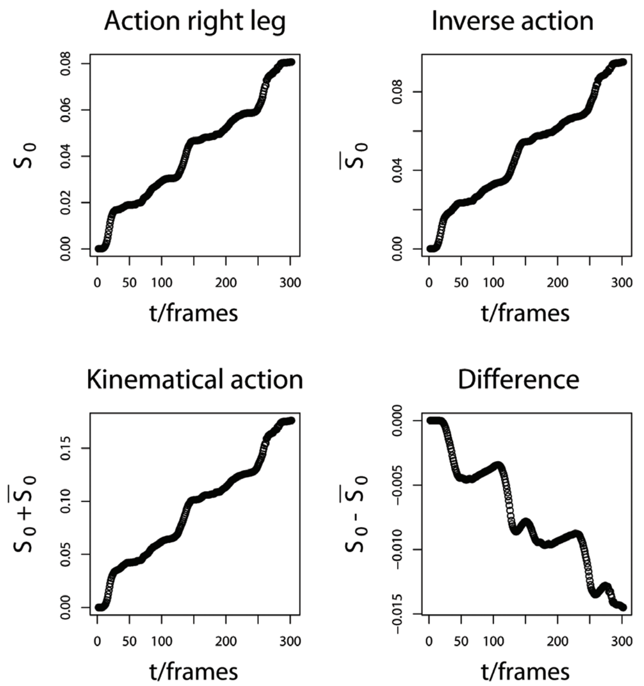

Fig. 5. Upper part: The minimal actions and of direct and inverse motion. Both functions display monotonous increase with time. Lower part: Their sum, the kinematical action. Although they look similar, the difference shows that they are not equal

The standard deviation $sd_t(S_K(t, t + T))$ is a function of T only. Since T is discrete, its minimum, the averaged period of the gait, can be found simply by selective sorting (Fig. 6, upper). This function shows a minimum at $T = T_0$. The minimum can again be selected (Fig. 6, lower). The resulting period has 122 frames which, using a sample rate of 120 Hz, equals a duration of 1.02 s. The initial time with the least action is at frame 134. Using these summation limits, we calculated the resulting stationary points measured in units of femur length.

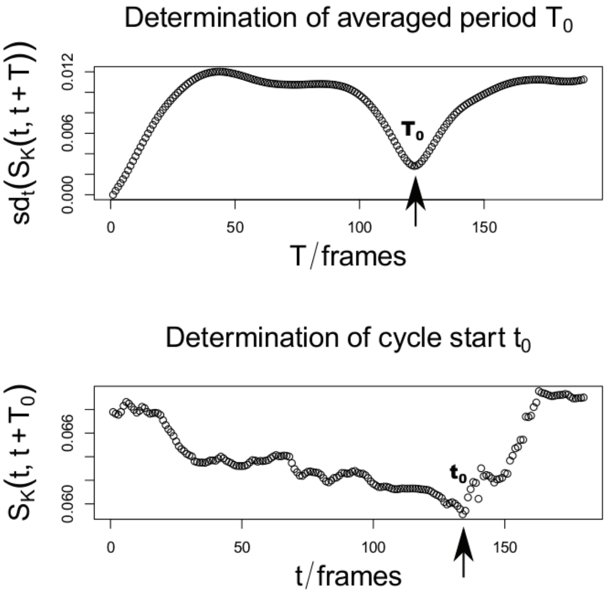

Fig. 6. The method for determining the period and the initial time by minimization is illustrated using the data from our example. Note: the standard deviation using the whole data set starts linearly from zero for small values, so that the minimum at the arrow is only local

Since we set the femur length to unit, the characteristic points of the collective (10 right and 10 left legs) could be evaluated. For inverse motion, we obtained characteristic points in the fixed system, see Table 1. The mean and standard deviation of their components were

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0.221 \pm 0.045 \\ -0.865 \pm 0.038 \end{pmatrix}.$$

The characteristic points lay below and anterior to the hip joint, and close above the knee marker in the mid of the stance phase (Fig. 7).

For the direct movement, we obtained the components of the characteristic points of the moving system. Means and standard deviations were found to be:

$$\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} -0.042 \pm 0.031 \\ -0.053 \pm 0.017 \end{pmatrix}.$$

These characteristic points were also found to be close to the knee point.

These means and standard deviations are expected to be different (in the sense of a statistical test) from those coming from ill subjects in order to distinguish the corresponding groups.

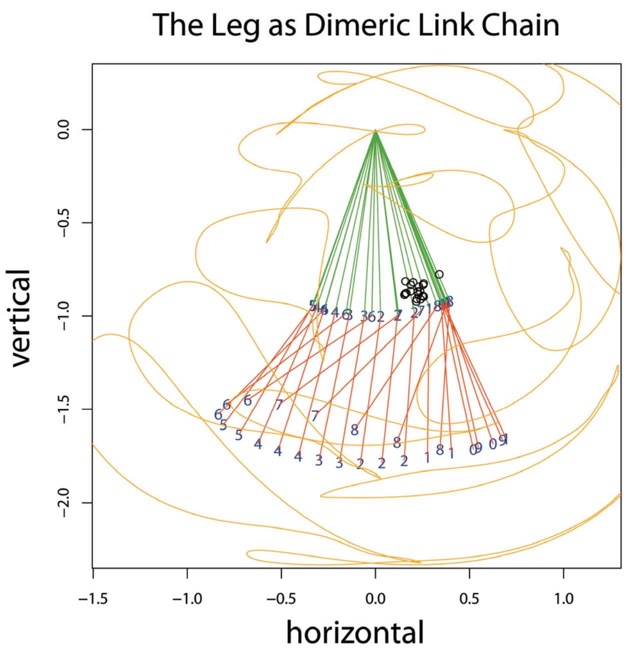

Fig. 7. The average motion of a human leg. Green: femur (with length 1). Red: tibia and foot (moving system against fixed hip). Blue: Integer part of stage in percent 10. Orange: Paths of arbitrarily selected points. Compare these arbitrary paths with those in Fig. 1 (magenta). Black circles: calculated characteristic points of the legs of 10 volunteers

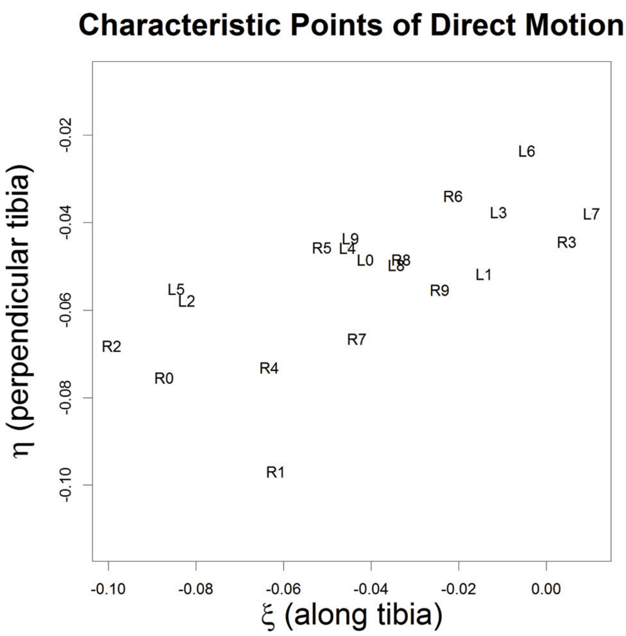

Fig. 8. The location of the characteristic points of the direct motion on the tibia depending on the side as plot symbol

Therefore we compared the 10 right with the 10 left legs of the collective measured. In Fig. 8, the respective characteristic points of direct motion are drawn in the moving reference system. We looked at the coordinate η and calculated the difference $\eta_R - \eta_L$ of each subject. A Wilcoxon signed rank test of these differences $\eta_R - \eta_L$ against zero resulted in a p-value of 0.0137. Hence, locations of the characteristic points partially depended on the side of the leg used.

7. Discussion

We presented a reading of the least action principle in the context of kinematics. It is not focused on the derivation of the equations of motion, because the motion in question is experimentally given. At first sight, this is the reason why this principle seems not to be applicable to kinematics. Notwithstanding, we use it to find some characteristic features of the motion under consideration such as the characteristic or stationary points, respectively. Anyhow, such points exist as soon as some integration limits (e.g., the initial and final time of the cycle of interest) are given. Our analytical results on the characteristic points (equations (16) and (25)) are independent of a linear transformation of the time, but they depend in general on the parametrization. Since our theory is related to experiments, time is the natural quantity for parametrizing the measured velocity and acceleration. Therefore, we use this as parametrization for the action instead of the "classical" kinematical parameter, namely the angle between the fixed and the moving frame. Nevertheless, we leave the question for a better parametrization open for further investigations.

The main advantage of this new method is that it is based on a physical principle and is therefore applicable to all kinds of movements. The calculations are mathematically simple, their results are unique and easy to implement but somehow abstract. For hinge movements, the algorithm may detect the hinge axes. There is no more dependence on a particular reference system: The two sides of a joint are equally represented.

We understand direct and inverse motion as two aspects of the same phenomenon. Although both should contain the whole information, their mathematical description is different. For example, in the first differential order, the fixed and moving centrodes belong to the direct and inverse motion. But, both together can be used to reconstruct the motion, since they roll on each other. The benefit is then:

We need not to descend too deep in the differential order.

Since the necessary calculations have to be carried out by computer, we are able to process larger samples of kinematical data. The cycles of minimal action thus determined can then be used to elaborate further characteristics for the given set of motions as, for example, Fourier coefficients or statistical moments. Since the problem of determining the period can only be treated numerically, it is possible to use other dispersion measures than standard deviation, for example, the interquartile range. The different outcomes might be considered in a future work.

As a first test of the applicability of our approach to experiments, we choosed data coming from the human gait. We found a partial dependence of the stationary points, on the side of the leg used (Fig. 8). This finding may be explained by the fact that approximately 90% of adult people are right-footed [7]. Hence, the characteristic points seem to be proper attributes to characterize (quasi-)periodic movements. We see an interesting experimental finding that, for our volunteers, the difference $S_0 - \overline{S}_0$ is much less than S_0 or $\overline{S}_0$ itself.

By that we are encouraged to investigate further collectives with well-defined diagnoses as nominal variables to distinguish subsets with statistical methods. We refer to different movements as coming from different situations such as medical diagnoses, biological variations and so on. One of the next goals in discriminating between different subsets of such a sample would be to find a set of suitable characteristic parameters of the motion with invariance properties like ours. In this regard, our paper is only the first step. The method of characteristic points can be extended to three dimensions. Because real-world motions have to be described by three-dimensional kinematics, our example requires either this extension, or an approximation of virtually planar motion to precise planar motion. In our case, the goodness of this approximation may be verified by inspecting the lateral components of the marker coordinates measured. An approach that may be used in coming applications would be to set up an algorithm to extract the planar component of virtually planar motion and to control this process.

Motion analysis like gait analysis uses typically a subset of kinematical, kinetical and morphometrical parameters. Our contribution adds a new brick to the kinematical ones. We expect it to be useful for all types of movements.

Appendix: Time variation of the characteristic points

Calculating the time variation of the characteristic points ξ_0 and $\mathbf{x}_0$, note that ξ_0 and $\mathbf{x}_0$, in contrast to the points ξ and $\mathbf{x}$, may not be considered as constant, since they depend on the upper and lower integration limits (equations (16), (25)). By studying the relationships $\dot{S}_0(\xi_0^2)$ and $\dot{\overline{S}}_0(\mathbf{x}_0^2)$ the time derivative of the kinematical action (equation (27)) can be calculated: We choose for all terms $t_1 = t_A$ and $t_2 = t$.

$\dot{S}_0(\xi_0^2)$: Differentiation of equation (15) yields

$$0 = \dot{\xi}_0 I_R + \xi_0 \dot{I}_R + \dot{\mathbf{R}}^T \cdot \dot{\mathbf{d}} = \dot{\xi}_0 I_R + \xi_0 \dot{\alpha}^2 + \dot{\mathbf{R}}^T \cdot \dot{\mathbf{d}} ,$$

and

$$\dot{\xi}_0(t_1,t_2) = -\frac{1}{I_R(t_1,t_2)}(\xi_0(t_1,t_2)\dot{\alpha}^2(t_2) + \dot{\mathbf{R}}^T(t_2)\cdot \dot{\mathbf{d}}(t_2) . \tag{A1}$$

By using $\dot{I}_\xi^2 + \dot{I}_\eta^2 = \dot{I}_R + \dot{I}_T$, one can show that

$$\frac{I_R}{\dot{I}_R}\dot{\xi}_0^2 = 2\dot{S}_0 . \tag{A2}$$

$\dot{\overline{S}}_0(\mathbf{x}_0^2)$: Differentiation of equation (25) yields

$$\mathbf{x}_0 = \frac{1}{I_R^2}((\dot{\alpha}^2\mathbf{d} + \dot{\alpha}\ \mathbf{J}\cdot\mathbf{d})I_R - \dot{\alpha}^2[\int \dot{\alpha}^2\mathbf{d}\ dt + J\int \dot{\alpha}\ \dot{\mathbf{d}}\ dt]) .$$

The term in the square brackets divided by I_R is again equal to $\mathbf{x_0}$. Therefore the last expression can be rewritten as

$$\dot{\mathbf{x}}_0(t_1,t_2) := \overline{\mathbf{v}}_0(t_1,t_2) = \frac{1}{I_R(t_1,t_2)}(\dot{\alpha}^2(t_2)\mathbf{d}(t_2) + \dot{\alpha}^2(t_2)\mathbf{J}\cdot\mathbf{d}(t_2) - \dot{\alpha}^2(t_2)\mathbf{x}_0(t_1,t_2)). \tag{A3}$$

By using $\dot{I}_x^2 + \dot{I}_y^2 = \dot{I}_R(\dot{I}_T + \dot{I}_D - 2\dot{I}_I)$, it holds

$$\frac{I_R^2}{\dot{I}_R}\dot{\mathbf{x}}_0^2 = 2\dot{\overline{S}}_0 . \tag{A4}$$

Adding equations (A2) and (A4) yields equation (27).

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