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Optimality conditions for a class of relaxed quasiconvex minimax problems

by

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Abstract: A class of minimax problems is considered. We approach it with the techniques of quasiconvex optimization, which includes most important nonsmooth and relaxed convex problems and has been intensively developed. Observing that there have been many contributions to various themes of minimax problems, but surprisingly very few on optimality conditions, the most traditional and developed topic in optimization, we establish both necessary and sufficient conditions for solutions and unique solutions. A main feature of this work is that the involved functions are relaxed quasiconvex in the sense that the sublevel sets need to be convex only at the considered point. We use star subdifferentials, which are slightly bigger than other subdifferentials applied in many existing results for minimization problems, but may be empty or too small in various situations. Hence, when applied to the special case of minimization problems, our results may be more suitable. Many examples are provided to illustrate the applications of the results and also to discuss the imposed assumptions.

Keywords: minimax problems, optimality conditions, convex sublevel sets, normal cones, star subdifferentials, adjusted subdifferentials

1. Introduction

Optimality conditions occupy central position in optimization theory. They are closely related to other fundamental topics in optimization, such as duality, stability and numerical methods. Among the earliest results of modern mathematics, we can see those on optimality conditions using classical derivatives, i.e., Gateaux and Fréchet derivatives. In the middle of the last century, nonsmooth optimization began to be developed, since most of the practical problems are nonsmooth, i.e., the involved functions and mappings do not have classical derivatives. Nonsmooth optimization has been very intensively investigated for at least three decades. Its problems are often also nonconvex. In a different direction of research, nonsmooth problems are assumed to satisfy relaxed convexity assumptions. Here, generalized derivatives are defined based on relaxed convexity properties. This paper is devoted to optimality conditions for relaxed-quasiconvex problems. Notice that quasiconvex problems constitute the most important class of generalized convex problems, since such problems are often met in practice and we still can apply the powerful tools of convex analysis in suitable ways to study them. There have been a number of works devoted to optimality conditions for quasiconvex minimization problems, see, e.g., Aussel and Hadjisavvas (2005), Aussel and Ye (2006), Daniilidis, Hadjisavvas and Martínez–Legaz (2001), Linh and Penot (2006), Penot (2003a, 2003b), and references therein. A function f from a normed space X to $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called quasiconvex if its sublevel set $L_f(x) := \{u \in X : f(u) \leq f(x)\}$ at x is convex for all $x \in X$ or, equivalently, if for each $r \in \mathbb{R}$ the strict sublevel set $L_f^<(r) := \{u \in X : f(u) < r\}$ is convex. Hence, f is quasiconvex if and only if the strict sublevel set $L_f^<(x) := \{u \in X : f(u) < f(x)\}$ is convex for all $x \in X$. Another equivalent statement, which is often met in the literature, is that f is quasiconvex if for all $x, y \in \mathrm{dom} f := \{x \in X : f(x) < +\infty\}$, all $t \in [0, 1], f((1-t)x + ty) \leq \max\{f(x), f(y)\}$. A minimization problem is quasiconvex if the objective is quasiconvex and the constraint set is convex. In Khanh, Quyen and Yao (2011), many optimality conditions for quasiconvex minimization problems are extended to relaxed quasiconvex cases. In this paper, we deal with such relaxed quasiconvex functions, too. Namely, we need to assume that sublevel sets L_f or $L_f^<$ are convex only at the point $\bar{x}$ under consideration, which is much weaker than the assumed quasiconvexity in Aussel and Hadjisavvas (2005), Aussel and Ye (2006), Daniilidis, Hadjisavvas and Martínez-Legaz (2001), Linh and Penot (2006), Penot (2003a, 2003b). (Function f_2 in all our Examples 1-7 is not quasiconvex.)

Besides minimization problems, minimax problems also draw much attention of researchers all over the world. These problems occur in game theory, economic equilibrium study, and also in many themes of minimization problems like duality, dual or primal-dual algorithms, Lagrange multipliers, etc. However, unlike for minimization problems, there have been surprisingly few papers dealing with optimality conditions for minimax problems (in fact, we see only Bhatia and Mehra, 2001, Chen and Lai, 2004, and some references therein). This motivates the aim of this paper, which is establishing optimality conditions for minimax problems. But, at this step we restrict ourselves to a particular class of minimax problems satisfying relaxed quasiconvexity assumptions. Our results are different from those of Bhatia and Mehra (2001) and Chen and Lai (2004), since the assumptions, conclusions, and techniques are different. They are also different from the results of Khanh, Quyen and Yao (2011) when applied to the special case of minimization problems. But, due to our relaxed quasiconvexity assumptions and our use of star subdifferentials, the results of this paper are more advantageous than those of Aussel and Hadjisavvas (2005), Aussel and Ye

(2006), and Linh and Penot (2006) in some cases.

The layout of the paper is as follows. Section 2 is devoted to preliminaries needed in the sequel. There we recall the notions of generalized subdifferentials appropriate for our study in this paper and define our minimax problem. In Section 3, we establish the necessary and sufficient conditions for our minimax problem, followed by many illustrative examples. The case of unique solutions is presented as well.

2. Preliminaries

Throughout the paper, let X be a normed space, $\mathbb{R}$ be the set of the real numbers and $\mathbb{R}_+ := [0, \infty)$. For $A \subseteq X$, int$A$, cl$A$ and cone$A$ denote the interior, closure and conical hull (called also the cone generated by A), i.e., $\text{cone}A := \{\lambda x : x \in A, \lambda \in \mathbb{R}_+\}$, respectively. The distance from $x \in X$ to A is $\text{dist}(x, A) := \inf\{\|x - y\| : y \in A\}$. X^* is the topological dual of X and $\langle ., . \rangle$ is the duality pairing. The normal cone at x to A, denoted by $N(A, x)$, is defined by

$$N(A, x) := \{x^* \in X^* : \forall u \in A, \langle x^*, u - x\rangle \leq 0\}.$$

If $x \notin \text{cl}A$, we adopt that $N(A, x) = \emptyset$. The contingent cone of A at $x \in X$, denoted by $T(A, x)$, is the following cone

$$T(A, x) := \{v \in X : \exists (r_n) \to 0_+, \exists (v_n) \to v, \forall n, x + r_n v_n \in A\}.$$

To see the relationships between $N(A, x)$ and $T(A, x)$, recall that the polar cones of cones $B \subseteq X$ and $D \subseteq X^*$ are

$$B^- := \{x^* \in X^* : \forall x \in B, \langle x^*, x\rangle \leq 0\},$$

$$D^- := \{x \in X : \forall x^* \in D, \langle x^*, x\rangle \leq 0\}.$$

Clearly, $N(A, x) = [\text{clcone}(A - x)]^-$. Setting, in the definition of $T(A, x)$, $x_n = x + r_n v_n$, we see that

$$T(A, x) = \{v : \exists (r_n) \to 0, \exists (x_n) \subseteq A \to x, \ \lim \frac{x_n - x}{r_n} = v\} \subseteq \text{clcone}(A - x).$$

Hence, $T(A, x)^- \supseteq N(A, x)$. Furthermore, if $v \in T(A, x)$, i.e., v is of the form $\lim \frac{x_n - x}{r_n}$, and $x^* \in N(A, x)$, then $\langle x^*, v\rangle \leq 0$. Therefore, $T(A, x) \subseteq N(A, x)^-$. Moreover, if A is convex, then the above containments become equalities.

Let $f : X \to \bar{\mathbb{R}}$ be finite at $\bar{x}$. f is said to be upper semicontinuous (shortly u.s.c.) at $\bar{x}$ if $\text{limsup}_{x \to \bar{x}} f(x) \leq f(\bar{x})$, and lower semicontinuous (shortly l.s.c.) at $\bar{x}$ if $\text{liminf}_{x \to \bar{x}} f(x) \geq f(\bar{x})$. We recall now the definitions of subdifferentials, needed in the sequel. The lower subdifferential or Plastria subdifferential from Plastria (1985) is defined by

$$\partial^< f(\bar{x}) := \left\{x^* \in X^* : \forall x \in L_f^<(\bar{x}), f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle\right\}.$$

The infradifferential or Gutiérrez subdifferential from Gutiérrez (1984) is

$$\partial^{\leq} f(\bar{x}) := \{x^* \in X^* : \forall x \in L_f(\bar{x}), f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle\}.$$

The Greenberg-Pierskalla subdifferential from Greenberg and Pierskalla (1973), which is akin to the normal cone, is defined by

$$\partial^* f(\bar{x}) := \left\{x^* \in X^* : \forall x \in L_f^<(\bar{x}), \langle x^*, x - \bar{x}\rangle < 0\right\}.$$

So, we say that it is a kind of normal-cone subdifferential. The star subdifferentials defined in Borde and Crouzeix (1990) and Penot and Zalinescu (2000) are the following normal-cone subdifferentials

$$\partial^{\nu} f(\bar{x}) := N(L_f(\bar{x}), \bar{x}),$$

$$\partial^{\circledast} f(\bar{x}) := N(L_f^<(\bar{x}), \bar{x}).$$

The adjusted sublevel set of f at $\bar{x}$ in Aussel and Ye (2006) is

$$L_f^a(\bar{x}) = L_f(\bar{x}) \cap \mathrm{cl}B(L_f^<(\bar{x}), \rho_{\bar{x}})$$

if $\bar{x}$ is not a global minimizer of f and $L_f^a(\bar{x}) = L_f(\bar{x})$ otherwise, where $B(A, \rho) := \{x \in X : \mathrm{dist}(x, A) < \rho\}$ and $\rho_x := \mathrm{dist}(x, L_f^<(x))$. The adjusted subdifferential in Aussel and Hadjisavvas (2005) is

$$\partial^a f(\bar{x}) := N(L_f^a(\bar{x}), \bar{x}).$$

It is obvious that

$$\partial^< f(\bar{x}) \subseteq \partial^* f(\bar{x}) \subseteq \partial^{\circledast} f(\bar{x}),$$

$$\partial^{\leq} f(\bar{x}) \subseteq \partial^{\nu} f(\bar{x}) \subseteq \partial^a f(\bar{x}) \subseteq \partial^{\circledast} f(\bar{x}).$$

For details about the calculus of these subdifferentials, the reader is referred to Aussel and Hadjisavvas (2005) and Penot (2003b). Although they are defined for arbitrary functions (finite at $\bar{x}$), they possess good properties only under additional conditions. In the literature, the sublevel sets are usually assumed to be convex, i.e., the functions are quasiconvex. In this paper, like in Khanh, Quyen and Yao (2011), we relax remarkably this assumption to the convexity only at $\bar{x}$.

By the above chains of inclusions of the subdifferentials, it is clear that $\mathbb{R}_+\partial^< f(\bar{x}) \subseteq \partial^{\circledast} f(\bar{x})$ and $\mathbb{R}_+\partial^{\leq} f(\bar{x}) \subseteq \partial^{\nu} f(\bar{x})$. Therefore, the following definitions are natural. A function f is said to be a Plastria function at $\bar{x}$ if its strict sublevel set $L_f^<(\bar{x})$ is convex and

$$\mathbb{R}_+\partial^< f(\bar{x}) = \partial^{\circledast} f(\bar{x}),$$

and to be a Gutiérrez function at $\bar{x}$ if $L_f(\bar{x})$ is convex and

$$\mathbb{R}_+\partial^{\leq} f(\bar{x}) = \partial^{\nu} f(\bar{x}).$$

In this paper, we consider optimality conditions for the following minimax problem

$$\text{(P)} \qquad \min_{x\in X}\max_{1\leq i\leq k} f_i(x),$$

$$g_j(x) \leq 0,\ j = 1, ...m,$$

where X is a normed space, $f_i : X \to \bar{\mathbb{R}}$, $g_j : X \to \bar{\mathbb{R}}$ for $i \in I := \{1, ..., k\}$ and $j \in J := \{1, ...m\}$.

Remark 1 *(i) In nonsmooth optimization-related problems, many kinds of generalized derivatives and subdifferentials have been proposed and used for studying optimality conditions and other themes. Each of them is effective for some problems, but none is universal for all situations. Like almost all existing contributions to types of quasiconvex problems, we invoke the above-mentioned subdifferentials, since they are designed just for such problems. They enable applications of powerful tools of convex analysis to the involved convex lower level sets. Note also that types of quasiconvex problems are the most important class of nonconvex problems.*

(ii) Following a suggestion of a referee, it is worth noticing that, in general, problem settings can be converted to each other in many cases. In particular, consider the following general minimax problem

$$\min_{x\in X}\max_{y\in Y} f(x, y),$$

$$g_j(x) \leq 0,\ j = 1, 2, ...,$$

where Y is an arbitrary index set. We can reformulate it to the following set-constrained optimization problem

$$\min_{(x,t)\in D}(t + \langle 0, x\rangle),$$

where $D := \{(x, t) \in X \times \mathbb{R} : f(x, y) - t \leq 0 \text{ for } y \in Y, g_i(x) \leq 0 \text{ for } i = 1, 2, ...\}$.

However, such a formulation may be convenient for some consideration purposes and not for others. In this paper, we choose the option of working directly on (P). In fact, the results we obtained here are different from those of Khanh, Quyen and Yao (2011) when applied to optimization problems, and in particular, Theorem 1 improves over Theorem 10 in Linh and Penot (2006), devoted to quasiconvex optimization problems (as explained in Remark 2 (iv) and Examples 1-4).

3. Optimality conditions for the minimax problem (P)

Consider problem (P) stated in the previous section. Denote
$g := \max_{j\in J} g_j$, $C := g^{-1}(-\infty, 0]$, $J(\bar{x}) := \{j \in J : g_j(\bar{x}) = g(\bar{x})\}$,
$I(\bar{x}) := \{i \in I : f_i(\bar{x}) = \max_{i\in I} f_i(\bar{x})\}$, $h := \max_{j\in J(\bar{x})} g_j$,
and $\rho_{f_i}(\bar{x}) := \text{dist}(\bar{x}, L^{<}_{f_i}(\bar{x}))$.

Theorem 1 *(necessary optimality condition). Let $\bar{x}$ be a solution of* (P) *with the corresponding objective value $\bar{q}$ such that $\bar{x}$ is not a local solution of the unconstrained problem corresponding to* (P). *Let the following conditions hold.*

(i) $L^{<}_{f_i}(\bar{x})$ *and* $L_{g_j}(\bar{x})$ *are convex for all $i \in I$ and $j \in J$; $\rho_{f_i}(\bar{x}) = 0$ for all $i \in I(\bar{x})$.*

(ii) *f_i and g_j are u.s.c. at $\bar{x}$ for all $i \notin I(\bar{x})$ and $j \notin J(\bar{x})$.*

(iii) (constraint qualification) *There exists $t \in J(\bar{x})$ such that*

$$L_{g_t}(\bar{x}) \cap \left(\cap_{j \neq t, j \in J(\bar{x})} \mathrm{int} L_{g_j}(\bar{x})\right) \neq \emptyset.$$

(iv) (regularity of the objective functions)

$$\cap_{i \in I} \mathrm{int} L^{<}_{f_i}(\bar{q}) \neq \emptyset.$$

Then, (besides zero) the following intersection contains also a nonzero point in X^*

$$\sum_{i \in I(\bar{x})} \partial^{\circledast} f_i(\bar{x}) \cap \left(- \sum_{j \in J(\bar{x})} \partial^{\nu} g_j(\bar{x}) \right). \tag{1}$$

Therefore, there exist $\lambda_i, \mu_j \in \mathbb{R}_+$ for $i \in I$ and $j \in J$ such that

$$\sum_{i \in I(\bar{x})} \lambda_i^2 > 0, \quad \sum_{j \in J(\bar{x})} \mu_j^2 > 0, \tag{2}$$

$$0 \in \sum_{i=1}^{k} \lambda_i \partial^{\circledast} f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \partial^{\nu} g_j(\bar{x}), \tag{3}$$

$$\lambda_i \left(f_i(\bar{x}) - \bar{q}\right) = 0, i = 1, ..., k, \tag{4}$$

$$\mu_j g_j(\bar{x}) = 0, j = 1, 2, .., m. \tag{5}$$

If, additionally, f_i is l.s.c. at $\bar{x}$, then we can replace $\partial^{\circledast} f_i$ by the smaller subdifferential $\partial^{a} f_i$ in (1) and (3).

Proof. It is clear that the fact that $\bar{x}$ is a solution of (P) with the objective value $\bar{q}$ means that, for all $x \in C$, there is $i_0 = i_0(x)$ such that $f_{i_0}(x) \geq \max_{i \in I} f_i(\bar{x}) = \bar{q}$. This is equivalent to the emptiness of the set $\{x \in C : \ f_i(x) < \bar{q} \text{ for all } i \in I\}$, i.e., $C \cap \bigcap_{i \in I} L^{<}_{f_i}(\bar{q}) = \emptyset$. By (i), these two disjoint sets are convex. Hence, due to (iv), the separation theorem yields $x^* \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, x - \bar{x} \rangle \geq \alpha \geq \langle x^*, w - \bar{x} \rangle \tag{6}$$

for all $x \in C$ and $w \in \cap_{i \in I} L^{<}_{f_i}(\bar{q})$. Setting $x = \bar{x}$ in the leftmost expression, one sees that $0 \geq \alpha$. Observe that the fact of $\bar{x}$ not being a local unconstrained solution means that for any neighborhood $B(\bar{x}, \frac{1}{n})$ of $\bar{x}$, there exists $x_n \in B(\bar{x}, \frac{1}{n})$ with $f_i(x_n) < \bar{q}$ for all $i \in I$. Hence, $x_n \in \cap_{i \in I} L^{<}_{f_i}(\bar{q})$ and $(x_n) \to \bar{x}$. By replacing w by x_n in (6), one gets $\alpha \geq 0$. Thus, $\alpha = 0$, and (6) becomes

$$\langle x^*, x - \bar{x} \rangle \geq 0 \geq \langle x^*, w - \bar{x} \rangle. \tag{7}$$

The left inequality means that $x^* \in -N(C,\bar{x})$. We now prove that $N(C,\bar{x}) \subseteq N(L_h(\bar{x}),\bar{x})$ by checking that $T(C,\bar{x}) \supseteq T(L_h(\bar{x}),\bar{x})$. Due to the convexity, one has $T(L_h(\bar{x}),\bar{x}) = \mathrm{clcone}(L_h(\bar{x}) - \bar{x})$, i.e., each point of $T(L_h(\bar{x}),\bar{x})$ is of the form $\lim t_l(x_l - \bar{x})$ as $l \to \infty$, for $t_l > 0$ and $x_l \in L_h(\bar{x})$. On the other hand, for any $x \in L_h(\bar{x})$ and $t > 0$, the point $t(x - \bar{x})$ has the following property. If $j \in J(\bar{x})$ and $x_t := \bar{x} + t(x - \bar{x})$, then $g_j(x_t) \leq 0$ for $t \in [0,1]$ by the convexity. If $j \notin J(\bar{x})$, $g_j(\bar{x}) < 0$. By the upper semicontinuity of g_j, $g_j(x) < 0$ for all x in a neighborhood of $\bar{x}$. So, $g_j(x_t) \leq 0$ for all small t. Thus, $x_t \in C$ for small t, and $t(x - \bar{x}) \in \mathrm{cone}(C - \bar{x})$ for all positive t. Therefore, $\lim t_l(x_l - \bar{x}) \in \mathrm{clcone}(C - \bar{x}) = T(C,\bar{x})$. Hence, $T(L_h(\bar{x}),\bar{x}) \subseteq T(C,\bar{x})$ and then $N(C,\bar{x}) \subseteq N(L_h(\bar{x}),\bar{x})$. Consequently,

$$x^* \in -N(L_h(\bar{x}),\bar{x}) = -N(\cap_{j\in J(\bar{x})} L_{g_j}(\bar{x}),\bar{x})$$

$$= -\sum_{j\in J(\bar{x})} N(L_{g_j}(\bar{x}),\bar{x}) = -\sum_{j\in J(\bar{x})} \partial^{\nu} g_j(\bar{x})$$

(the last but one equality is due to (iii) and the Moreau-Rockafellar theorem).

The right inequality of (7) means that $x^* \in N(\cap_{i\in I} L^{<}_{f_i}(\bar{q}),\bar{x})$. Since f_i is u.s.c. at $\bar{x}$ for $i \notin I(\bar{x})$, one has

$$N(\cap_{i\in I} L^{<}_{f_i}(\bar{q}),\bar{x}) = N(\cap_{i\in I(\bar{x})} L^{<}_{f_i}(\bar{q}),\bar{x}).$$

As $\rho_{f_i}(\bar{x}) = 0$ for all $i \in I(\bar{x})$, $N(\cap_{i\in I(\bar{x})} L^{<}_{f_i}(\bar{q}),\bar{x}) = N(\cap_{i\in I(\bar{x})} L^{<}_{f_i}(\bar{x}),\bar{x})$. By virtue of this and (iv), the Moreau-Rockafellar theorem gives

$$x^* \in N(\cap_{i\in I} L^{<}_{f_i}(\bar{q}),\bar{x}) = N(\cap_{i\in I(\bar{x})} L^{<}_{f_i}(\bar{x}),\bar{x})$$

$$= \sum_{i\in I(\bar{x})} N(L^{<}_{f_i}(\bar{x}),\bar{x}) = \sum_{i\in I(\bar{x})} \partial^{\circledast} f_i(\bar{x}).$$

Hence, we obtain (1). For $\lambda_i = 0$ and $\mu_j = 0$ for $i \notin I(\bar{x})$ and $j \notin J(\bar{x})$ and the other λ_i, μ_j being arbitrary such that

$$x^* \in \sum_{i\in I(\bar{x})} \lambda_i \partial^{\circledast} f_i(\bar{x}) \text{ and } x^* \in -\sum_{j\in J(\bar{x})} \mu_j \partial^{\nu} g_j(\bar{x}),$$

we have (2)-(5).

For $i \in I(\bar{x})$, as $\rho_{f_i}(\bar{x}) = 0$, we have $L^{a}_{f_i}(\bar{x}) = L_{f_i}(\bar{x}) \cap \mathrm{cl}L^{<}_{f_i}(\bar{x})$. Hence, $L^{a}_{f_i}(\bar{x}) \subseteq \mathrm{cl}L^{<}_{f_i}(\bar{x})$. Now, assuming that f_i are l.s.c. at $\bar{x}$, we show the reverse inclusion. Let $x_n \in L^{<}_{f_i}(\bar{x})$ and $(x_n) \to x$. By the lower semicontinuity, $f_i(\bar{x}) \geq \liminf f_i(x_n) \geq f_i(x)$, i.e., $x \in L_f(\bar{x})$. Hence, $x \in L^{a}_{f}(\bar{x})$. Looking at (7), we see that it is satisfied also for $w \in \mathrm{cl}\cap_{i\in I} L^{<}_{f_i}(\bar{q})$. Therefore,

$$x^* \in N(\mathrm{cl}\cap_{i\in I} L^{<}_{f_i}(\bar{q}),\bar{x}) = N(\mathrm{cl}\cap_{i\in I(\bar{x})} L^{<}_{f_i}(\bar{x}),\bar{x})$$

$$= N(\cap_{i\in I(\bar{x})} \mathrm{cl}L^{<}_{f_i}(\bar{x}),\bar{x}) = \sum_{i\in I(\bar{x})} N(\mathrm{cl}L^{<}_{f_i}(\bar{x}),\bar{x})$$

$$= \sum_{i\in I(\bar{x})} N(L^a_{f_i}(\bar{x}), \bar{x}) = \sum_{i\in I(\bar{x})} \partial^a f_i(\bar{x})$$

(for the third equality, we use (iv)). □

Remark 2 *(i) Note that in Theorem 1 as well as in our results below, only the sublevel sets at $\bar{x}$ are assumed to be convex.*

(ii) If f_i are Plastria functions at $\bar{x}$, and g_j are Gutiérrez functions at $\bar{x}$ for all $i \in I(\bar{x})$ and $j \in J(\bar{x})$, then, in Theorem 1, the intersection in (1) collapses to $\sum_{i\in I(\bar{x})} \partial^< f_i(\bar{x}) \cap (-\sum_{j\in J(\bar{x})} \partial^\leq g_j(\bar{x}))$. If, more specifically, f_i and g_j are Gateaux differentiable at $\bar{x}$ with nonzero derivatives, then, by Proposition 2.2 of Khanh, Quyen and Yao (2011), the conclusion of Theorem 1 becomes the assertion that there are λ_i and μ_j satisfying (2) such that

$$0 = \sum_{i\in I(\bar{x})} \lambda_i f_i'(\bar{x}) + \sum_{j\in J(\bar{x})} \mu_j g_j'(\bar{x}),$$

i.e., the classical Kuhn-Tucker condition.

(iii) Note also that the preceding statement, as well as the forthcoming ones, are different from the results in Bhatia and Mehra (2001) and Chen and Lai (2004), since the problem setting, assumptions and techniques are different.

(iv) Using the star subdifferentials $\partial^{\circledast}$ and ∂^ν makes our results more suitable in many cases. Even for the special case with $k = 1$, i.e., (P) collapses to a mathematical programming problem, Theorem 1 is more advantageous than Theorem 10 in Linh and Penot (2006), considering this case, since there f needs to be a Plastria function and g_j Gutiérrez functions. These are very restrictive conditions on $\partial^< f$ and $\partial^\leq g_j$. In all our Examples 1-4, $\partial^< f(\bar{x})$ is even empty, for $f = f_2$.

In the following example, Theorem 1 is used to reject a candidate for a solution of (P).

Example 1 Let $f_1, f_2, g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = -x$,

$$f_1(x) = \begin{cases} x & \text{if } x < 0, \\ 0 & \text{if } 0 \leq x \leq 1, \\ x-1 & \text{if } x > 1, \end{cases} \quad f_2(x) = \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x = 0 \text{ or } x > 2, \\ \frac{1}{4}x^2 - x + 2 & \text{if } 0 < x \leq 2, \end{cases}$$

and $\bar{x} = 1$ be a candidate to be considered. To verify assumption (i), we have $L^<_{f_1}(\bar{x}) = (-\infty, 0)$, $L^<_{f_2}(\bar{x}) = (1, \infty)$, $L_g(\bar{x}) = [1, \infty)$, $\bar{q} = f_2(1) = 5/4$, $I(\bar{x}) = \{2\}$, and $\rho_{f_2}(1) = 0$. Hence, (i) is fulfilled. So is (ii), since f_1 is u.s.c. at $\bar{x}$. (iii) and (iv) are obviously satisfied. To check the necessary condition (1), we have $\partial^{\circledast} f_1(\bar{x}) = \{x^* : \forall x \in (1, \infty), \langle x^*, x-1\rangle \leq 0\} = (-\infty, 0]$ and $\partial^\nu g(\bar{x}) = \{x^* : \forall x \geq 1, \langle x^*, x-1\rangle \leq 0\} = (-\infty, 0]$, and hence (1) is violated and Theorem 1 rejects $\bar{x} = 1$. We verify directly to see that $\bar{x} = 1$ is not a solution of (P) (the solutions are 0 and 2).

Example 2 (the condition $\rho_{f_i}(\bar{x}) = 0$ for all $i \in I(\bar{x})$ in (i) is essential). Let f_1 and g be as in Example 1. Let $\bar{x} = 2$ and

$$f_2(x) = \begin{cases} 1/2 & \text{if } x \leq 0, \\ \frac{1}{4}x^2 - x + 2 & \text{if } 0 < x \leq 2, \\ 1 & \text{if } x > 2. \end{cases}$$

To check (i) we have $L^{<}_{f_1}(\bar{x}) = (-\infty, 2)$, $L^{<}_{f_2}(\bar{x}) = (-\infty, 0)$, $L_g(\bar{x}) = [2, \infty)$, $\bar{q} = f_1(\bar{x}) = f_2(\bar{x}) = 1$, $I(\bar{x}) = \{1, 2\}$, $\rho_{f_1}(\bar{x}) = 0$, and $\rho_{f_1}(\bar{x}) = 2$. Thus, the last equality is not as required in (i). It is clear that (ii)-(iv) are satisfied. Condition (1) is fulfilled, since $\partial^{\circledast} f_1(\bar{x}) = \{x^* : \forall x \in (-\infty, 2), \langle x^*, x - 2\rangle \leq 0\} = [0, \infty)$, $\partial^{\circledast} f_2(\bar{x}) = \{x^* : \forall x \in (-\infty, 0], \langle x^*, x - 2\rangle \leq 0\} = [0, \infty)$, and $\partial^{\nu} g(\bar{x}) = \{x^* : \forall x \in [2, \infty), \langle x^*, x - 2\rangle \leq 0\} = (-\infty, 0]$. Clearly, $\bar{x} = 2$ has been shown not to be a solution.

Example 3 (the condition $\rho_{f_i}(\bar{x}) = 0$ for all $i \in I(\bar{x})$ in (i) is sufficient, but not necessary). Let f_1 and g be as in Example 1. Let $\bar{x} = 2$ and

$$f_2(x) = \begin{cases} 1/2 & \text{if } x < 0, \\ 1 & \text{if } x = 0 \text{ or } x > 2, \\ \frac{1}{4}x^2 - x + 2 & \text{if } 0 < x \leq 2. \end{cases}$$

For (i), we have $L^{<}_{f_1}(\bar{x}) = (-\infty, 2)$, $L^{<}_{f_2}(\bar{x}) = (-\infty, 0)$, $L_g(\bar{x}) = [2, \infty)$, $\bar{q} = f_1(\bar{x}) = f_2(\bar{x}) = 1$, $I(\bar{x}) = \{1, 2\}$, $\rho_{f_1}(\bar{x}) = 0$, and $\rho_{f_2}(\bar{x}) = 2$, and hence the last equality violates the condition in (i). (ii)-(iv) are trivially satisfied. Condition (1) is satisfied, since $\partial^{\circledast} f_1(\bar{x}) = \{x^* : \forall x \in (-\infty, 2), \langle x^*, x - 2\rangle \leq 0\} = [0, \infty)$, $\partial^{\circledast} f_2(\bar{x}) = \{x^* : \forall x \in (-\infty, 0), \langle x^*, x - 2\rangle \leq 0\} = [0, \infty)$, and $\partial^{\nu} g(\bar{x}) = \{x^* : \forall x \in [2, \infty), \langle x^*, x - 2\rangle \leq 0\} = (-\infty, 0]$. We see directly that $\bar{x} = 2$ is a solution.

Theorem 2 *(sufficient optimality condition). Let $\bar{x}$ be a feasible point of* (P). *Let, for $i \in I(\bar{x})$ and $j \in J(\bar{x})$, $L^{<}_{f_i}(\bar{x})$ and $L_{g_j}(\bar{x})$ be convex and f_i u.s.c. in $L^{<}_{f_i}(\bar{x})$. Then, the conclusion of Theorem 1 becomes a sufficient condition for optimality.*

Proof. Let $E = h^{-1}(-\infty, 0]$. Then $C \subseteq E$. We know that $\bar{x}$ is a solution of (P) if and only if $C \cap \bigcap_{i \in I} L^{<}_{f_i}(\bar{q}) = \emptyset$. We will prove the stronger conclusion that

$$E \cap \bigcap_{i \in I} L^{<}_{f_i}(\bar{x}) = \emptyset \tag{8}$$

(note that, for $i \in I(\bar{x})$, $L^{<}_{f_i}(\bar{q}) = L^{<}_{f_i}(\bar{x})$). Observe that (1) implies the existence of $x^* \in X^* \setminus \{0\}$ such that

$$x^* \in \sum_{i \in I(\bar{x})} N(L^{<}_{f_i}(\bar{x}), \bar{x}) \subseteq N(\cap_{i \in I(\bar{x})} L^{<}_{f_i}(\bar{x}), \bar{x}),$$

and

$$x^* \in -\sum_{j\in J(\bar{x})} N(L_{g_j}(\bar{x}),\bar{x}) \subseteq -N(\cap_{j\in J(\bar{x})}L_{g_j}(\bar{x}),\bar{x}) = -N(E,\bar{x})$$

(the inclusions follow from the convexity; regularity conditions are not needed). Hence, for all $x \in E$ and $w \in \cap_{i\in I(\bar{x})}L^{<}_{f_i}(\bar{x})$,

$$\langle x^*, x-\bar{x}\rangle \geq 0 \geq \langle x^*, w-\bar{x}\rangle .$$

Now suppose the contrary to (8), i.e. that there exists $v \in E\cap\bigcap_{i\in I} L^{<}_{f_i}(\bar{x})$. Then, $\langle x^*, v-\bar{x}\rangle = 0$. Note that $\cap_{i\in I}L^{<}_{f_i}(\bar{x})$ is open. Indeed, let $x \in L^{<}_{f_i}(\bar{x})$ and $i \in I(\bar{x})$. Then, $f_i(x) < f_i(\bar{x})$. As f_i is u.s.c. at x, there is a neighborhood of x, where f_i is also strictly less than $f_i(\bar{x})$, i.e., $L^{<}_{f_i}(\bar{x})$ is open and so is the mentioned intersection. For any $d \in X$ and small positive t, $v+td \in \cap_{i\in I}L^{<}_{f_i}(\bar{x})$. Consequently,

$$t < x^*, d >=< x^*, v-\bar{x}+td > - < x^*, v-\bar{x} >\leq 0.$$

Thus, $x^* = 0$, a contradiction. So, (8) holds and hence $\bar{x}$ is a solution of (P). □

The following example illustrates the above sufficient condition.

EXAMPLE 4 Let all the data be as in Example 3, except that now $\bar{x} = 0$. To check the assumptions, we see that $\bar{q} = f_2(\bar{x}) = 1$, $I(\bar{x}) = \{2\}$, $L^{<}_{f_2}(\bar{x}) = (-\infty, 0)$, $L_g(\bar{x}) = [0,\infty)$, and f_2 is u.s.c. in $L^{<}_{f_2}(\bar{x})$. Furthermore, (1) holds since $\partial^{\circledast} f_2(\bar{x}) = \{x^* : \forall x \in (-\infty,0), \langle x^*, x\rangle \leq 0\} = [0,\infty)$, and $\partial^{\nu} g(\bar{x}) = \{x^* : \forall x \in [0,\infty), \langle x^*, x\rangle \leq 0\} = (-\infty, 0]$. Theorem 2 asserts that $\bar{x} = 0$ is a solution, and this can also be seen directly.

Now we pass to considering unique solutions.

THEOREM 3 *(necessary optimality condition, unique solution). Let the assumptions of Theorem 1 be fulfilled. If, additionally, $\bar{x}$ is a unique solution and $C\setminus\{\bar{x}\} \neq \emptyset$, then $\partial^{\circledast} f_i$, $i \in I(\bar{x})$, used in Theorem 1, can be replaced by the smaller subdifferential $\partial^{\nu} f_i$.*

Proof. That $\bar{x}$ is a unique solution of (P) means that $C \cap \bigcap_{i\in I} L_{f_i}(\bar{q}) = \{\bar{x}\}$. Hence, $\bar{x}$ cannot be in the interior of $\cap_{i\in I}L_{f_i}(\bar{q})$, which is nonempty by (iv). Therefore, the separation theorem yields $x^* \in X^*\setminus\{0\}$ such that, for $x \in C$ and $w \in \cap_{i\in I(\bar{x})}L_{f_i}(\bar{q})$,

$$\langle x^*, x-\bar{x}\rangle \geq 0 \geq \langle x^*, w-\bar{x}\rangle . \tag{9}$$

The rest of the proof is similar to that for Theorem 1. But here, we are working with L_{f_i} instead of $L^{<}_{f_i}$ as in Theorem 1, and hence we obtain that $\partial^{\nu} f_i$ replaces $\partial^{\circledast} f_i$ in the conclusion. □

The following example says that we cannot replace $\partial^{\circledast} f_i$, $i \in I(\bar{x})$, by $\partial^{\nu} f_i$ in Theorem 1 (as we do in Theorem 3).

EXAMPLE 5 Let all the data be as in Example 3, except that now $\bar{x} = 0$. It is not hard to verify that the assumptions of Theorem 1 are satisfied. Since $L_{f_2}(\bar{x}) = (-\infty, 0] \cup [2, \infty)$ and hence $\partial^\nu f_2(\bar{x}) = \{0\}$, we have $\partial^\nu f_2(\bar{x}) \cap (-\partial^\nu g(\bar{x})) = \{0\}$. It is seen in this case that the sublevel set $L_{f_2}(\bar{x})$ at the considered point is not convex, but Theorem 1 still works.

THEOREM 4 *(sufficient optimality condition, unique solution). Let the assumptions of Theorem 2 be satisfied. Suppose, furthermore, that $f_i^{-1}(f_i(\bar{x})) = \{\bar{x}\}$ for all $i \in I(\bar{x})$. Then, the necessary condition stated in Theorem 3 is a sufficient one.*

Proof. The necessary condition mentioned implies the existence of $x^* \in X^* \setminus \{0\}$ satisfying (9). Suppose that $\bar{x}$ is not a unique solution, i.e., one finds $v \in C \cap \bigcap_{i \in I} L_{f_i}(\bar{q})$ different from $\bar{x}$. Then, $v \in C \cap \bigcap_{i \in I(\bar{x})} L_{f_i}(\bar{x})$. By (9), $\langle x^*, v - \bar{x} \rangle = 0$. On the other hand, as $f_i^{-1}(f_i(\bar{x})) = \{\bar{x}\}$, $L_{f_i}(\bar{x}) \setminus \{\bar{x}\} = L^<_{f_i}(\bar{x})$ for $i \in I(\bar{x})$. By the upper semicontinuity of f_i, $L^<_{f_i}(\bar{x})$ is open and so is $(\cap_{i \in I(\bar{x})} L_{f_i}(\bar{x})) \setminus \{\bar{x}\}$. Consequently, for any $d \in X$ and small positive t, $v + td \in \cap_{i \in I} L_{f_i}(\bar{x})$. Therefore,

$$t < x^*, d > = < x^*, v - \bar{x} + td > - < x^*, v - \bar{x} > \leq 0.$$

Thus, $x^* = 0$, a contradiction. □

Theorem 4 confirms that a candidate is a solution in the next example.

EXAMPLE 6 Let f_1 and g be as in Example 1. Let $\bar{x} = 0$ and

$$f_2(x) = \begin{cases} 1/2(x+1) & \text{if } x \leq 0, \\ \frac{1}{4}x^2 - x + 2 & \text{if } 0 < x \leq 2, \\ 1 & \text{if } x > 2. \end{cases}$$

Clearly, the assumptions of Theorem 2 are fulfilled. We have $I(\bar{x}) = \{2\}$, $\bar{q} = 1/2$, and $f_2^{-1}(f_2(\bar{x})) = \{\bar{x}\}$. Furthermore, as $\partial^\nu f_2(\bar{x}) \cap (-\partial^\nu g(\bar{x})) = [0, \infty) \neq \{0\}$, Theorem 4 asserts that $\bar{x} = 0$ is the unique solution, which is also seen directly.

In Theorem 4, the condition that $f_i^{-1}(f_i(\bar{x})) = \{\bar{x}\}$ for all $i \in I(\bar{x})$ is essential as shown by the following

EXAMPLE 7 Let f_1, g be as in Example 1, $\bar{x} = 2$, and

$$f_2(x) = \begin{cases} x + 1 & \text{if } x \leq 0, \\ \frac{1}{4}x^2 - x + 2 & \text{if } x > 0. \end{cases}$$

We have $I(\bar{x}) = \{1, 2\}$, $\bar{q} = 1$, $L^<_{f_1}(\bar{x}) = (-\infty, 2)$, $L^<_{f_2}(\bar{x}) = (-\infty, 0)$, $L_g(\bar{x}) = [2, \infty)$, and f_i is u.s.c in $L^<_{f_i}(\bar{x})$ for $i = 1, 2$. Furthermore, $L_{f_1}(\bar{x}) = (-\infty; 2]$, $\partial^\nu f_1(\bar{x}) = \mathbb{R}_+$, $L_{f_2}(\bar{x}) = (-\infty; 0] \cup \{2\}$, $\partial^\nu f_2(\bar{x}) = \mathbb{R}_+$, $\partial^\nu g(\bar{x}) = \mathbb{R}_-$, and hence $\sum_{i=1,2} \partial^\nu f_i(\bar{x}) \cap (-\partial^\nu g(\bar{x})) = [0, \infty)$, as Theorem 4 requires. But, $\bar{x} = 2$ is not a unique solution (0 is another one). The cause is that $f_2^{-1}(f_2(\bar{x})) = \{0; 2\} \neq \{\bar{x}\}$.

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