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Reliability modelling of complex systems - Part 2

Keywords

reliability, large system, asymptotic approach, limit reliability function

Abstract

Series-“*m* out of *n*” systems and “*m* out of *n*”-series systems are defined and exemplary theorems on their limit reliability functions are presented and applied to the reliability evaluation of a piping transportation system and a rope elevator. Applications of the asymptotic approach in large series systems reliability improvement are also presented. The paper is completed by showing the possibility of applying the asymptotic approach to the reliability analysis of large systems placed in their variable operation processes. In this scope, the asymptotic approach to reliability evaluation for a port grain transportation system related to its operation process is performed.

1. Reliability of large series-“*m* out of *n*” systems

Definition 1. A two-state system is called a series-“*m* out of k_n” system if its lifetime T is given by

$$T = T_{(k_n - m+1)}, \ m = 1,2,...,k_n,$$

where $T_{(k_n - m+1)}$ is the mth maximal order statistic in the set of random variables

$$T_i = \min_{1 \le j \le l_i} \{T_{ij}\}, \ i = 1,2,...,k_n.$$

The above definition means that the series-“*m* out of k_n” system is composed of k_n series subsystems and it is not failed if and only if at least m out of its k_n series subsystems are not failed.
The series-“*m* out of k_n” system is a series-parallel for $m = 1$ and it becomes a series system for $m = k_n$.
The reliability function of the two-state series-“*m* out of k_n” system is given either by

$$\boldsymbol{R}^{(m)}_{k_n, l_1, l_2, ... l_{k_n}}(t) = 1$$

$$- \sum_{\substack{r_1, r_2, ..., r_{k_n} = 0 \\ r_1 + r_2 + ... + r_{k_n} \le m-1}}^{1} \prod_{i=1}^{k_n} [\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i} [1 - \prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i},$$

for $t \in (-\infty, \infty)$ or by

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n, l_1, l_2, ... l_{k_n}}(t)$$

$$= \sum_{\substack{r_1, r_2, ..., r_{k_n} = 0 \\ r_1 + r_2 + ... + r_{k_n} \le \overline{m}}}^{1} \prod_{i=1}^{k_n} [1 - \prod_{j=1}^{l_i} R_{ij}(t)]^{r_i} [\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i},$$

for $t \in (-\infty, \infty)$, where $\overline{m} = k_n - m$.

Definition 2. The series-“*m* out of k_n” system is called regular if

$$l_1 = l_2 = ... = l_{k_n} = l_n, \ \ l_n \in N.$$

Definition.3. The series-“*m* out of k_n” system is called homogeneous if its component lifetimes T_{ij} have an identical distribution function

$$F(t) = P(T_{ij} \le t), \ t \in (-\infty, \infty), \ i = 1,2,...,k_n, \ j = 1,2,...,l_i,$$

i.e. if its components E_{ij} have the same reliability function

$$R(t) = 1 - F(t), \ t \in (-\infty, \infty).$$

From the above definitions it follows that the reliability function of the homogeneous and regular series-“*m* out of k_n” system is given either by

$$\boldsymbol{R}^{(m)}_{k_n,l_n}(t) = 1 - \sum_{i=0}^{m-1}\binom{k_n}{i}[R^{l_n}(t)]^i[1-R^{l_n}(t)]^{k_n-i}$$

for $t \in (-\infty,\infty)$ or by

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t) = \sum_{i=0}^{\overline{m}}\binom{k_n}{i}[1-R^{l_n}(t)]^i[R^{l_n}(t)]^{k_n-i}$$

for $t \in (-\infty,\infty)$, $\overline{m} = k_n - m$, where k_n is the number of series subsystems in the "m out of k_n" system and l_n is the number of components of the series subsystems.

Corollary 1. If components of the homogeneous and regular two-state series-"m out of k_n" system have Weibull reliability function

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \ge 0,\ \alpha > 0,\ \beta > 0,$$

then its reliability function is given either by

$$\boldsymbol{R}^{(m)}_{k_n,l_n}(t)$$

$$= 1 - \sum_{i=0}^{m-1}\binom{k_n}{i}[\exp[-il_n\beta t^{\alpha}]][1-\exp[-l_n\beta t^{\alpha}]]^{k_n-i}$$

for $t \ge 0$ or by

$$\overline{\boldsymbol{R}}^{(\overline{m})}_{k_n,l_n}(t)$$

$$= \sum_{i=0}^{\overline{m}}\binom{k_n}{i}[1-\exp[-l_n\beta t^{\alpha}]]^i[\exp[-(k_n-i)l_n\beta t^{\alpha}]] \quad (2)$$

for $t \ge 0$, $\overline{m} = k_n - m$.

Proposition 1. If components of the two-state homogeneous and regular series-"m out of k_n" system have Weibull reliability function

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \ge 0,\ \alpha > 0,\ \beta > 0,$$

and

$$\lim_{n\to\infty} k_n = k,\ k > 0,\ 0 < m \le k,\ \lim_{n\to\infty} l_n = \infty,$$

$$a_n = (\beta l_n)^{-\frac{1}{\alpha}},\quad b_n = 0,$$

then

$$\boldsymbol{\mathfrak{R}}^{(2)}_{9}(t)$$

$$= 1 - \sum_{i=0}^{m-1}\binom{k}{i}\exp[-it^{\alpha}][1-\exp[-t^{\alpha}]]^{k-i} \text{ for } t \ge 0$$

is its limit reliability function, i.e., for $t \ge 0$, we have

$$\boldsymbol{R}^{(m)}_{k_n,l_n}(t) \cong \boldsymbol{\mathfrak{R}}^{(2)}_{9}\left(\frac{t-b_n}{a_n}\right)$$

$$= 1 - \sum_{i=0}^{m-1}\binom{k}{i}\exp[-i\beta l_n t^{\alpha}][1-\exp[-\beta l_n t^{\alpha}]]^{k-i}. \quad (3)$$

Example 1. The piping transportation system is set up to receive from ships, store and send by carriages or cars oil products such as petrol, driving oil and fuel oil. Three terminal parts A, B and C fulfil these purposes. They are linked by the piping transportation systems. The unloading of tankers is performed at the pier. The pier is connected to terminal part *A* through the transportation subsystem S_1 built of two piping lines. In part *A* there is a supporting station fortifying tankers' pumps and making possible further transport of oil by means of subsystem S_2 to terminal part *B*. Subsystem S_2 is built of two piping lines. Terminal part *B* is connected to terminal part *C* by subsystem S_3. Subsystem S_3 is built of three piping lines. Terminal part *C* is set up for loading the rail cisterns with oil products and for the wagon carrying these to the railway station.
We will analyse the reliability of the subsystem S_3 only. This subsystem consists of $k_n = 3$ identical piping lines, each composed of $l_n = 360$ steel pipe segments. In each of lines there are pipe segments with Weibull reliability function

$$R(t) = \exp[-0.0000000008t^4] \text{ for } t \ge 0.$$

We suppose that the system is good if at least 2 of its piping lines are not failed. Thus, according to *Definitions 2-3*, it may be considered as a homogeneous and regular series-"2 out of 3" system, and according to *Proposition 1*, assuming

$$a_n = \frac{1}{(\beta l_n)^{1/\alpha}} = \frac{1}{(0.000000288)^{1/4}},\ b_n = 0,$$

and using (3), its reliability function is given by

$$\boldsymbol{R}^{(2)}_{3,360}(t) \cong \boldsymbol{\mathfrak{R}}^{(2)}_{9}\left(\frac{t}{a_n}\right)$$

$$= \sum_{i=0}^{1}\binom{3}{i}\exp[-i\cdot 0.000000288t^4]$$

$$\cdot[1-\exp[-0.000000288t^4]]^{3-i} \text{ for } t \ge 0.$$

2. Reliability of large "*m* out of *n*"-series systems

Definition 4. A two-state system is called an "m_i out of l_i"-series system if its lifetime T is given by

$$T = \min_{1 \le i \le k_n} T_{(l_i - m_i + 1)},\ m_i = 1,2,\ldots,l_i,$$

where $T_{(l_i - m_i + 1)}$ is the m_ith maximal order statistic in the set of random variables

$$T_{i1},\ T_{i2},\ \ldots,\ T_{il_i},\quad i = 1,2,\ldots,k_n.$$

The above definition means that the "m_i out of l_i"-series system is composed of k_n subsystems that are "m_i out of l_i" systems and it is not failed if all its "m_i out of l_i" subsystems are not failed.
The "m_i out of l_i"-series system is a parallel-series system if $m_1 = m_2 = \ldots = m_{k_n} = 1$ and it becomes a series system if $m_i = l_i$ for all $i = 1,2, \ldots, k_n$.

The reliability function of the two-state "m_i out of l_i"-series system is given either by

$$\overline{\boldsymbol{R}}^{(m_1,m_2,\ldots,m_{k_n})}_{k_n,l_1,l_2,\ldots,l_{k_n}}(t)$$

$$= \prod_{i=1}^{k_n} [1 - \sum_{\substack{r_1,r_2,\ldots,r_{l_i}=0 \\ r_1+r_2+\ldots+r_{l_i} \le m_i - 1}}^{1} [\prod_{j=1}^{l_i} R_{ij}(t)]^{r_i} [1 - \prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i}]$$

for $t \in (-\infty,\infty)$ or by

$$\overline{\overline{\boldsymbol{R}}}^{(\bar{m}_1,\bar{m}_2,\ldots,\bar{m}_{k_n})}_{k_n,l_1,l_2,\ldots,l_{k_n}}(t)$$

$$= \prod_{i=1}^{k_n} [\sum_{\substack{r_1,r_2,\ldots,r_{l_i}=0 \\ r_1+r_2+\ldots+r_{l_i} \le \bar{m}_i}}^{1} [1 - \prod_{j=1}^{l_i} R_{ij}(t)]^{r_i} [\prod_{j=1}^{l_i} R_{ij}(t)]^{1-r_i}]$$

for $t \in (-\infty,\infty)$, where $\bar{m}_i = l_i - m_i,\ i = 1,2,\ldots,k_n$.

Definition 5. The two-state "m_i out of l_i"-series system is called homogeneous if its component lifetimes T_{ij} have an identical distribution function

$$F(t) = P(T_{ij} \le t),\ t \in (-\infty,\infty),\ i = 1,2,\ldots,k_n,\ j = 1,2,\ldots,l_i,$$

i.e. if its components E_{ij} have the same reliability function

$$R(t) = 1 - F(t),\ t \in (-\infty,\infty).$$

Definition 6. The "m_i out of l_i"-series system is called regular if

$$l_1 = l_2 = \ldots = l_{k_n} = l_n$$

and

$$m_1 = m_2 = \ldots = m_{k_n} = m, \text{ where } l_n,\ m \in N,\quad m \le l_n.$$

The reliability function of the two-state homogeneous and regular „m out of l_n"-series system is given either by

$$\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}(t) = [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} [R(t)]^i [1 - R(t)]^{l_n - i}]^{k_n}$$

for $t \in (-\infty,\infty)$ or by

$$\overline{\overline{\boldsymbol{R}}}^{(\bar{m})}_{k_n,l_n}(t) = [\sum_{i=0}^{\bar{m}} \binom{l_n}{i} [1 - R(t)]^i [R(t)]^{l_n - i}]^{k_n}$$

for $t \in (-\infty,\infty)$, $\bar{m} = l_n - m$ where k_n is the number of "m out of l_n" subsystems linked in series and l_n is the number of components in the "m out of l_n" subsystems.

Corollary 2. If the components of the two-state homogeneous and regular "m out of l_n"-series system have Weibull reliability function

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \ge 0,\ \alpha > 0,\ \beta > 0,$$

then its reliability function is given either by

$$\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}(t)$$

$$= [1 - \sum_{i=0}^{m-1} \binom{l_n}{i} \exp[-i\beta t^{\alpha}] [1 - \exp[-\beta t^{\alpha}]]^{l_n - i}]^{k_n} \tag{4}$$

for $t \ge 0$ or by

$$\overline{\overline{\boldsymbol{R}}}^{(\bar{m})}_{k_n,l_n}(t) = [\sum_{i=0}^{\bar{m}} \binom{l_n}{i} [1 - \exp[-\beta t^{\alpha}]]^i \exp[-(l_n - i)\beta t^{\alpha}]]^{k_n}$$

for $t \ge 0$, $\bar{m} = l_n - m$.
Proposition 2. If components of the two-state homogeneous and regular "m out of l_n"-series system have Weibull reliability function

$$R(t) = \exp[-\beta t^{\alpha}] \text{ for } t \ge 0,\ \alpha > 0,\ \beta > 0,$$

and

$$\lim_{n\to\infty} k_n = k,\ k > 0,\ \ 0 < m \le k,\ \ \lim_{n\to\infty} l_n = \infty,$$

$$a_n = \frac{b_n}{\alpha \log n},\ \ b_n = [\frac{\log n}{\beta}]^{\frac{1}{\alpha}}, \tag{5}$$

then

$$[\overline{\mathfrak{R}}_3^{(0)}(t)]^k = [1 - \exp[-\exp[-t]]\sum_{i=0}^{m-1}\frac{\exp[-it]}{i!}]^k$$

for $t \in (-\infty,\infty)$, is its limit reliability function, i.e.

$$\overline{\boldsymbol{R}}^{(m)}_{k_n,l_n}(t) \cong [\overline{\mathfrak{R}}_3^{(0)}(\frac{t-b_n}{a_n})]^k$$

$$= [1 - \exp[-\exp[-\frac{t-b_n}{a_n}]]\sum_{i=0}^{m-1}\frac{\exp[-i\frac{t-b_n}{a_n}]}{i!}]^k \tag{6}$$

for $t \in (-\infty,\infty)$, where a_n and b_n are defined by (5).

Example 2. Let us consider the ship-rope transportation system (elevator). The elevator is used to dock and undock ships coming in to shipyards for repairs. The elevator is composed of a steel platform-carriage placed in its syncline (hutch). The platform is moved vertically with 10 rope hoisting winches fed by separate electric motors. During ship docking the platform, with the ship settled in special supporting carriages on the platform, is raised to the wharf level (upper position). During undocking, the operation is reversed. While the ship is moving into or out of the syncline and while stopped in the upper position the platform is held on hooks and the loads in the ropes are relieved.
In our further analysis we will discuss the reliability of the rope system only. The system under consideration is in order if all its ropes do not fail. Thus we may assume that it is a series system composed of 10 components (ropes). Each of the ropes is composed of 22 strands. Thus, considering the strands as basic components of the system and assuming that each of the ropes is not failed if at least $m = 5$ out of its strands are not failed, according to *Definitions 5-6*, we conclude that the rope elevator is the two-state homogeneous and regular „5 out of 22"-series system. It is composed of k_n = 10 series-linked "5 out of 22" subsystems (ropes) with l_n = 22 components (strands). Assuming additionally that strands have Weibull reliability functions with parameters $\alpha = 2$, $\beta = 0.05$, i.e.

$$R(t) = \exp[-0.05t^2] \text{ for } t \ge 0,$$

from (4), we conclude that the elevator reliability function is given by

$$\overline{\boldsymbol{R}^{(5)}_{10,22}(t)} =$$

$$[1 - \sum_{i=0}^{4}\binom{22}{i}\exp[-i0.05t^2][1-\exp[-0.05t^2]]^{22-i}]^{10}$$

for $t \ge 0$.

Next, applying *Proposition 2* with

$$a_n = \frac{7.8626}{2\log 22} \cong 1.2718,\ \ b_n = [\frac{\log 22}{0.05}]^{\frac{1}{2}} \cong 7.8626,$$

and (6) we get the following approximate formula for the elevator reliability function

$$\overline{\boldsymbol{R}^{(5)}_{10,22}(t)} \cong [\overline{\mathfrak{R}}_3^{(0)}(0.7863t - 6.1821)]^{10}$$

$$= [1 - \exp[-\exp[-0.7863t + 6.1821]]$$

$$\cdot \sum_{i=0}^{4}\frac{\exp[-0.7863it + 6.1821i]}{i!}]^{10},\ t \in (-\infty,\infty).$$

3. Asymptotic approach to systems reliability improvement

We consider the homogeneous series system illustrated in *Figure 1*.

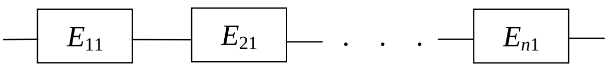

Figure 1. The scheme of a series system

It is composed of n components E_{i1}, $i = 1,2,\ldots,n$, having lifetimes T_{i1}, $i = 1,2,\ldots,n$, and exponential reliability functions

$$R(t) = \exp[-\lambda t] \text{ for } t \ge 0,\ \ \lambda > 0.$$

Its lifetime and its reliability function respectively are given by

$$T^{(0)} = \min_{1 \le i \le n}\{T_{i1}\},$$

$$\boldsymbol{R}_n(t) = [R(t)]^n = \exp[-\lambda nt],\ \ t \ge 0.$$

In order to improve of the reliability of this series system the following exemplary methods can be used:

– replacing the system components by the improved components with reduced failure rates by a factor ρ, $0 < \rho < 1$,
– a warm duplication (a single reservation) of system components,
– a cold duplication of system components,
– a mixed duplication of system components,
– a hot system duplication,
– a cold system duplication.

It is supposed here that the reserve components are identical to the basic ones.

The results of these methods of system reliability improvement are briefly presented below, giving the system schemes, lifetimes and reliability functions.

Case 1. Replacing the system components by the improved components $E_{i1}^{'}$ $i = 1,2,...,n,$ with reduced failure rates by a factor ρ, $0 < \rho < 1$, having lifetimes $T_{i1}^{'}$, $i = 1,2,...,n,$ and exponential reliability functions

$$R(\rho t) = \exp[-\rho\lambda t] \text{ for } t \geq 0, \ \lambda > 0.$$

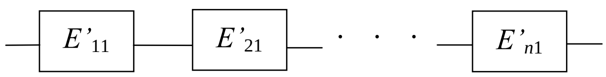

Figure 2. The scheme of a series system with improved components

$$T^{(1)} = \min_{1 \leq i \leq n}\{T_{i1}^{'}\},$$

$$\boldsymbol{R}_n^{(1)}(t) = [R(\rho t)]^n = \exp[-\rho\lambda n t], \ t \geq 0.$$

Case 2. A hot reservation of the system components

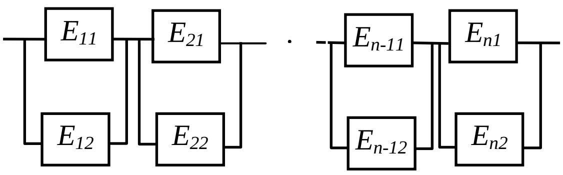

Figure 3. The scheme of a series system with components having hot reservation

$$T^{(2)} = \min_{1 \leq i \leq n}\{\max_{1 \leq j \leq 2}\{T_{ij}\}\},$$

$$\boldsymbol{R}_n^{(2)}(t) = [1 - [F(t)]^2]^n = [1 - [1 - \exp[-\lambda t]]^2]^n, \ t \geq 0.$$

Case 3. A cold reservation of the system components

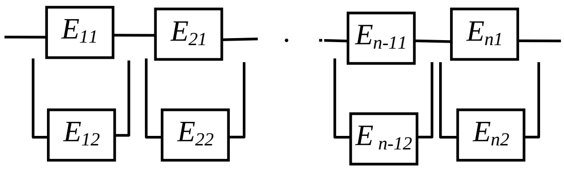

Figure 4. The scheme of a series system with components having cold reservation

$$T^{(3)} = \min_{1 \leq i \leq n}\{\sum_{j=1}^{2} T_{ij}\},$$

$$\boldsymbol{R}_n^{(3)}(t) = [1 - [F(t)] * [F(t)]]^n = [1 + \lambda t]^n \exp[-n\lambda t], \ t \geq 0.$$

Case 4. A mixed reservation of the system components

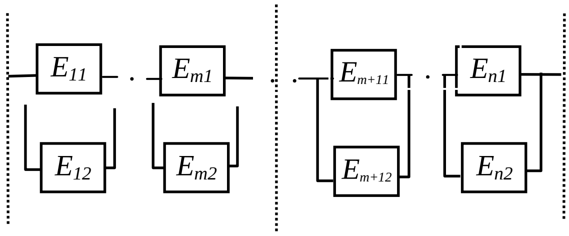

Figure 5. The scheme of a series system with components having mixed reservation

$$T^{(4)} = \min\{\min_{1 \leq i \leq m}\{\sum_{j=1}^{2} T_{ij}\}, \min_{m+1 \leq i \leq n}\{\max_{1 \leq j \leq 2}\{T_{ij}\}\}\},$$

$$\boldsymbol{R}_n^{(4)}(t) = [1 - [F(t)] * [F(t)]]^m [1 - R^2(t)]^{n-m}$$

$$= [1 + \lambda t]^m \exp[-\lambda n t][2 - \exp[-\lambda t]]^{n-m}, \ t \geq 0.$$

Case 5. A hot system reservation

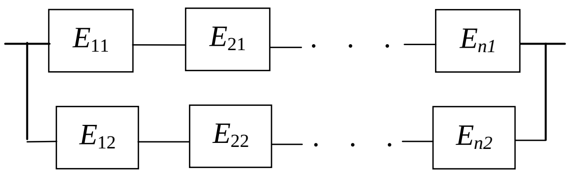

Figure 6. The scheme of a series system with hot reservation

$$T^{(5)} = \max_{1 \leq j \leq 2}\{\min_{1 \leq j \leq n}\{T_{ij}\}\},$$

$$\boldsymbol{R}_n^{(5)}(t) = 1 - [1 - [R(t)]^n]^2 = 1 - [1 - \exp[-n\lambda t]]^2, \ t \geq 0.$$

Case 6. A cold system reservation

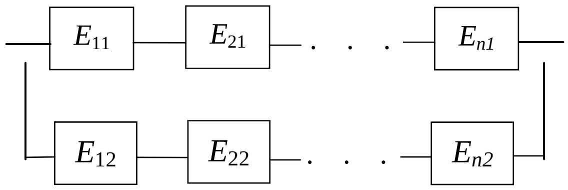

Figure 7. The scheme of a series system with cold reservation

$$T^{(6)} = \sum_{j=1}^{2} \min_{1 \le i \le n}\{T_{ij}\},$$

$$\boldsymbol{R}_n^{(6)}(t) = 1 - [1 - [R(t)]^n] * [1 - [R(t)]^n]$$

$$= [1 + n\lambda t]\exp[-n\lambda t], \quad t \ge 0.$$

The difficulty arises when selecting the right method of improvement of reliability for a large system. This problem may be simplified and approximately solved by the application of the asymptotic approach. Comparisons of the limit reliability functions of the systems with different types of reserve and such systems with improved components allow us to find the value of the components' decreasing failure rate factor ρ, which gives rise to an equivalent effect on the system reliability improvement. Similar results are obtained under comparison of the system lifetime mean values. As an example we will present the asymptotic approach to the above methods of improving reliability for homogeneous two-state series systems.

Proposition 3.Case 1. If

$$a_n = 1/\lambda\rho n, \; b_n = 0,$$

then

$$\mathfrak{R}^{(1)}(t) = \exp[-t] \;\; \text{for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with reduced failure rates of its components, i.e.

$$\boldsymbol{R}_n^{(1)}(t) = \mathfrak{R}^{(1)}(\lambda\rho n t) = \exp[-\lambda\rho n t] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(1)} = E[T^{(1)}] = \frac{1}{\lambda\rho n}.$$

Case 2. If

$$a_n = 1/\lambda\sqrt{n}, \; b_n = 0,$$

then

$$\mathfrak{R}^{(2)}(t) = \exp[-t^2] \;\; \text{for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with hot reservation of its components, i.e.

$$\boldsymbol{R}_n^{(2)}(t) \cong \mathfrak{R}^{(2)}(\lambda\sqrt{nt}) = \exp[-\lambda^2 n t^2] \;\; \text{for } t \ge 0$$

and

$$\boldsymbol{T}^{(2)} = E[T^{(2)}] \cong \Gamma(\frac{3}{2})\frac{1}{\lambda\sqrt{n}}.$$

Case 3. If

$$a_n = \sqrt{2}/\lambda\sqrt{n}, \; b_n = 0,$$

then

$$\mathfrak{R}^{(3)}(t) = \exp[-t^2] \text{ for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with cold reservation of its components, i.e.

$$\boldsymbol{R}_n^{(3)}(t) \cong \mathfrak{R}^{(3)}(\lambda\sqrt{\frac{nt}{2}}) = \exp[-\frac{1}{2}\lambda^2 n t^2] \;\; \text{for } t \ge 0$$

and

$$\boldsymbol{T}^{(3)} = E[T^{(3)}] \cong \Gamma(\frac{3}{2})\frac{1}{\lambda}\sqrt{\frac{2}{n}}.$$

Case 4. If

$$a_n = \frac{1}{\lambda}\sqrt{\frac{2}{2n-m}}, \; b_n = 0,$$

then

$$\mathfrak{R}^{(4)}(t) = \exp[-t^2] \;\; \text{for } t \ge 0,$$

is the limit reliability function of the homogeneous exponential series system with mixed reservation of its components, i.e.

$$\boldsymbol{R}_n^{(4)}(t) \cong \mathfrak{R}^{(4)}(\lambda\sqrt{\frac{2n-m}{2}})$$

$$= \exp[-\frac{2n-m}{2}\lambda^2 t^2] \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(4)} = E[T^{(4)}] \cong \Gamma(\frac{3}{2})\frac{1}{\lambda}\sqrt{\frac{2}{2n-m}}.$$

Case 5. If

$$a_n = \frac{1}{\lambda n},\ b_n = 0,$$

then

$$\mathfrak{R}^{(5)}(t) = 1-[1-\exp[-t]]^2 \text{ for } t \geq 0,$$

is the limit reliability function of the homogeneous exponential series system with hot reservation, i.e.

$$\boldsymbol{R}_n^{(5)}(t) = \mathfrak{R}^{(5)}(\lambda n t) = 1-[1-\exp[-\lambda n t]]^2 \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(5)} = E[T^{(5)}] = \frac{3}{2\lambda n}.$$

Case 6. If

$$a_n = \frac{1}{\lambda n},\ b_n = 0,$$

then

$$\mathfrak{R}^{(6)}(t) = [1+t]\exp[-t] \text{ for } t \geq 0,$$

is the limit reliability function of the homogeneous exponential series system with cold reservation, i.e.

$$\boldsymbol{R}_n^{(6)}(t) = \mathfrak{R}^{(6)}(\lambda n t) = [1+\lambda n t]\exp[-\lambda n t] \text{ for } t \geq 0$$

and

$$\boldsymbol{T}^{(6)} = E[T^{(6)}] = \frac{2}{\lambda n}.$$

Corollary 3. Comparison of the system reliability functions

$$\mathfrak{R}^{(i)}(t) = \mathfrak{R}^{(1)}(t),\ i = 2,3,\ldots,6,$$

results respectively in the following values of the factor ρ:

$$\rho = \rho(t) = \lambda t \text{ for } i = 2,$$

$$\rho = \rho(t) = \frac{1}{2}\lambda t \text{ for } i = 3,$$

$$\rho = \rho(t) = \frac{2n-m}{2n}\lambda t \text{ for } i = 4,$$

$$\rho = \rho(t) = 1-\log[2-\exp[-\lambda n t]] \text{ for } i = 5,$$

$$\rho = \rho(t) = 1-\frac{1}{\lambda n t}\log[1+\lambda n t] \text{ for } i = 6,$$

while comparison of the system lifetimes

$$\boldsymbol{T}^{(i)}(t) = \boldsymbol{T}^{(1)}(t),\ i = 2,3,\ldots,6$$

results respectively in the following values of the factor ρ:

$$\rho = \frac{1}{\Gamma(\frac{3}{2})\sqrt{n}} \text{ for } i = 2,$$

$$\rho = \frac{1}{\Gamma(\frac{3}{2})\sqrt{2n}} \text{ for } i = 3,$$

$$\rho = \frac{1}{\Gamma(\frac{3}{2})n\sqrt{\frac{2}{2n-m}}} \text{ for } i = 4,$$

$$\rho = \frac{2}{3} \text{ for } i = 5,$$

$$\rho = \frac{1}{2} \text{ for } i = 6.$$

Example 3. We consider a simplified bus service company composed of 81 communication lines. We suppose that there is one bus operating on each communication line and that all buses are of the same type with the exponential reliability function

$$R(t) = \exp[-\lambda t] \text{ for } t \geq 0,\ \lambda > 0.$$

Additionally we assume that this communication system is working when all its buses are not failed, i.e.

it is failed when any of the buses are failed. The failure rate of the buses evaluated on statistical data coming from the operational process of bus service company transportation system is assumed to be equal to 0.0049 h^{-1}.

Under these assumptions the considered transportation system is a homogeneous series system made up of components with a reliability function

$$R(t) = \exp[-0.0049t] \text{ for } t \ge 0.$$

Here we will use four sensible methods from those considered for system reliability improvement. Namely, we apply the four previously considered cases.

Case 1. Replacing the system components by the improved components with reduced failure rates by a factor ρ.

Applying *Proposition 3* with normalising constants

$$a_{81} = \frac{1}{0.0049 \cdot 81\rho} = \frac{1}{0.397\rho},\ b_{81} = 0,$$

we conclude that

$$\mathfrak{R}^{(1)}(t) = \exp[-t] \text{ for } t \ge 0,$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_n^{(1)}(t) = \mathfrak{R}^{(1)}(0.397\rho t) = \exp[-0.397\rho t] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(1)} = E[T^{(1)}] = \frac{1}{0.397\rho} \text{ h.}$$

Case 2. Improving the reliability of the system by a single hot reservation of its components.

This means that each of 81 communication lines has at its disposal two identical buses it can use and its task is performed if at least one of the buses is not failed.

Applying *Proposition 3* with normalising constants

$$a_{81} = \frac{1}{0.0049 \cdot \sqrt{81}} = \frac{1}{0.0441},\ b_{81} = 0,$$

we conclude that

$$\mathfrak{R}^{(2)}(t) = \exp[-t^2],\ \text{t} \ge 0.$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_{81}^{(2)}(t) \cong \mathfrak{R}^{(2)}(0.0441t) \cong \exp[-0.0019t^2],\ \ t \ge 0,$$

and

$$\boldsymbol{T}^{(2)} = E[T^{(2)}] \cong \Gamma(\frac{3}{2}) \frac{1}{0.0049\sqrt{81}} \cong 20.10\ h.$$

Case 4. Improving the reliability of the system by a single mixed reservation of its components.

This means that each of 81 communication lines has at its disposal two identical buses. There are $m = 50$ communication lines with small traffic which are using one bus permanently and after its failure it is replaced by the second bus (a cold reservation) and $n - m = 81 - 50 = 31$ communication lines with large traffic which are using two buses permanently (a hot reservation).

Applying *Proposition 3* with normalising constants

$$a_n = \frac{1}{0.0049}\sqrt{\frac{2}{112}} = \frac{1}{0.0367},\ b_n = 0,$$

we conclude that

$$\mathfrak{R}^{(4)}(t) = \exp[-t^2] \text{ for } t \ge 0,$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_n^{(4)}(t) \cong \mathfrak{R}^{(4)}(0.0367t) = \exp[-0.00135t^2] \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(4)} = E[T^{(4)}] \cong \Gamma(\frac{3}{2}) \frac{1}{0.0049}\sqrt{\frac{2}{112}} \cong 24.15\ h.$$

Case 5. Improving the reliability of the system by a single hot reservation.

This means that the transportation system is composed of two independent companies, each of them operating on the same 81 communication lines and having at their disposal one identical bus for use on each line.

Applying *Proposition 3* with normalising constants

$$a_{81} = \frac{1}{0.0049 \cdot 81} = \frac{1}{0.397},\ b_{81} = 0,$$

we conclude that

$$\mathfrak{R}^{(5)}(t) = 1 - [1 - \exp[-t]]^2 \text{ for } t \ge 0,$$

is the limit reliability function of the system, i.e.

$$\boldsymbol{R}_n^{(5)}(t) = \mathfrak{R}^{(5)}(0.397t)$$

$$= 1 - [1 - \exp[-0.397t]]^2 \text{ for } t \ge 0$$

and

$$\boldsymbol{T}^{(5)} = E[T^{(5)}] = \frac{3}{2 \cdot 0.0049 \cdot 81} \cong 3.78 \text{ h.}$$

Comparing the system reliability functions for considered cases of improvement, from *Corollary 3*, results in the following values of the factor ρ:

$$\rho = \rho(t) = 0.0049t \quad \text{for } i = 2,$$

$$\rho = 0.0340t \quad \text{for } i = 4,$$

$$\rho = \rho(t) = 1 - \log[2 - \exp[-0.397t]] \quad \text{for } i = 5,$$

while comparison of the system lifetimes results respectively in:

$$\rho = 0.1254 \quad \text{for } i = 2,$$

$$\rho = 0.1043 \quad \text{for } i = 4,$$

$$\rho = 0.6667 \quad \text{for } i = 5.$$

Methods of system reliability improvement presented here supply practitioners with simple mathematical tools, which can be used in everyday practice. The methods may be useful not only in the operation processes of real technical objects but also in designing new operation processes and especially in optimising these processes. Only the case of series systems made up of components having exponential reliability functions with single reservations of their components and subsystems is considered. It seems to be possible to extend these results to systems that have more complicated reliability structures, and made up of components with different from the exponential reliability functions.

4. Reliability of large systems in their operation processes

This section proposes an approach to the solution of the practically very important problem of linking systems' reliability and their operation processes. To connect the interactions between the systems' operation processes and their reliability structures that are changing in time a semi-markov model ([1]) of the system operation processes is applied. This approach gives a tool that is practically important and not difficult for everyday use for evaluating reliability of systems with changing reliability structures during their operation processes. Application of the proposed methods is illustrated here in the reliability evaluation of the port grain transportation system.

We assume that the system during its operation process is taking different operation states. We denote by $Z(t)$, $t \in <0, \infty>$, the system operation process that may assume v different operation states from the set

$$Z = \{z_1, z_2, \ldots, z_v\}.$$

In practice a convenient assumption is that $Z(t)$ is a semi-markov process ([1]) with its conditional sojourn times θ_{bl} at the operation state z_b when its next operation state is z_l, $b, l = 1,2,\ldots,v$, $b \neq l$. In this case this process may be described by:
- the vector of probabilities of the initial operation states $[p_b(0)]_{1xv}$,
- the matrix of the probabilities of its transitions between the states $[p_{bl}]_{vxv}$,
- the matrix of the conditional distribution functions $[H_{bl}(t)]_{vxv}$ of the sojourn times θ_{bl}, $b \neq l$, where

$$H_{bl}(t) = P(\theta_{bl} < t) \text{ for } b, l = 1,2,\ldots,v, \ b \neq l,$$

and

$$H_{bb}(t) = 0 \text{ for } b = 1,2,\ldots,v.$$

Under these assumptions, the lifetime θ_{bl} mean values are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t), \ b, l = 1,2,\ldots,v, \ b \neq l. \quad (7)$$

The unconditional distribution functions of the sojourn times θ_b of the process $Z(t)$ at the states z_b, $b = 1,2,\ldots,v$, are given by

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t), \ b = 1,2,\ldots,v.$$

The mean values $E[\theta_b]$ of the unconditional sojourn times θ_b are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl}, \ b = 1,2,\ldots,v, \quad (8)$$

where M_{bl} are defined by (7).
Limit values of the transient probabilities at the states

$$p_b(t) = P(Z(t) = z_b), \ t \in <0, \infty), \ b = 1,2,\ldots,v,$$

are given by ([1])

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v}\pi_l M_l}, \quad b=1,2,...,v, \tag{9}$$

where the probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases}[\pi_b]=[\pi_b][p_{bl}]\\ \sum_{l=1}^{v}\pi_l = 1.\end{cases}$$

We consider a series-parallel system and we assume that the changes of its operation process $Z(t)$ states have an influence on the system components E_{ij} reliability and on the system reliability structure as well. Thus, we denote ([13]) the conditional reliability function of the system component E_{ij} while the system is at the operational state z_b, $b=1,2,...,v,$ by

$$[R^{(i,j)}(t)]^{(b)} = P(T_{ij}^{(b)} \geq t / Z(t)=z_b),$$

for $t \in <0,\infty)$, $b=1,2,...,v,$ and the conditional reliability function of the non-homogeneous regular series-parallel system while the system is at the operational state z_b, $b=1,2,...,v,$ by

$$[\boldsymbol{R}_{k_n,l_n}(t)]^{(b)} = P(T^{(b)} \geq t / Z(t)=z_b)$$

$$=1-\prod_{i=1}^{a}[1-[[R^{(i)}(t)]^{(b)}]^{l_n}]^{q_i k_n} \tag{10}$$

for $t \in <0,\infty)$ and

$$[R^{(i)}(t)]^{(b)} = \prod_{j=1}^{e_i}[[R^{(i,j)}(t)]^{(b)}]^{p_{ij}}, \; i = 1,2,...,a. \tag{11}$$

The reliability function $[R^{(i,j)}(t)]^{(b)}$ is the conditional probability that the component E_{ij} lifetime $T_{ij}^{(b)}$ in the is not less than t, while the process $Z(t)$ is at the operation state z_b. Similarly, the reliability function $[\boldsymbol{R}_{k_n,l_n}(t)]^{(b)}$ is the conditional probability that the series-parallel system lifetime $T^{(b)}$ is not less than t, while the process $Z(t)$ is at the operation state z_b. In the case when the system operation time is large enough, the unconditional reliability function of the series-parallel system is given by

$$\boldsymbol{R}_{k_n,l_n}(t) = P(T>t) \cong \sum_{b=1}^{v} p_b[\boldsymbol{R}_{k_n,l_n}(t)]^{(b)} \tag{12}$$

for $t \geq 0$ and T is the unconditional lifetime of the series-parallel system.
The mean values and variances of the series-parallel system lifetimes are

$$M \cong \sum_{b=1}^{v} p_b M_b, \tag{13}$$

where

$$M_b = \int_0^\infty [\boldsymbol{R}_{\boldsymbol{k_n},l_n}(t)]^{(b)}\, dt, \tag{14}$$

and

$$D[T^{(b)}] = 2\int_0^\infty t[\boldsymbol{R}_{k_n l_n}(t)]^{(b)}\, dt - [M_b]^2, \tag{15}$$

for $b=1,2,...,v.$

Example 5. We analyse the reliability of one of the subsystems of the port grain elevator. The considered system is composed of four two-state non-homogeneous series-parallel transportation subsystems assigned to handle and clearing of exported and imported grain. One of the basic elevator functions is loading railway trucks with grain.
In loading the railway trucks with grain the following elevator transportation subsystems take part: S_1 – horizontal conveyors of the first type, S_2 – vertical bucket elevators, S_3 – horizontal conveyors of the second type, S_4 – worm conveyors.
We will analyze the reliability of the subsystem S_4 only.
Taking into account experts opinion in the operation process, $Z(t), t \geq 0$ of the considered transportation subsystem we distinguish the following as its three operation states:
an operation state z_1 – the system operation with the largest efficiency when all components of the subsystem S_4 are used,
an operation state z_2 – the system operation with less efficiency system when the first and second conveyors of subsystem S_4 are used,
an operation state z_3 – the system operation with least efficiency when the first conveyor of subsystem S_4 is used.

On the basis of data coming from experts, the probabilities of transitions between the subsystem S_4 operation states are given by

$$[p_{bl}] = \begin{bmatrix} 0 & 0.357 & 0.643 \\ 0.8 & 0 & 0.2 \\ 0.385 & 0.615 & 0 \end{bmatrix},$$

and their mean values, from (8), are

$$M_1 = E[\theta_1] = 0.357 \cdot 0.36 + 0.643 \cdot 0.2 \cong 0.257,$$

$$M_2 = E[\theta_2] = 0.8 \cdot 0.05 + 0.2 \cdot 0.2 \cong 0.08,$$

$$M_3 = E[\theta_3] = 0.385 \cdot 0.08 + 0.615 \cdot 0.05 \cong 0.062.$$

Since from the system of equations

$$\begin{cases} [\pi_1, \pi_2, \pi_3] = [\pi_1, \pi_2, \pi_3] \begin{bmatrix} 0 & 0.357 & 0.643 \\ 0.8 & 0 & 0.2 \\ 0.385 & 0.615 & 0 \end{bmatrix}, \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

we get

$$\pi_1 = 0.374, \ \pi_2 = 0.321, \ \pi_3 = 0.305,$$

then the limit values of the transient probabilities $p_b(t)$ at the operation states z_b, according to (9), are given by

$$p_1 = 0.684, \ p_2 = 0.183, \ p_3 = 0.133. \qquad (16)$$

The subsystem S_4 consists of three chain conveyors. Two of these are composed of 162 components and the remaining one is composed of 242 components. Thus it is a non-regular series-parallel system. In order to make it a regular system we conventionally complete two first conveyors having 162 components with 80 components that do not fail. After this supplement subsystem S_4 consists of k_n = 3 conveyors, each composed of l_n = 242 components. In two of them there are:

- two driving wheels with reliability functions

$R^{(1,1)}(t)$ = exp[–0.0798t],

- 160 links with reliability functions

$R^{(1,2)}(t)$ = exp[–0.124t],

- 80 components with “reliability functions”

$R^{(1,3)}(t) = \exp[-\lambda_1(1)\, t]$, where $\lambda_1(1) = 0$.

The third conveyer is composed of:

- two driving wheels with reliability functions

$R^{(2,1)}(t)$ = exp[–0.167t]

- 240 links with reliability functions

$R^{(2,2)}(t)$ = exp[–0.208t].

At the operation state z_1 the subsystem S_4 becomes a non-homogeneous regular series-parallel system with parameters

k_n = 3, l_n = 242, a = 2, q_1 = 2/3, q_2 = 1/3,

e_1 = 3, e_2 = 2,

p_{11} = 2/242, p_{12} = 160/242, p_{13} = 80/242,

p_{21} = 2/242, p_{22} = 240/242,

and from (10)-(11) the reliability function of this system is given by

$$[\boldsymbol{R}_{3,242}(t)]^{(1)}$$

$$= 1 - [1 - \exp[-19.9892t]]^2[1 - \exp[-50.2628t]]$$

$$= 2\exp[-19.9892t] - 2\exp[-70.252t]$$

$$+ \exp[-50.2628t] + \exp[-90.2412t]$$

$$- \exp[-39.9784t] \text{ for } t \geq 0. \qquad (17)$$

According to (14)-(15), the subsystem lifetime mean value and the standard deviation are

$$M_1 \cong 0.078, \ \sigma_1 \cong 0.054. \qquad (18)$$

At the operation state z_2 the subsystem S_4 becomes a non-homogeneous regular series-parallel system with parameters

k_n = 2, l_n = 162, a = 1, q_1 = 1, e_1 = 2,

p_{11} = 2/162, p_{12} = 160/162.

and from (10)-(11) the reliability function of this system is given by

$$[\boldsymbol{R}_{2,162}(t)]^{(2)} = 1 - [1 - \exp[-20.007t]]^2$$

$$= 2\exp[-20.007t] - \exp[-40.014t] \text{ for } t \ge 0. \quad (19)$$

According to (14)-(15), the subsystem lifetime mean value and the standard deviation are

$$M_2 \cong 0.075, \ \sigma_2 \cong 0.056. \quad (20)$$

At the operation state z_3 the subsystem S_4 becomes a non-homogeneous regular series-parallel (series) system with parameters

$$k_n = 1, \ l_n = 162, \ q_1 = 1, \ e_1 = 3,$$

$$p_{11} = 2/162, \ p_{12} = 160/162,$$

and from (10)-(11) the reliability function of this system is given by

$$[\boldsymbol{R}_{1,162}(t)]^{(3)} = \exp[-19.999t] \text{ for } t \ge 0. \quad (21)$$

According to (14)-(15), the system lifetime mean value and the standard deviation are

$$M_3 \cong 0.050, \ \sigma_3 \cong 0.050. \quad (22)$$

Finally, considering (12), the subsystem S_4 unconditional reliability is given by

$$\boldsymbol{R}(t) \cong 0.684 \cdot [\boldsymbol{R}_{3,242}(t)]^{(1)} + 0.183 \cdot [\boldsymbol{R}_{2,162}(t)]^{(2)}$$

$$+ 0.133 \cdot [\boldsymbol{R}_{1,162}(t)]^{(3)}, \quad (23)$$

where $[\boldsymbol{R}_{3,242}(t)]^{(1)}$, $[\boldsymbol{R}_{2,162}(t)]^{(2)}$, $[\overline{\boldsymbol{R}}_{1,162}(t)]^{(3)}$, are given by (17), (19), (21).
Hence, applying (16) and (18), (20), (22), we get the mean values and standard deviations of the subsystem unconditional lifetimes given by

$$M \cong 0.684 \cdot 0.078 + 0.183 \cdot 0.075$$

$$+ 0.133 \cdot 0.050 \cong 0.074, \quad (24)$$

$$\sigma(1) \cong 0.054. \quad (25)$$

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