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Computation of a canonical form for neutral delay-differential systems*

by

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Abstract: In this paper, symbolic computation techniques are used to obtain a canonical form for polynomial matrices arising from linear delay-differential systems of the neutral type. The canonical form can be regarded as an extension of the companion form, often encountered in the theory of linear systems, described by ordinary differential equations. Using the Smith normal form, the exact connection between the original polynomial matrix and the reduced canonical form is set out. An example is given to illustrate the computational aspects involved.

Keywords: delay-differential systems, polynomial matrices, Smith form, unimodular-equivalence, OreModules.

1. Introduction

Canonical forms play an important role in the modern theory of linear systems. In particular, the so-called companion matrix has been used by many authors in the analysis and synthesis of 1-D linear control systems. For instance, Barnett (1976) showed that many of the concepts encountered in 1-D linear systems theory such as controllability, observability, stability and pole assignment, can be nicely linked via the companion matrix. Boudellioua (2007) suggested a matrix form, which can be regarded as a generalization of the companion form for a class of bivariate polynomials. These polynomials arise in the study of linear neutral delay-differential systems, as suggested by Byrnes, Spong and Tarn (1984). However, in that paper, the author did not establish the exact connection between the original matrix and the reduced canonical form. In this paper, using symbolic computation based on the OreModules, Maple package (see Chyzak, Quadrat and Robertz, 2007) the connection between the original polynomial matrix and the canonical form is established.

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2. Polynomial matrices arising from linear neutral delay-differential systems

Consider the following linear system of delay-differential equations as given in Byrnes, Spong and Tarn (1984):

$$\sum_{i=0}^{p} E_i \dot{x}(t-ih) = \sum_{i=0}^{p} A_i x(t-ih) + \sum_{j=0}^{q} B_j u(t-jh) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the vector of state variables, $u(t) \in \mathbb{R}^l$ is the vector of controlled variables, A_i, B_j, C_k, D_q are real constant matrices of appropriate dimensions and $h \in \mathbb{R}^+$ is a constant. The system (1) can be written in the polynomial form:

$$[sE(z) - A(z)]\, x(t) = B(z)u(t) \tag{2}$$

where $E(z), A(z), B(z)$ are, respectively, $n \times n, n \times n, n \times l$ over $\mathbb{R}[z]$, $s = d/dt$ denotes a differential operator and z is a backward shift operator, i.e., $zx(t) = x(t-h)$. The polynomial matrix over $\mathbb{R}[s,z]$,

$$T(s,z) = sE(z) - A(z) \tag{3}$$

is the characteristic matrix associated with (1). The system (1) is called a neutral system because, it contains delays in the derivatives of the states or inputs. Neutral delay-differential equations of the type (1) have many applications and may arise, for example, in the study of lumped parameter networks interconnected by transmission lines (see Byrnes, Spong and Tarn, 1984). It is assumed in equations (2) that $E(z)$ is atomic at zero i.e. $\det(E(0)) \neq 0$. This is necessary, in general, to guarantee causality.

Throughout this paper, unless specified otherwise, $D = K[x_1, \ldots, x_n]$ denotes the polynomial ring in the indeterminates $x_1, \ldots, x_n$ with coefficients in an arbitrary, but fixed, field K. First, we present a few definitions that will be needed later in the paper.

3. Definitions

Definition 1 *Let $D = K[x_1, \ldots, x_n]$. The general linear group $GL_p(D)$ is defined by*

$$GL_p(D) = \left\{ M \in D^{p\times p} \ \mid \ \exists N \in D^{p\times p} : MN = NM = I_p \right\}.$$

An element $M \in GL_p(D)$ is called a unimodular matrix. It follows that M is unimodular if and only if the determinant of M is invertible in D, i.e., is a non-zero element of K.

One of standard tasks, which is performed in systems theory, is to transform a given system representation into a simpler form before applying any analytical or numerical method. The transformation involved must, of course, preserve the relevant system properties, if conclusions about the reduced system are to remain valid with respect to the original one. An equivalence transformation used in the context of multidimensional systems is unimodular equivalence. This transformation can be regarded as an extension of Rosenbrock's equivalence (Rosenbrock, 1970) from the univariate to the multivariate setting and is defined by the following:

Definition 2 *Let T_1 and T_2 denote two $q \times p$ matrices with elements in D: then T_1 and T_2 are said to be unimodular-equivalent if there exist two matrices $M \in GL_q(D)$ and $N \in GL_p(D)$ such that*

$$T_2 = MT_1N. \tag{4}$$

4. Equivalence to Smith form over $D = K[x_1, \ldots, x_n]$

The Smith form S of a $p \times q$ matrix T with elements in a domain D is usually the result of an equivalence transformation, i.e. a transformation of the form

$$S = MTN. \tag{5}$$

where M and N are unimodular matrices with elements in D, i.e., square with determinant being a unit of D. The resulting Smith form S is given by

$$\begin{array}{cll} \begin{bmatrix} \Gamma & 0 \end{bmatrix} & ; & (p < q) \\ \Gamma & ; & (p = q) \\ \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} & ; & (p > q) \end{array} \tag{6}$$

where Γ is a $t \times t$ diagonal matrix given by

$$\Gamma = diag(\Phi_1, \Phi_2, \ldots, \Phi_r, 0, 0, \ldots, 0), \tag{7}$$

$t = \min(p, q)$, r =rank of T; the invariant polynomials Φ_i in (7) are given by

$$\Phi_i = d_i/d_{i-1}, i = 1, 2, \ldots, r, \tag{8}$$

$d_0 = 1$ and d_i is the greatest common divisor of the ith order minors of T. In order to show that any matrix can be brought by an equivalence transformation to its Smith form, it is usually required that D be a principal ideal domain or a Euclidean domain. The problem of equivalence of a multivariate polynomial matrix to its Smith form was first studied by Frost and Storey (1979) who proposed only necessary conditions. Later on, Frost and Boudellioua (1986) presented the necessary and sufficient conditions for a class of bivariate polynomial matrices. Lee and Zak (1983) also gave some necessary and sufficient conditions in terms of solutions of some polynomial equations. However these conditions are difficult to test. Lin, Boudellioua and Xu (2006) extended result from Frost and Boudellioua (1986) to the multivariate case and Boudellioua and Quadrat (2010) generalized them to a larger class of matrices using a module theoretic approach. The establishment of the equivalence to the Smith form is based on the application of the well known Quillen-Suslin Theorem. For an implementation of Quillen-Suslin Theorem with Maple and applications to multidimensional systems theory, the reader is referred to the paper by Fabianska and Quadrat (2007).

Theorem 1 *(Quillen, 1976; Suslin, 1976) Let K be a principal ideal domain and $D = K[x_1, \ldots, x_n]$ and let $R \in D^{q\times p}$ be a matrix which admits a right-inverse $\tilde{R} \in D^{p\times q}$, i.e., $R\tilde{R} = I_q$. Then there exists a unimodular matrix $N \in GL_p(D)$ such that*

$$RN = \begin{pmatrix} I_q & 0 \end{pmatrix}. \tag{9}$$

Now we state the necessary and sufficient conditions for the reduction of a class of polynomial matrices to the Smith form.

Theorem 2 *(Frost and Boudellioua, 1986; Lin, Boudellioua and Xu, 2006; Boudellioua and Quadrat, 2010) Let K be a principal ideal domain and $D = K[x_1, \ldots, x_n]$ and let $T \in D^{p\times p}$, with full row rank, then T is unimodular-equivalent to the Smith form*

$$S = \begin{pmatrix} I_{p-1} & 0 \\ 0 & \det(T) \end{pmatrix} \tag{10}$$

if and only if there exists a vector $U \in D^p$ which admits a left inverse over D such that the matrix $\begin{pmatrix} T & U \end{pmatrix}$ has a right inverse over D.

The problem of finding a vector $U \in D$, when it exists, such that the condition in Theorem 2 is satisfied is neither trivial nor random. On simple examples over a commutative polynomial ring $D = K[x_1, \ldots, x_n]$ with coefficients in a computable field K (e.g., $K = \mathbb{Q}$), one may take a generic vector $U \in D^q$ with a fixed total degree in the x_i's and compute the D-module $\mathrm{ext}^1_D(E, D) = D^{1\times q}/\left(D^{1\times(p+1)}\,(T \quad U)^T\right)$ by means of a Gröbner basis computation and check whether or not the D-module $\mathrm{ext}^1_D(E, D)$ vanishes on certain branches of the corresponding *tree of integrability conditions*, see Pommaret and Quadrat (2000) or on certain *obstructions to genericity* (i.e., constructible sets of the K-parameters of U), see Levandovskyy and Zerz (2007) for a survey, explaining these techniques and their implementations in Singular.

5. Canonical form for linear neutral delay-differential systems

Now, let $D = \mathbb{R}[s, z]$ and suppose now there exists a vector $U \in D^{n+m}$ such that the condition in Theorem 2 is satisfied. Then it follows that the matrix

$T \in D^{p\times p}$ is equivalent over D to the Smith form

$$S = \begin{pmatrix} I_{p-1} & 0 \\ 0 & \det(T) \end{pmatrix} \tag{11}$$

where

$$\det(T) = |sE(z) - A(z)|$$

is the 2-D characteristic polynomial of the matrix pair $(A(z), E(z))$.

Introducing the canonical form given in Boudellioua (2007) for a matrix T in the form (3) and letting

$$\det(T) \equiv \det(sE(z) - A(z)) = \sum_{i=0}^{n} e_i(z)s^{n-i}, \qquad (e_0(z) \text{ monic}) \tag{12}$$

We are considering now the matrix $T_F \in D^{n\times n}$ in the canonical form:

$$T_F = s\bar{E}(z) - F(z) \tag{13}$$

where $\bar{E}(z)$ is given by

$$\bar{E}(z) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & e_0(z) \end{pmatrix} \tag{14}$$

and $F(z)$ is the companion matrix associated with the polynomial

$$k(s,z) = \det(T) + s^n - e_0(z)s^n,$$

i.e.,

$$F(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -e_n(z) & -e_{n-1}(z) & -e_{n-2}(z) & \cdots & -e_1(z) \end{pmatrix}. \tag{15}$$

The matrix $T_F = s\bar{E}(z) - F(z)$ takes the form:

$$T_F = \begin{pmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ e_n(z) & e_{n-1}(z) & e_{n-2}(z) & \cdots & se_0(z) + e_1(z) \end{pmatrix}. \tag{16}$$

It can be easily verified that $\det(T) = \det(s\bar{E}(z) - F(z)$.

It should be noted here that the unimodular equivalence of a system described by the polynomial matrix T in (3), satisfying the condition in Theorem

2 means that it can be reduced to an equivalent presentation, involving only one single equation in one unknown function. Furthermore, the class of neutral delay-differential systems in (1), amenable to be reduced to the canonical form described above are those which are strongly controllable as studied by Zerz (2000), page 75.

Lemma 1 *The matrix in the canonical form T_F in* (16) *is unimodular-equivalent to the Smith form* (11).

Proof. Consider the matrix T_F in the canonical form (16), the vector $U = E_n$, where E_n is the n-th column of the identity matrix I_n and let $m^{2,3,\ldots,n}_{1,2,\ldots n}$ be the highest order minor formed from the rows $1, 2, \ldots n$, and columns $2, 3, \ldots, n+1$ of the matrix $(T_F \quad E_n)$, i.e.,

$$m^{2,3,\ldots,n+1}_{1,2,\ldots n} = \begin{vmatrix} -1 & 0 & \cdots & 0 \\ * & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{vmatrix}. \tag{17}$$

Clearly, the matrix $(T_F \quad U) \equiv (T_F \quad E_n)$ has a right inverse over D, since it has a highest order minor $m^{2,3,\ldots,n+1}_{1,2,\ldots n}$, which is lower triangular with all diagonal elements equal to ± 1, i.e. $m^{2,3,\ldots,n+1}_{1,2,\ldots n} = (-1)^{n-1}$. Therefore, by Theorem 2, the matrix T_F is unimodular-equivalent to the Smith form (11):

$$S = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \det(T_F) \end{pmatrix}. \tag{18}$$

The following result, based on the Smith form, establishes the connection between a polynomial matrix T in the form (3) and its equivalent canonical form T_F in (16).

Theorem 3 *Let T be a polynomial matrix in the form* (3) *satisfying the condition given in Theorem 2, then T is unimodular-equivalent over D to the canonical form T_F in* (16). *Furthermore*

$$T_F = M_2^{-1} M_1 T N_1 N_2^{-1} \tag{19}$$

where the Smith form $S = M_1 T N_1 = M_2 T_F N_2$.

Proof. By Theorem 2, T is equivalent to the Smith form S_1, where $S_1(n, n) = \det(T)$ and by Lemma 1, T_F is equivalent to the Smith form S_2, with $S_2(n, n) = \det(T_F)$. Since $\det(T) = \det(T_F)$, it follows that T and T_F are equivalent to the same Smith form , i.e. $S_1 = S_2$, i.e. there exist matrices $M_1, N_1, M_2, N_2 \in GL_n(D)$ such that $S = M_1 T N_1 = M_2 T_F N_2$. By transitivity of the unimodular equivalence, T and T_F are also equivalent with

$$T_F = M_2^{-1} M_1 T N_1 N_2^{-1}.$$

6. Illustrative example

Let $D = \mathbb{R}[s, z]$ and

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \tag{20}$$

with

$$\begin{aligned}
t_{11} &= (-2\,z^3 + z^2 + 2\,z)s - z^3 + 2\,z - 1 \\
t_{12} &= (2\,z^3 - z^2 + 2\,z - 1)s - 3\,z^3 + z^2 + 6\,z - 2 \\
t_{13} &= (2\,z^4 - 5\,z^3 + 2\,z^2 + 3\,z)s + z^4 - 2\,z^3 - 5\,z^2 + 2\,z - 1 \\
t_{21} &= (2\,z^3 - z^2 - 2\,z + 1)s + z^3 - z \\
t_{22} &= (-2\,z^3 + z^2 - 2\,z + 1)s + 3\,z^3 - z^2 - 6\,z + 1 \\
t_{23} &= (-2\,z^4 + 5\,z^3 - 2\,z^2 - 4\,z + 2)s - z^4 + 2\,z^3 + 4\,z^2 + z - 1 \\
t_{31} &= (-2\,z^2 + z + 1)s - z^2 + 1 \\
t_{32} &= (2\,z^2 - z + 2)s - 3\,z^2 + z + 5 \\
t_{33} &= (2\,z^3 - 5\,z^2 + 3\,z + 1)s + z^3 - 2\,z^2 - 4\,z - 1
\end{aligned}$$

where

$$\det(T) = (2\,z - 1)\,s^3 + (-z - 2)\,s^2 + (z + 2)\,s + z - 1.$$

Using the equations in (14) and (15), the matrix in canonical form T_F, associated with the polynomial $\det(T)$, is obtained as:

$$T_F \equiv s\bar{E}(z) - F(z) = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ z-1 & z+2 & (2\,z-1)\,s - z - 2 \end{pmatrix}. \tag{21}$$

First, we reduce the matrix T to the Smith form S, i.e., compute $M_1 \in GL_3(D)$ and $N_1 \in GL_3(D)$ such that $S = M_1 T N_1$ where S is given by (11).

Using the method given by Boudellioua and Quadrat (2010), consider the vector

$$U_1 = \begin{pmatrix} z & -z & 1 \end{pmatrix}^T \in D^3$$

and $P_1 = (T \quad U_1) \in D^{3\times 4}$. Using the package OreModules in Maple, see Chyzak, Quadrat and Robertz (2007), we can check that P_1 admits a right inverse over D and we can compute a minimal parametrization $Q_m \in D^4$ of P_1, where $Q_m = (Q_1^T \quad Q_2^T)^T$ and $P_1 Q_m = 0$,

$$Q_m = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \left(\begin{array}{c} (-z+2)\,s^2 + \left(z^2 - 3\,z + 2\right)s + z^3 - 4\,z^2 + 5\,z - 3 \\ -s - z + 1 \\ -s^2 + (z-1)\,s + z^2 - 2\,z + 1 \\ \hline (2\,z-1)\,s^3 + (-z-2)\,s^2 + (z+2)\,s + z - 1 \end{array}\right). \tag{22}$$

Computing the SyzygyModule $K_1 \in D^{2\times 3}$ of Q_1, i.e., $K_1 Q_1 = 0$ gives

$$K_1 = \begin{pmatrix} -s - z + 1 & 1 & (z-2)\,s + z^2 - 3\,z + 2 \\ -z^2 + 2\,z - 1 & -s + 2\,z - 2 & z^3 - 4\,z^2 + 5\,z - 1 \end{pmatrix} \tag{23}$$

where the matrix $Q_3 \in D^{3\times 2}$ is the right inverse of K_1, i.e.,

$$Q_3 = \begin{pmatrix} (z-2)\,s - 2\,z^2 + 6\,z - 4 & z - 2 \\ 1 & 0 \\ s - 2\,z + 2 & 1 \end{pmatrix} \tag{24}$$

Thus, the matrix $N_1 = (Q_3 \quad Q_1) \in GL_3(D)$ is given by:

$$N_1 = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix} \tag{25}$$

where

$$\begin{aligned} n_{11} =&\quad (z-2)s - 2z^2 + 6z - 4 \\ n_{12} =&\quad z - 2 \\ n_{13} =&\quad (-z+2)s^2 + (z^2 - 3z + 2)s + z^3 - 4z^2 + 5z - 3 \\ n_{21} =&\quad 1 \\ n_{22} =&\quad 0 \\ n_{23} =&\quad -s - z + 1 \\ n_{31} =&\quad s - 2z + 2 \\ n_{32} =&\quad 1 \\ n_{33} =&\quad -s^2 + (z-1)s + z^2 - 2z + 1. \end{aligned}$$

The matrix $M_1 = (TQ_3 \quad -U)^{-1} \in GL_3(D)$ is given by:

$$M_1 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \tag{26}$$

where

$$\begin{aligned} m_{11} =&\quad -1 \\ m_{12} =&\quad -1 \\ m_{13} =&\quad 0 \\ m_{21} =&\quad -z + 1 \\ m_{22} =&\quad -z \\ m_{23} =&\quad -z \\ m_{31} =&\quad (-2\,z + 1)s^2 + (z+2)s - z - 2 \\ m_{32} =&\quad (-2\,z + 1)s^2 + (-z+3)s + 2\,z + 1 \\ m_{33} =&\quad (-2\,z^2 + z)s + 3\,z^2 + 3\,z - 1 \end{aligned}$$

and it can be easily verified that the matrix $M_1 T N_1$ yields the Smith form

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & Q_2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (2\,z-1)\,s^3 + (-z-2)\,s^2 + (z+2)\,s + z - 1 \end{pmatrix} \tag{27}$$

where $Q_2 = \det(T)$. Similarly, the matrix T_F is reduced to the Smith form S, i.e., computing $M_2 \in GL_3(D)$ and $N_2 \in GL_3(D)$ such that $S = M_2 T_F N_2$, where S is given by (11). Now, consider the vector $U_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \in D^3$ and $P_2 = (T_F \quad U_2) \in D^{3\times 4}$. Using the package OreModules in Maple, we can check that P_2 admits a right inverse over D and we can compute a minimal parametrization $\bar{Q}_m \in D^4$ of P_2, where $\bar{Q}_m = (\bar{Q}_1^T \quad \bar{Q}_2^T)^T$ and $P_2 \bar{Q}_m = 0$,

$$\bar{Q}_m \equiv \begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -s \\ -s^2 \\ \hline (2\,z-1)\,s^3 + (-z-2)\,s^2 + (z+2)\,s + z - 1 \end{pmatrix}. \tag{28}$$

Computing the SyzygyModule $K_2 \in D^{2\times 3}$ of $\bar{Q}_1$, i.e., $K_2 \bar{Q}_1 = 0$ gives

$$K_2 = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \end{pmatrix} \tag{29}$$

where the matrix $\bar{Q}_3 \in D^{3\times 2}$ is the right inverse of K_2, i.e.,

$$\bar{Q}_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ -s & -1 \end{pmatrix}. \tag{30}$$

Thus, the matrix $N_2 = \begin{pmatrix} \bar{Q}_3 & \bar{Q}_1 \end{pmatrix} \in GL_3(D)$ is given by:

$$N_2 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -s \\ -s & -1 & -s^2 \end{pmatrix} \tag{31}$$

The matrix $M_2 = (T\bar{Q}_3 \quad -U_2)^{-1} \in GL_3(D)$ is given by:

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (-2\,z+1)\,s^2 + (z+2)\,s - z - 2 & (-2\,z+1)\,s + z + 2 & -1 \end{pmatrix} \tag{32}$$

and it can be easily verified the matrix $M_2 T_F N_2$ yields the Smith form

$$S=\begin{pmatrix} I_2 & 0 \\ 0 & \bar{Q}_2 \end{pmatrix}\equiv\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (2\,z-1)\,s^3+(-z-2)\,s^2+(z+2)\,s+z-1 \end{pmatrix} \tag{33}$$

where $\bar{Q}_2 = \det(T_F) = \det(T)$. It follows that the matrix T_F is related to the matrix T by the following transformation:

$$T_F = MTN \tag{34}$$

where

$$\begin{array}{lllll} M & = & M_2^{-1}M_1 & \equiv & \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} \in GL_3(D), \\ N & = & N_1 N_2^{-1} & \equiv & \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \in GL_3(D) \end{array}$$

and

$$m_1=\begin{pmatrix} -1 \\ -z+1 \\ (4z-2)\,s^2+(-2z-4+(-2z+1)\,(-z+1))\,s+2z+4+(z+2)(-z+1) \end{pmatrix}, \tag{35}$$

$$m_2 = \begin{pmatrix} -1 \\ -z \\ (4\,z-2)\,s^2+(-5-(-2\,z+1)\,z)\,s-z+1-(z+2)\,z \end{pmatrix}, \tag{36}$$

$$m_3 = \begin{pmatrix} 0 \\ -z \\ \left(-(-2\,z+1)\,z+2\,z^2-z\right)s+1-(z+2)\,z-3\,z^2-3\,z \end{pmatrix}, \tag{37}$$

$$n_1=\begin{pmatrix} \left((z-2)\,s-2z^2+6z-4\right)s+3-(-z+2)\,s^2-\left(z^2-3z+2\right)s-z^3+4z^2-5z \\ 2\,s+z-1 \\ (s-2\,z+2)\,s-1+s^2-(z-1)\,s-z^2+2\,z \end{pmatrix}, \tag{38}$$

$$n_2 = \begin{pmatrix} 2\,z^2-6\,z+4 \\ -1 \\ 2\,z-2 \end{pmatrix}, \tag{39}$$

and

$$n_3 = \begin{pmatrix} -z+2 \\ 0 \\ -1 \end{pmatrix}. \tag{40}$$

7. Conclusions

In this paper, the Smith normal form of a bivariate polynomial matrix, together with symbolic computation techniques, are used effectively to compute the equivalence transformations that reduce a class of bivariate polynomial matrices to a canonical form. The classes of matrices considered are those amenable to be reduced by unimodular equivalence to a single equation in one unknown function. These matrices arise from neutral delay-differential systems, which are strongly controllable. Furthermore the results given in the paper can be extended to delay-differential systems with non-commensurate delays.

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