BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 68, No. 2, 2020
DOI: 10.24425/bpasts.2020.133112

CONTROL, INFORMATICS AND ROBOTICS

Global stability of positive standard and fractional nonlinear feedback systems

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Abstract. The global stability of positive continuous-time standard and fractional order nonlinear feedback systems is investigated. New sufficient conditions for the global stability of these classes of positive nonlinear systems are established. The effectiveness of these new stability conditions is demonstrated on simple examples of positive nonlinear systems.

Key words: global stability, fractional order, positive, nonlinear system.

1. Introduction

In positive system inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1–3]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models displaying positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in monographs [1–5].

Mathematical fundamentals of the fractional calculus are given in monographs [4–7]. The positive and fractional linear systems have been investigated in [4, 5, 8–16]. Positive linear systems with different fractional orders have been addressed in [11, 16]. Descriptor positive systems have been analyzed in [12, 17]. Linear positive electrical circuits with state feedbacks have been addressed in [5, 17]. The stability of nonlinear systems has been investigated in [18, 19] and the global stability of nonlinear systems with negative feedbacks and positive and not necessarily asymptotically stable parts in [20, 21].

In this paper the global stability of nonlinear standard and fractional positive feedback systems will be addressed.

The paper is organized as follows. In Section 2, basic definitions and theorems concerning positive standard and fractional linear systems and their transfer matrices are recalled. New sufficient conditions for the global positive standard nonlinear systems are established in Section 3. Similar sufficient conditions for fractional positive nonlinear systems are given in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – $n\times n$ identity matrix.

2. Preliminaries

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 1. [4, 5] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 1. [4, 5] The continuous-time linear system (1) is positive if and only if

$$A \in M_n, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}. \tag{2}$$

Definition 2. [4, 5] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for any} \quad x(0) \in \Re_+^n. \tag{3}$$

Theorem 2. [4, 5] The positive continuous-time linear system (1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

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Manuscript submitted 2019-10-18, revised 2020-01-15, initially accepted for publication 2020-01-18, published in April 2020

2. There exists a strictly positive vector $\lambda^T = \begin{bmatrix}\lambda_1 & \cdots & \lambda_n\end{bmatrix}^T$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$A\lambda < 0 \qquad \text{or} \qquad \lambda^T A < 0. \tag{5}$$

If matrix A is nonsingular then we can choose $\lambda = A^{-1}c$, where $c \in \Re^n$ is strictly positive.

In this paper the following Caputo definition of the fractional derivative of α order will be used [4–7]:

$${}_0D_t^\alpha f(t) = \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot f(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, \quad 0<\alpha<1, \tag{6}$$

where $\dot f(\tau) = \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$, $\mathrm{Re}(x) > 0$ is the Euler gamma function.

Consider the fractional continuous-time linear system

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \tag{7a}$$

$$y(t) = Cx(t), \tag{7b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 3. [4, 5] The fractional system (7) is called (internally) positive if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 3. [4, 5] The fractional system (7) is positive if and only if

$$A \in M_n, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}. \tag{8}$$

The fractional positive linear system (7) is called asymptotically stable (and matrix A Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for all} \quad x(0) \in \Re_+^n. \tag{9}$$

The positive fractional system (7) is asymptotically stable if and only if the real parts of all eigenvalues s_k of the matrix A are negative, i.e. $\mathrm{Re}s_k < 0$ for $k = 1,\ldots,n$ [4, 7].

Theorem 4. The positive fractional system (7) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{10}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,\ldots,n-1$.

2. There exists a strictly positive vector $\lambda = \begin{bmatrix}\lambda_1 & \cdots & \lambda_n\end{bmatrix}$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$A\lambda < 0 \qquad \text{or} \qquad \lambda^T A < 0. \tag{11}$$

The transfer matrix of the system (7) is given by

$$T(s^\alpha) = C[I_n s^\alpha - A]^{-1}B. \tag{12}$$

3. Global stability of standard nonlinear feedback systems

Consider the nonlinear feedback system shown in Fig. 1 which consists of a positive linear part and a nonlinear element with characteristic $u = f(e)$. The linear part is described by the equations

$$\begin{aligned}\dot x &= Ax + Bu,\\ y &= Cx,\end{aligned} \tag{13}$$

where $x = x(t) \in \Re_+^n$, $u = u(t) \in \Re_+$, $y = y(t) \in \Re_+$ is the state vector, input and output and $A \in M_n$, $B \in \Re_+^{n\times 1}$, $C \in \Re_+^{1\times n}$.

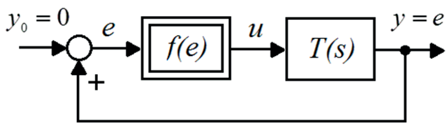

Fig. 1. The nonlinear feedback system

The characteristic of the nonlinear element is shown in Fig. 2 and it satisfies the condition

$$0 \leq \frac{f(e)}{e} \leq k < \infty. \tag{14}$$

It is assumed that the positive linear part is asymptotically stable (the matrix $A \in M_n$ is Hurwitz).

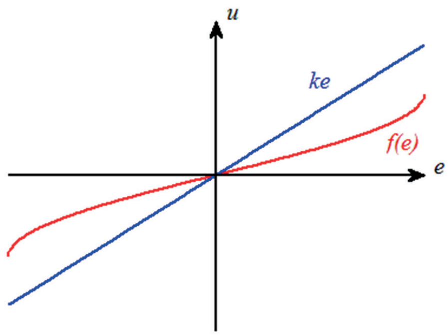

Fig. 2. Characteristic of the nonlinear element

Definition 4. The nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $x(0) \in \Re_+$.

The following theorem gives sufficient conditions for the global stability of the positive nonlinear system.

Theorem 5. The nonlinear system consisting of the positive and asymptotically stable linear part and the nonlinear element satisfying the condition (14) is globally stable if the matrix

$$A + kBC \in M_n \tag{15}$$

is asymptotically stable.

Proof. The proof will be accomplished by the use of the Lyapunov method [22, 23]. As the Lyapunov function $V(x)$ we choose

$$V(x) = \lambda^T x \geq 0 \qquad \text{for} \quad x \in \Re_+^n, \tag{16}$$

where λ is strictly positive vector, i.e. $\lambda_k > 0$, $k = 1, \ldots, n$.

Using (16) and (13) we obtain

$$\begin{aligned} \dot{V}(x) &= \lambda^T \dot{x} = \lambda^T (Ax + Bu) \\ &= \lambda^T (Ax + Bf(e)) \leq \lambda^T (A + kBC)x \end{aligned} \tag{17}$$

since $u = f(e) \leq ke = kCx$.

From (17) it follows that $\dot{V}(t) < 0$ if the condition (15) is satisfied and the nonlinear system is globally stable. □

Example 1. Consider the nonlinear system with the positive linear part with the matrices

$$A = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{18}$$

and the nonlinear element satisfying the condition (14) for $k = 1$ and $k = 2$.

Using (15) and (18) for $k = 1$ we obtain

$$\begin{aligned} A_1 = A + kBC &= \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \in M_2 . \end{aligned} \tag{19}$$

The matrix (19) is Hurwitz since the characteristic polynomial

$$\det(I_2 s - A_1) = \begin{vmatrix} s+3 & -2 \\ -2 & s+3 \end{vmatrix} = s^2 + 6s + 5 \tag{20}$$

has the zeros $s_1 = -1$, $s_2 = -5$.

We obtain the same result using Theorem 2, since for $\lambda^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$ we have

$$A_1 \lambda = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{21}$$

For $k = 2$ we obtain

$$\begin{aligned} A_2 = A + kBC &= \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 10 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}. \end{aligned} \tag{22}$$

The matrix (22) is not Hurwitz since

$$\det(I_2 s - A_2) = \begin{vmatrix} s+2 & -2 \\ -3 & s+3 \end{vmatrix} = s(s+5) \tag{23}$$

and the nonlinear system for $k = 2$ does not satisfy the conditions of Theorem 5.

4. Global stability of fractional nonlinear feedback systems

Consider the nonlinear feedback system shown in Fig. 1, which consists of a fractional positive linear part and a nonlinear element with characteristic $u = f(e)$ shown in Fig. 2. The fractional linear part is described by equations

$$\begin{aligned} \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{24}$$

where $x = x(t) \in \Re_+^n$, $u = u(t) \in \Re_+$, $y = y(t) \in \Re_+$ are the state vector, input and output, the fractional derivative $\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha}$ is defined by (6) and $A \in M_n$, $B \in \Re_+^{n \times 1}$, $C \in \Re_+^{1 \times n}$.

The characteristic of the nonlinear element shown in Fig. 2 satisfies the condition

$$0 \leq \frac{f(e)}{e} \leq k < \infty. \tag{25}$$

It is assumed that the fractional positive linear part is asymptotically stable (the matrix $A \in M_n$ is Hurwitz).

Definition 5. A fractional nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $x(0) \in \Re_+$.

The following theorem gives sufficient conditions for the global stability of the fractional positive nonlinear system.

Theorem 6. The fractional nonlinear system consisting of the positive and asymptotically stable linear part and the nonlinear element satisfying the condition (14) is globally stable if the matrix

$$A + kBC \in M_n . \tag{26}$$

is asymptotically stable.

Proof. The proof will be accomplished by the use of the Lyapunov method [22, 23]. As the Lyapunov function $V(x)$ we choose the scalar function defined by (16).

Using (16) and (13) we obtain

$$\begin{aligned} \frac{\mathrm{d}^\alpha V}{\mathrm{d}t^\alpha} &= \lambda^T \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \lambda^T (Ax + Bu) \\ &= \lambda^T (Ax + Bf(e)) \leq \lambda^T (A + kBC)x \end{aligned} \tag{27}$$

since $u = f(e) \leq ke = kCx$.

From (27) it follows that $\frac{\mathrm{d}^\alpha V}{\mathrm{d}t^\alpha} < 0$ if the condition (26) is satisfied and the fractional nonlinear system is globally stable. □

Example 2. Consider a nonlinear system with a fractional positive linear part with matrices

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \tag{28}$$

and a nonlinear element satisfying the condition (14) for $k = 0.8$. Using (26) and (28) we obtain

$$\begin{aligned} A_1 = A + kBC &= \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} + 0.8 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1.6 & 1.8 \\ 1.2 & -2.6 \end{bmatrix} \in M_2 . \end{aligned} \tag{29}$$

The matrix (29) is Hurwitz since the characteristic polynomial

$$\det(I_2 s - A_1) = \begin{vmatrix} s+1.6 & -1.8 \\ -1.2 & s+2.6 \end{vmatrix} = s^2 + 4.2s + 2 \tag{30}$$

has positive coefficients. Therefore, the fractional positive system is globally stable.

From comparison of Theorems 5 and 6, it follows that the same condition (15) is the sufficient condition for the global stability of the standard and fractional positive nonlinear systems. Therefore, we have the following important conclusion.

Conclusion 1. The global stability of the positive nonlinear systems is independent of its standard and fractional orders.

5. Conclusions

The global stability of positive continuous-time standard and fractional orders nonlinear systems has been investigated. New sufficient conditions for the global stability of the positive standard (Theorem 5) and fractional order (Theorem 6) nonlinear systems have been established. The effectiveness of the new stability conditions has been demonstrated on simples examples. The considerations can be extended to discrete-time positive nonlinear systems and to positive nonlinear systems with different orders.

Acknowledgements. This work was supported by National Science Centre in Poland under work No. 2017/27/B/ST7/02443.

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