Malinowski Jacek 0000-0002-4413-1868
Systems Research Institute, Polish Academy of Sciences, Warsaw, Poland,
jacek.malinowski{at}ibspan.waw.pl

Estimation of Weibull distribution parameters based on sequences of minimal repairs

Keywords

Weibull distribution, minimal repair, maximum likelihood estimation, gamma function, Euler constant

Abstract

A new method of estimating the scale and shape parameters of the Weibull distribution is presented. According to this method, a Weibull distributed time-to-failure (TTF) of a test item is measured m times. It undergoes a minimal repair after each of the first m–1 failures, and is put out of use after the m-th failure. This procedure is repeated n times. Based on m TTFs of one test item, which are neither independent nor identically distributed (IID), the maximum likelihood estimators (MLE) of the scale and shape parameters, called m-sample estimators, are obtained. The accuracy of the m-sample estimators is low, however, it can be improved by using the mean values of their n IID realizations as more precise estimators. The latter are called n·m-sample estimators, have the same biases as the respective m-sample ones, but their variances are n times smaller. Interestingly enough, the n·m-sample estimators of the scale and shape parameters, as well as their biases, are given by relatively simple explicit formulas. This is somewhat unexpected in view of the fact that the standard MLE of the shape parameter, based on IID TTFs of non-repairable test items, is obtained from an equation that cannot be solved analytically.

1. Introduction

The current chapter deals with the problem of estimating the scale and shape parameters of the Weibull distribution. This topic has been thoroughly investigated by multiple statisticians (Almazah & Ismail, 2021; Chikr el-Mezouar, 2010; Dodson, 2006). Weibull estimation has many practical applications demonstrated, inter alia, in (Evans et al., 2019; Lei, 2008; Wu et al., 2021). Nevertheless, a new method, stemming from the reliability theory, has been developed and is presented here. It is a well-known fact that the time-to-failure (TTF) of many technical devices (or their components) is a Weibull distributed random variable. Therefore, in order to estimate its parameters, the usual procedure is to measure the TTF's of a number of test items, and calculate the required estimates from the values of the random sample. Such an approach is pursued in (Almazah & Ismail, 2021; Alizadeh et al., 2015; Wu et al., 2021), to name a few. Sometimes, due to restrictions imposed on the sampling time, only censored data are available. Weibull estimation with such data is discussed in (Alkutubi & Ali, 2011). No matter whether the sample is complete or censored, the standard procedure has one essential disadvantage – if failed objects are no longer usable then a large number of test items are needed in order to achieve high estimation accuracy, which may lead to unacceptable cost. However, if the test items are repairable, then a different approach can be used to reduce this cost. According to the proposed method each item undergoes $m-1$ minimal repairs, where the i-th repair follows the i-th failure, $1 \le i \le m-1$, and is put out of use after the m-th failure. It is natural for m

Safety and Reliability of Systems and Processes, Summer Safety and Reliability Seminar 2022.

DOI: 10.26408/srsp-2022-11.

to be the minimum number of failures that one tested object can survive; usually m is not large. The above procedure is repeated n times, which amounts to destructive testing of n items. Let t_{ij} be the i-th operation period of the j-th item, i.e. the time elapsed between the $(i-1)$-th repair and the i-th failure of this item, $1 \le i \le m$, $1 \le j \le n$ (the 0-th repair takes place when a new item is put into operation). The time periods t_{ij}, $1 \le i \le m$, $1 \le j \le n$, constitute a random sample composed of n vectors of length m, where the j-th vector contains successive operation periods for the j-th item. Based upon the collected data, and the appropriately constructed estimators, the sought parameters can be evaluated. Following this scheme, the required estimation accuracy can be reached for the number n significantly smaller than in the case of a non-repairable test item that becomes unusable after the first failure.

The chapter is organized as follows. In Section 2 the notation used in the chapter is defined. Section 3 is a reminder on the standard MLEs of the Weibull distribution parameters, based on multiple i.i.d. realizations of a Weibull distributed r.v. In Section 4 the MLEs of the scale and shape parameters of the Weibull distribution, based on a sequence of $m-1$ minimal repairs, are constructed. It is then explained how the expected values of those estimators can be approximated using n independent test items. In Section 5 the sought parameters are expressed in terms of the expected values of the respective estimators. The derived formulas allow to calculate, in a simple way, the biases of the estimators constructed in Section 3. Finally, Section 6, summarizes the obtained results and indicates topics for further research.

The proposed method allows to express the MLE of the scale and shape parameters in analytical form. These estimators occur to be biased, but their biases can also be computed analytically. Thus, the newly developed method of estimating the Weibull distribution parameters is based solely on analytical formulas. For comparison, the standard MLE estimator of the shape parameter, using an IID sample obtained from a large number of test items, is derived from an equation that cannot be solved analytically.

2. Notation

- CDF – cumulative distribution function,
- MGF – moment generating function,
- MLE – maximum likelihood estimator/estimation,
- PDF – probability density function,
- TTF – time-to-failure (a continuous, nonnegative random variable),
- $r(t)$ – failure (hazard) rate;
  i.e. $r(t) = f(t)/[1-F(t)]$, where $f(t)$ and $F(t)$ are the PDF and CDF of the TTF under consideration,
- i.i.d. – an abbreviation meaning ***independent and identically distributed***,
- r.v. – an abbreviation meaning ***random variable***,
- $T_1,\dots,T_m$ – the sequence of successive TTFs of a repairable item subjected to $m-1$ minimal repairs following the first $m-1$ failures of this item,
- $S_1,\dots,S_m$ – moments of successive failures of a repairable item, i.e. $S_i = T_1+\dots+T_i$, $1 \le i \le m$,
- m-sample estimator – the MLE based on the sequence of $m-1$ minimal repairs performed on one item (m TTFs),
- $n{\cdot}m$-sample estimator – the MLE based on n sequences of $m-1$ minimal repairs performed on n independent items ($n{\cdot}m$ TTFs).

3. Standard MLEs based on multiple i.i.d. realizations of Weibull r.v.

The PDF of a Weibull r.v. with the scale and shape parameters λ and α is given by

$$f(t) = \lambda\alpha(\lambda t)^{\alpha-1}\exp[-(\lambda t)^{\alpha}] \quad (1)$$

Thus, the following formula defines the PDF of a vector of N i.i.d. Weibull r.v.:

$$f_N(t_1,\dots,t_N) = \prod_{j=1}^{N} f(t_j)$$

$$= (\lambda\alpha)^N \prod_{j=1}^{N}(\lambda t_j)^{\alpha-1}\exp\left[-(\lambda t_j)^{\alpha}\right] \quad (2)$$

The logarithm of $f_N(t_1,\dots,t_N)$ regarded as a function of λ and α, where $t_1,\dots,t_N$ constitute an i.i.d. random sample, is the log-likelihood function used to obtain MLEs of λ and α. This function is usually denoted as $L(\lambda,\alpha|t_1,\dots,t_N)$ and the respective estimators as $\hat{\lambda}$ and $\hat{\alpha}$. These estimators are

equal to the optimal values of λ and α for which $L(\lambda,\alpha|t_1,\dots,t_N)$ attains its maximum, and are found by equating the partial derivatives of $L(\lambda,\alpha|t_1,\dots,t_N)$ to zero (necessary but not sufficient condition for a maximum to exist).
From (2) we obtain:

$$L(\lambda,\alpha|t_1,\dots,t_N)$$

$$= N\cdot\ln(\lambda\alpha)+\sum_{j=1}^{N}\left[(\alpha-1)\ln(\lambda t_j)-(\lambda t_j)^{\alpha}\right]$$

$$= N\cdot[\ln(\alpha)+\alpha\cdot\ln(\lambda)]+$$

$$+\sum_{j=1}^{N}\left[(\alpha-1)\ln(t_j)-(\lambda t_j)^{\alpha}\right]. \qquad (3)$$

Differentiating the log-likelihood function w.r.t. λ and α yields:

$$\frac{\partial L(\lambda,\alpha|t_1,\dots,t_N)}{\partial\lambda}=\frac{\alpha}{\lambda}\left(N-\lambda^{\alpha}\sum_{j=1}^{N}t_j^{\alpha}\right) \qquad (4)$$

$$\frac{\partial L(\lambda,\alpha|t_1,\dots,t_N)}{\partial\alpha}=\frac{N}{\alpha}+N\cdot\ln(\lambda)+\sum_{j=1}^{N}\ln(t_j)+$$

$$-\lambda^{\alpha}\ln(\lambda)\sum_{j=1}^{N}t_j^{\alpha}-\lambda^{\alpha}\sum_{j=1}^{N}t_j^{\alpha}\ln(t_j). \qquad (5)$$

Let $\hat{\lambda}$ and $\hat{\alpha}$ be the values of λ and α for which the right-hand sides of (4) and (5) are equal to zero. From (4) we have:

$$\hat{\lambda}=\left(\frac{N}{\sum_{j=1}^{N}t_j^{\hat{\alpha}}}\right)^{1/\hat{\alpha}}. \qquad (6)$$

Substituting λ in (5) with $\hat{\lambda}$ given by (6) we obtain:

$$\frac{1}{\hat{\alpha}}+\frac{1}{N}\sum_{j=1}^{N}\ln(t_j)-\frac{\sum_{j=1}^{N}t_j^{\hat{\alpha}}\ln(t_j)}{\sum_{j=1}^{N}t_j^{\hat{\alpha}}}=0. \qquad (7)$$

The above equation is used to determine $\hat{\alpha}$. It can only be solved numerically, e.g. by the Newton-Raphson method. However, once $\hat{\alpha}$ is found, $\hat{\lambda}$ is easily computed from (6).
The estimators $\hat{\lambda}$ and $\hat{\alpha}$ satisfying (6) and (7) are biased and there exists no simple method of computing the respective biases (Chen et al., 2017). On the contrary, the new estimators presented in Sections 4 and 5, as well as their biases, are given by relatively simple explicit formulas.

4. New MLEs based on sequences of minimal repairs

Lemma 1

Let $f(t)$ be the PDF of the test item's TTF. Let the item undergo $m-1$ minimal repairs, where $S_1,S_2,\dots,S_m$ are the moments of successive failures, i.e. the new item is put to operation at $S_0=0$, the i-th minimal repair is performed at S_i, $1\le i\le m-1$, and S_m is the time of the last failure after which no more repairs are performed. Clearly, $S_i=T_1+\dots+T_i$, $1\le i\le m$. Under these assumptions, the PDF of the vector r.v. $[T_1,\dots,T_m]^T$, denoted by $f^{(m)}(t_1,\dots,t_m)$, is given by the following formula:

$$f^{(m)}(t_1,\dots,t_m)$$

$$=\prod_{i=1}^{m-1}r(t_1+\dots+t_i)\,f(t_1+\dots+t_m)$$

$$=\prod_{i=1}^{m-1}r(s_i)\,f(s_m) \qquad (8)$$

Proof:

For $m\ge 2$ it holds that

$$\Pr\begin{pmatrix}T_1\in(t_1,\ t_1+\Delta t_1],\\ \vdots\\ T_m\in(t_m,\ t_m+\Delta t_m]\end{pmatrix}$$

$$=\Pr(T_1\in(t_1,t_1+\Delta t_1])\times$$

$$\times\prod_{i=2}^{m}\Pr\begin{pmatrix}T_i\in(t_i,t_i+\Delta t_i]\ |\\ T_{i-1}\in(t_{i-1},t_{i-1}+\Delta t_{i-1}],\\ \vdots\\ T_1\in(t_1,t_1+\Delta t_1]\end{pmatrix}$$

$$\approx[F(t_1+\Delta t_1)-F(t_1)]\times$$

$$\times\prod_{i=2}^{m}\Pr\begin{pmatrix}T_i\in(t_i,t_i+\Delta t_i]\ |\\ T_1+\dots+T_{i-1}=t_1+\dots+t_{i-1}\end{pmatrix}$$

$$=[F(t_1+\Delta t_1)-F(t_1)]\times$$

$$\times\prod_{i=2}^{m}\frac{F(t_1+\dots+t_{i-1}+t_i+\Delta t_i)-F(t_1+\dots+t_{i-1}+t_i)}{1-F(t_1+\dots+t_{i-1})}$$

$$=\prod_{i=1}^{m-1}\frac{F(t_1+\dots+t_i+\Delta t_i)-F(t_1+\dots+t_i)}{1-F(t_1+\dots+t_i)}\times$$

$$\times[F(t_1+\dots+t_m+\Delta t_m)-F(t_1+\dots+t_m)]. \qquad (9)$$

The penultimate equality in (2) holds due to the fact that if T_a is the residual TTF of an operable item after it has reached the age a, then

$$\Pr(T_a \in (t+\Delta t]) = \frac{F(a+t+\Delta t)-F(a+t)}{1-F(a)}.$$

We have:

$$f^{(m)}(t_1,\dots,t_m) = \frac{\partial^m F^{(m)}(t_1,\dots,t_m)}{\partial t_1 \dots \partial t_m}$$

$$= \lim_{\Delta t_1 \to 0,\dots,\Delta t_m \to 0} \frac{\Pr\begin{pmatrix} T_1 \in (t_1, t_1+\Delta t_1] \\ \vdots \\ T_m \in (t_m, t_m+\Delta t_m] \end{pmatrix}}{\Delta t_1 \dots \Delta t_m} \quad (10)$$

where $F^{(m)}(t_1,\dots,t_m)$ is the CDF of $[T_1,\dots,T_m]^T$. From (9) and (10) it follows that

$$f^{(m)}(t_1,\dots,t_m)$$

$$= \prod_{i=1}^{m-1} \frac{f(t_1+\dots+t_i)}{1-F(t_1+\dots+t_i)} f(t_1+\dots+t_m)$$

$$= \prod_{i=1}^{m-1} r(s_i)\, f(s_m). \quad (11)$$

This ends the proof.
In the case of a two-parameter Weibull distribution, i.e.

$$f(t) = \alpha\lambda(\lambda t)^{\alpha-1}\exp[-(\lambda t)^\alpha],$$

$$r(t) = \alpha\lambda(\lambda t)^{\alpha-1} \quad (12)$$

where α and λ are the shape and scale parameters, we have:

$$f^{(m)}(t_1,\dots,t_m)$$

$$= \alpha^m \lambda^{\alpha m} \prod_{i=1}^{m} (t_1+\dots+t_i)^{\alpha-1} \times$$

$$\times \exp\left[-(\lambda)^\alpha (t_1+\dots+t_m)^\alpha\right]. \quad (13)$$

The above formula is obtained from (8) combined with (12). The density $f^{(m)}(t_1,\dots,t_n)$, interpreted as the function of λ and α, is the likelihood function for the sample $t_1,\dots,t_m$, denoted by $L(\alpha,\lambda|t_1,\dots,t_m)$. From (13) it follows that

$$\ln[L(\alpha,\lambda|t_1,\dots,t_m)]$$

$$= m\cdot\ln(\alpha) + \alpha\cdot m\cdot\ln(\lambda) +$$

$$+(\alpha-1)\sum_{i=1}^{m}\ln(t_1+\dots+t_i) +$$

$$-\lambda^\alpha (t_1+\dots+t_m)^\alpha. \quad (14)$$

We have thus derived the expression for the log-likelihood function (the logarithm of the likelihood function), which will play fundamental role in finding MLE of the parameters λ and α.
The standard way to find the maximum likelihood estimates of unknown parameters – the arguments of a likelihood function – is to compute the first partial derivatives of the log-likelihood function w.r.t. these parameters, and equate them to zero, while the variables (in this case $t_1,\dots,t_n$), which constitute a random sample, are considered to be fixed. It should also be checked if the likelihood function actually reaches a maximum where the derivatives are equal to zero, but this check is often omitted. Applying this standard procedure to our case we obtain:

$$\frac{\partial \ln[L(\alpha,\lambda|t_1,\dots,t_m)]}{\partial\alpha}$$

$$= \frac{m}{\alpha} + m\cdot\ln(\lambda) + \sum_{i=1}^{m}\ln(t_1+\dots+t_i) +$$

$$-\lambda^\alpha (t_1+\dots+t_m)^\alpha \ln[\lambda(t_1+\dots+t_m)] \quad (15)$$

and

$$\frac{\partial \ln[L(\alpha,\lambda|t_1,\dots,t_m)]}{\partial\lambda}$$

$$= \frac{\alpha\cdot m}{\lambda} - \alpha\cdot\lambda^{\alpha-1}(t_1+\dots+t_m)^\alpha. \quad (16)$$

In order to find $\hat{\lambda}$ and $\hat{\alpha}$ for which the above derivatives are equal to zero, we first equate the right-hand side of (16) to zero, which yields:

$$\hat{\lambda} = \frac{m^{1/\hat{\alpha}}}{t_1+\dots+t_m} \quad (17)$$

We then substitute λ in (15) with the quotient on the right-hand side of (17) and equate the right-hand side of (15) to zero, obtaining:

$$\frac{m}{\hat{\alpha}} + \frac{m}{\hat{\alpha}}\cdot\ln(m) - m\cdot\ln(t_1+\dots+t_m) +$$

$$+\sum_{i=1}^{m}\ln(t_1+\dots+t_i) - \frac{m}{\hat{\alpha}}\cdot\ln(m) = 0 \quad (18)$$

which yields:

$$\hat{\alpha} = \frac{m}{m \cdot \ln(t_1 + \dots + t_m) - \sum_{i=1}^{m} \ln(t_1 + \dots + t_i)}. \quad (19)$$

Note that if $\hat{\alpha}$ in (17) is substituted by the quotient on the right-hand side of (19), then $\hat{\lambda}$ becomes a function of m and $t_1,\dots,t_m$ alone.

The formulas (17) and (19) define the m-sample estimators based on one sequence of TTFs following the successive minimal repairs. Here, $t_1,\dots,t_m$ denote the values of a random sample obtained by recording the respective TTFs. Clearly, $\hat{\alpha}$ and $\hat{\lambda}$ are dependent on m, but for simplicity they are written without m. A natural question arises as to the accuracy of these estimators. It can be judged by two criteria – the estimators' biases and variances. The biases are given by the differences $\lambda - E(\hat{\lambda})$ and $\alpha - E(\hat{\alpha})$, where $\hat{\lambda}$ and $\hat{\alpha}$ are treated as random variables, i.e. in (17) and (19) the values $t_1,\dots,t_n$ have to be replaced by the random variables $T_1,\dots,T_n$. To be more precise, we will find the biases of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$ rather than of $\hat{\lambda}$ and $\hat{\alpha}$, i.e. formulas for $\ln(\lambda) - E[\ln(\hat{\lambda})]$ and $1/\alpha - E(1/\hat{\alpha})$ will be derived. This will be done in the next section along with the respective justification for this workaround. In turn, due to the encountered computational difficulties, the estimators' variances are not considered in this chapter, but will be a topic of future research.

In the case of estimation based on a sample composed of i.i.d. realizations of some random variable, its accuracy is usually determined by the confidence level along with the length of the confidence interval, and the sample size has to be sufficiently large in order to obtain the required accuracy. Obviously, this size is related to the variance of the considered random variable. However, in our case the $T_1,\dots,T_m$ are not independent, have different CDF's, and in practice it is not possible to perform a large number of minimal repairs on one object. Frequently, the object becomes unusable after several such repairs. For the above reasons our parameters will be estimated by taking n identical and independent objects, performing $m - 1$ minimal repairs on each of them (m not being large) to obtain n sample values of both $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$, and calculating the respective sample means to approximate $E[\ln(\hat{\lambda})]$ and $E(1/\hat{\alpha})$. Thus, $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ defined as follows

$$\widehat{\Lambda} = \frac{\ln(\hat{\lambda}_1) + \dots + \ln(\hat{\lambda}_n)}{n}, \widehat{\mathrm{A}} = \frac{1/\hat{\alpha}_1 + \dots + 1/\hat{\alpha}_n}{n}, \quad (20)$$

where $\hat{\lambda}_1,\dots,\hat{\lambda}_n$ and $\hat{\alpha}_1,\dots,\hat{\alpha}_n$ are i.i.d. instances of $\hat{\lambda}$ and $\hat{\alpha}$ respectively, will be used as estimators of $\ln(\lambda)$ and $1/\alpha$. They will be called n·m-sample estimators, because the number of TTFs needed for their computation equals $n \cdot m$. The law of large numbers yields that

$$\widehat{\Lambda} \approx E[\ln(\hat{\lambda})], \widehat{\mathrm{A}} \approx E(1/\hat{\alpha}), \quad (21)$$

thus our estimation task consists in approximating expected values with means of i.i.d. samples. Clearly, the number n for which the required accuracy is achieved is proportional to $Var(\ln(\hat{\lambda}))$ or $Var(1/\hat{\alpha})$. Let us note that in view of (20) we have:

$$Var(\widehat{\Lambda}) = \frac{Var(\ln(\hat{\lambda}))}{n}, Var(\widehat{\mathrm{A}}) = \frac{Var(1/\hat{\alpha})}{n}. \quad (22)$$

It should also be remembered that $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ are biased estimators of $\ln(\lambda)$ and $1/\alpha$. Let us note that

$$E(\widehat{\Lambda}) = E[\ln(\hat{\lambda})], E(\widehat{\mathrm{A}}) = E(1/\hat{\alpha}), \quad (23)$$

thus the biases of $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$ are equal to those of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$. In view of (22) and (23) we can say that the $n \cdot m$-sample estimators are n times more accurate than the m-sample ones. The formulas for the respective biases will be derived in the next section, while finding the variances and confidence intervals will be the subject of further research.

5. Expressing $\ln(\lambda)$ and $1/\alpha$ in terms of $E[\ln(\hat{\lambda})]$ and $E(1/\hat{\alpha})$, and finding biases of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$

For further considerations we will need two auxiliary lemmas.

Lemma 2

For $m \geq 1$ it holds that

$$\Pr(T_1 + \dots + T_m > t)$$

$$= \exp[-(\lambda t)^{\alpha}] \sum_{k=0}^{m-1} \frac{(\lambda t)^{k\alpha}}{k!}. \quad (24)$$

Proof:

Clearly, (24) holds for $m = 1$. For $m \geq 2$, in view of (1), we have:

$Pr(T_1 + \cdots + T_m \le t)$

$= \int_{t_1+\cdots+t_m \le t} r(t_1) \ldots r(t_1 + \cdots + t_{m-1}) \times$

$\times f(t_1 + \cdots + t_m) dt_m \ldots dt_1$

$= \int_0^t r(t_1) \ldots \int_0^{t-t_1-\cdots-t_{m-2}} r\,(t_1 + \cdots + t_{m-1})$

$\int_0^{t-t_1-\cdots-t_{m-1}} f\,(t_1 + \cdots + t_m) dt_m \ldots dt_1$

$= \int_0^t r(t_1) \ldots \int_0^{t-t_1-\cdots-t_{m-2}} r\,(t_1 + \cdots + t_{m-1}) \times$

$\times [F(t) - F(t_1 + \cdots + t_{m-1})] dt_{m-1} \ldots dt_1$

$= F(t) \times \int_0^t r(t_1) \ldots$

$\ldots F \int_0^{t-t_1-\cdots-t_{m-2}} r\,(t_1 + \cdots + t_{m-1}) dt_{m-1} \ldots dt_1$

$+ \int_0^t r(t_1) \ldots \int_0^{t-t_1-\cdots-t_{m-2}} r(t_1 + \cdots + t_{m-1}) \times$

$\times [1 - F(t_1 + \cdots + t_{m-1}) - 1] dt_{m-1} \ldots dt_1$

$= \int_0^t r(t_1) \ldots$

$\ldots \int_0^{t-t_1-\cdots-t_{m-2}} f(t_1 + \cdots + t_{m-1})\, dt_{m-1} \ldots dt_1$

$-[1 - F(t)] \times \int_0^t r(t_1) \ldots$

$\ldots \int_0^{t-t_1-\cdots-t_{m-2}} r(t_1 + \ldots + t_{m-1})\, dt_{m-1} \ldots dt_1$

$= Pr(T_1 + \cdots + T_{m-1} \le t) - [1 - F(t)] \times$

$\times \int_0^t r(t_1) \ldots$

$\ldots \int_0^{t-t_1-\cdots-t_{m-2}} r(t_1 + \cdots + t_{m-1})\, dt_{m-1} \ldots dt_1.$ (25)

In the last two steps of the above derivation the equality $r(t_1+\ldots+t_{m-1})[1 - F(t_1+\ldots+t_{m-1})] = f(t_1+\ldots+t_{m-1})$ and formula (1) were used. The last integral in (25) satisfies the following equality:

$\int_0^t r(t_1) \ldots \int_0^{t-t_1-\cdots-t_{m-2}} r(t_1 + \cdots + t_{m-1})$

$dt_{m-1} \ldots dt_1 = \frac{1}{(m-1)!} (\lambda t)^{(m-1)\alpha}.$ (26)

In order to prove (26) let us put

$s_0 = 0, s_1 = t_1, \ldots, s_{m-1} = t_1 + \cdots + t_{m-1}.$

We thus have

$\int_0^t r(t_1) \ldots$

$\ldots \int_0^{t-t_1-\cdots-t_{m-2}} r(t_1 + \cdots + t_{m-1})\, dt_{m-1} \ldots dt_1 =$

$\int_{s_0}^t r(s_1) \int_{s_1}^t r(s_2) \ldots \int_{s_{m-2}}^t r(s_{m-1})\, ds_{m-1} \ldots ds_1.$

By the *reverse* induction it can be shown that

$\int_{s_{k-1}}^t r(s_k) \ldots \int_{s_{m-2}}^t r(s_{m-1})\, ds_{m-1} \ldots ds_k$

$= \frac{1}{(m-k)!} [(\lambda t)^\alpha - (\lambda s_{k-1})^\alpha]^{m-k}$ (27)

for $k = m - 1,\ldots,1$. Clearly, (27) holds for $k = m - 1$. The induction step is based on the following derivation:

$\int_{s_{k-1}}^t r(s_k) \frac{1}{(m-k-1)!} \times$

$\times [(\lambda t)^\alpha - (\lambda s_k)^\alpha]^{m-k-1} ds_k$

$= \frac{1}{(m-k-1)!} \int_{s_{k-1}}^t [(\lambda t)^\alpha - (\lambda s_k)^\alpha]^{m-k-1} \times$

$\times \frac{d(\lambda s_k)^\alpha}{ds_k} ds_k$

$= \frac{1}{(m-k-1)!} \int_{u(s_{k-1})}^{u(t)} [(\lambda t)^\alpha - u]^{m-k-1} du$

$= -\frac{1}{(m-k-1)!(m-k)} [(\lambda t)^\alpha - u]^{m-k} \Big|_{u(s_{k-1})}^{u(t)}$

$= \frac{1}{(m-k)!} [(\lambda t)^\alpha - (\lambda s_{k-1})^\alpha]^{m-k}.$

Above, we used the following substitution:

$u(s) = (\lambda s)^\alpha.$

Now (26) is a direct consequence of (27). The formulas (25) and (26) yield the following recursive equation that holds for $m \ge 2$:

$Pr(T_1 + \cdots + T_m \le t)$

$$= Pr(T_1 + \cdots + T_{m-1} \le t) +$$

$$- \exp[-(\lambda t)^{\alpha}] \frac{1}{(m-1)!} (\lambda t)^{(m-1)\alpha}. \qquad (28)$$

Since (24) follows directly from (28), the proof is completed.

Lemma 3
If X is a continuous random variable then

$$E(|X|^r) = r \int_0^{\infty} x^{r-1} \Pr(|X| > x)\, dx. \qquad (29)$$

The proof can be found in (Feller, 1971).

Lemmas 2 and 3 are needed to prove the following theorem.

Theorem 1

$$E[\ln(T_1 + \cdots + T_i)]$$

$$= \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{k=1}^{i-1} \frac{1}{k}\right] - \ln(\lambda),\ i \ge 1 \qquad (30)$$

where Γ is the Euler's gamma function, and Γ' is its first derivative. The sum in brackets is assumed to be equal to zero for i = 1.

Proof:
We will first derive an expression for the MGF of $\ln(S_i) = \ln(T_1+\ldots+T_i)$. Let X be a continuous non-negative random variable. Let $G_{\ln(X)}(t)$ denote the MGF of ln(X). We have

$$G_{\ln(X)}(u) = E\left[e^{u\cdot\ln(X)}\right] = E\left[\left(e^{\ln(X)}\right)^{u}\right]$$

$$= E[X^u] = u \int_0^{\infty} x^{u-1} \Pr(X > x)\, dx \qquad (31)$$

where the last equality follows from (29). Combining (31) and (24) yields

$$G_{\ln(S_i)}(u)$$

$$= u \int_0^{\infty} x^{u-1} \exp[-(\lambda x)^{\alpha}] \sum_{k=0}^{i-1} \frac{(\lambda x)^{k\alpha}}{k!}\, dx. \qquad (32)$$

If we put $y = (\lambda x)^{\alpha}$, then

$$x = \frac{1}{\lambda} y^{\frac{1}{\alpha}},\ \frac{dy}{dx} = \alpha\lambda(\lambda x)^{\alpha-1} = \alpha\lambda y^{1-\frac{1}{\alpha}} \qquad (33)$$

Applying integration by substitution, we obtain

$$u \int_0^{\infty} x^{u-1} \exp[-(\lambda x)^{\alpha}] \frac{(\lambda x)^{k\alpha}}{k!}\, dx$$

$$= u \int_0^{\infty} \frac{y^{\frac{1}{\alpha}(u-1)}}{\lambda^{u-1}} \exp(-y) \frac{y^k}{k!} \frac{y^{\frac{1}{\alpha}-1}}{\alpha\lambda}\, dy$$

$$= \frac{u}{k!\alpha\lambda^u} \int_0^{\infty} y^{\frac{u}{\alpha}+k-1} \exp(-y)\, dy$$

$$= \frac{u}{k!\alpha\lambda^u} \Gamma\left(\frac{u}{\alpha} + k\right). \qquad (34)$$

Formulas (32) and (34) yield:

$$G_{\ln(S_i)}(u) = \frac{u}{\alpha\lambda^u} \sum_{k=0}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!}$$

$$= \frac{1}{\lambda^u}\frac{u}{\alpha}\Gamma\left(\frac{u}{\alpha}\right) + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!}$$

$$= \frac{1}{\lambda^u}\Gamma\left(\frac{u}{\alpha}+1\right) + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!}$$

$$= G_1(u) + G_2(u). \qquad (35)$$

The penultimate equality in (35) is a consequence of the fact that $v\Gamma(v) = \Gamma(v+1)$. The sum from k = 1 to i – 1 is assumed to be zero for i = 1. Differentiating $G_1(u)$ and $G_2(u)$ with respect to u we obtain:

$$G_1'(u) = \frac{d\left(\frac{1}{\lambda^u}\right)}{du}\Gamma\left(\frac{u}{\alpha}+1\right) + \frac{1}{\lambda^u}\Gamma'\left(\frac{u}{\alpha}+1\right)\frac{1}{\alpha}$$

$$= \frac{1}{\alpha\lambda^u}\Gamma'\left(\frac{u}{\alpha}+1\right) - \frac{\lambda^u\ln(\lambda)}{\lambda^{2u}}\Gamma\left(\frac{u}{\alpha}+1\right) \qquad (36)$$

and

$$G_2'(u) = \frac{d\left(\frac{u}{\alpha\lambda^u}\right)}{du} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!}$$

$$= \frac{\alpha\lambda^u - u\alpha\lambda^u\ln(\lambda)}{\alpha^2\lambda^{2u}} \sum_{k=1}^{i-1} \frac{\Gamma\left(\frac{u}{\alpha}+k\right)}{k!} + \frac{u}{\alpha\lambda^u} \sum_{k=1}^{i-1} \frac{\Gamma'\left(\frac{u}{\alpha}+k\right)}{k!} \qquad (37)$$

From the properties of the MGF combined with (36) and (37) we have

$$E[\ln(T_1 + \cdots + T_i)] = \left.\frac{dG_{\ln(S_i)}(u)}{du}\right|_{u=0}$$

$= \frac{1}{\alpha}\Gamma'(1) - \ln(\lambda)\,\Gamma(1) + \frac{1}{\alpha}\sum_{k=1}^{i-1}\frac{\Gamma(k)}{k!}$

$$= \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{k=1}^{i-1}\frac{1}{k}\right] - \ln(\lambda). \quad (38)$$

The proof of Theorem 1 is thus completed.

Applying Theorem 1 we will express α in terms of $\hat{\alpha}$, and λ in terms of $\hat{\lambda}$, thus solving the problem of finding the unknown parameters of the Weibull distribution. More precisely, for technical reasons explained further on, $1/\alpha$ will be expressed as a function of $E(1/\hat{\alpha})$, and $\ln(\lambda)$ – as a function of $E[\ln(\hat{\lambda})]$. In turn, $E(1/\hat{\alpha})$ and $E[\ln(\hat{\lambda})]$ can be easily approximated using an *n*-sized random sample of the vector random variable $[T_1,\ldots,T_m]$, where *n* is sufficiently large. First, we derive the relation between $1/\alpha$ and $E(1/\hat{\alpha})$. From (19) we obtain:

$$E(1/\hat{\alpha}) = \frac{1}{m}\begin{Bmatrix} m\cdot E[\ln(t_1+\cdots+t_m)] + \\ -\sum_{i=1}^{m} E[\ln(t_1+\cdots+t_i)] \end{Bmatrix}$$

$$= \frac{1}{m}\sum_{i=1}^{m-1}\begin{Bmatrix} E[\ln(t_1+\cdots+t_m)] + \\ -E[\ln(t_1+\cdots+t_i)] \end{Bmatrix}$$

$$= \frac{1}{m\cdot\alpha}\sum_{i=1}^{m-1}\left[\sum_{j=1}^{m-1}\frac{1}{j} - \sum_{j=1}^{i-1}\frac{1}{j}\right]$$

$$= \frac{1}{m\cdot\alpha}\sum_{i=1}^{m-1}\left[\sum_{j=i}^{m-1}\frac{1}{j}\right]. \quad (39)$$

The penultimate equality in (39) follows directly from Theorem 1. It holds that

$$\sum_{i=1}^{m-1}\left[\sum_{j=i}^{m-1}\frac{1}{j}\right] = m-1 \quad (40)$$

which is easily proved by induction. As a consequence of (39) and (40) we have:

$$\frac{1}{\alpha} = \frac{m}{m-1}E\left(\frac{1}{\hat{\alpha}}\right). \quad (41)$$

Hence, the bias of $1/\hat{\alpha}$ is given by

$$1/\alpha - E(1/\hat{\alpha}) = \frac{1}{m-1}E(1/\hat{\alpha}) \quad (42)$$

which yields that $1/\hat{\alpha}$ is an asymptotically unbiased estimator of $1/\alpha$ with respect to m. With regard to (20), the approximate value of $E(1/\hat{\alpha})$ can be found from the following formula:

$$E(1/\hat{\alpha}) \approx \hat{\mathrm{A}} = \left(\frac{1}{\hat{\alpha}_1} + \cdots + \frac{1}{\hat{\alpha}_n}\right)/n \quad (43)$$

where, according to (19),

$$\frac{1}{\hat{\alpha}_j} = \frac{m-1}{m}\ln\left(t_{1j}+\cdots+t_{mj}\right) +$$

$$-\frac{1}{m}\sum_{i=1}^{m-1}\ln\left(t_{1j}+\cdots+t_{ij}\right),\ 1 \le j \le n. \quad (44)$$

In the above formula $t_{1j},\ldots,t_{mj}$ are the TTFs constituting the *j*-th vector of the random sample composed of *n* realizations of the vector random variable $[T_1,\ldots,T_m]$.

Note that in formula (42) we have $1/\alpha$ and $E(1/\hat{\alpha})$ rather than α and $E(\hat{\alpha})$. This is caused by the fact that finding the relation between α and $E(\hat{\alpha})$ would involve the analytical computation of $E(m/[m\cdot\ln(s_m) - \sum_{i=1}^{m}\ln(s_i)])$, which is an impossible task if only the formulas for $E[\ln(s_i)]$ are known. The reason is that having a formula for $E(X)$ is not sufficient to compute $E(1/X)$.

Let us now express $\ln(\lambda)$ as a function of $E[\ln(\hat{\lambda})]$. From (17) it follows that

$$\ln(\hat{\lambda}) = \frac{1}{\hat{\alpha}}\ln(m) - \ln(t_1+\cdots+t_m). \quad (45)$$

Considering $\ln(\hat{\lambda})$ as a random variable and using (30), from (45) we obtain:

$$E[\ln(\hat{\lambda})]$$

$$= E\left(\frac{1}{\hat{\alpha}}\right)\ln(m) - \frac{1}{\alpha}\left[\Gamma'(1) + \sum_{j=1}^{m-1}\frac{1}{j}\right] + \ln(\lambda) \quad (46)$$

In view of (41) the above formula is converted to:

$$\ln(\lambda) = E[\ln(\hat{\lambda})] - \ln(m)\,E\left(\frac{1}{\hat{\alpha}}\right)$$

$$+\frac{m}{m-1}E\left(\frac{1}{\hat{\alpha}}\right)\left[\Gamma'(1) + \sum_{j=1}^{m-1}\frac{1}{j}\right]$$

$$= E[\ln(\hat{\lambda})] + \frac{m}{m-1}E\left(\frac{1}{\hat{\alpha}}\right)\cdot$$

$$\cdot\left[-\ln(m)\frac{m-1}{m} + \Gamma'(1) - \frac{1}{m} + \sum_{j=1}^{m}\frac{1}{j}\right]. \quad (47)$$

Hence, the bias of $\ln(\hat{\lambda})$ is given by

$$\ln(\lambda) - E[\ln(\hat{\lambda})]$$

$$= \frac{m}{m-1} E\left(\frac{1}{\hat{\alpha}}\right) \cdot$$

$$\cdot \left[\frac{\ln(m)}{m} - \frac{1}{m} + \Gamma'(1) - \ln(m) + \sum_{j=1}^{m} \frac{1}{j}\right]. \quad (48)$$

From the special functions theory it is known that $\Gamma'(1) = -\gamma$, where γ is the so-called Euler-Mascheroni constant defined as

$$\gamma = \lim_{m\to\infty} \left[\sum_{j=1}^{m} \frac{1}{j} - \ln(m)\right] \cong 0.577. \quad (49)$$

We also have:

$$\lim_{m\to\infty} \frac{m}{m-1} = 1, \lim_{m\to\infty} \frac{\ln(m)}{m} = 0, \lim_{m\to\infty} \frac{1}{m} = 0. \quad (50)$$

Thus, $\ln(\hat{\lambda})$ is an asymptotically unbiased estimator of $\ln(\lambda)$ with respect to $\boldsymbol{m}$. In view of (20), the approximate value of $E[\ln(\hat{\lambda})]$ can be found from the following formula:

$$E[\ln(\hat{\lambda})] \approx \widehat{\Lambda} = \frac{[\ln(\hat{\lambda}_1) + \cdots + \ln(\hat{\lambda}_n)]}{n} \quad (51)$$

where, according to (17) and (44),

$$\ln(\hat{\lambda}_j) = \frac{1}{\hat{\alpha}_j} \ln(m) - \ln(t_{1j} + \cdots + t_{mj})$$

$$= \left[\frac{m-1}{m} \ln(m) - 1\right] \ln(t_{1j} + \cdots + t_{mj}) +$$

$$- \frac{\ln(m)}{m} \sum_{i=1}^{m-1} \ln(t_{1j} + \cdots + t_{ij}). \quad (52)$$

Note that $\ln(\lambda)$ and $E[\ln(\hat{\lambda})]$ rather than λ and $E(\hat{\lambda})$ are used in (48), because it is easier to operate on logarithms than directly on λ and its estimator $\hat{\lambda}$.

The fact that the estimators $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$ are asymptotically unbiased with respect to $\boldsymbol{m}$ has rather theoretical significance, as in practice $\boldsymbol{m}$ is not large enough for these estimators to be close enough to $\ln(\lambda)$ and $1/\alpha$.

6. Conclusion

A new approach to estimating the parameters of the Weibull distribution has been presented. It uses the $\boldsymbol{m}$-sample estimators based on non-independent TTFs of a repairable item subjected to $\boldsymbol{m} - 1$ minimal repairs following the first $\boldsymbol{m} - 1$ failures. These estimators are defined by (17) and (19). It occurs that they are biased, the respective biases being explicitly given by (42) and (48). As explained in Section 5, the biases are calculated for the estimators $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$ rather than $\hat{\lambda}$ and $\hat{\alpha}$. The more accurate n·m-sample estimators $\widehat{\Lambda}$ and $\widehat{\mathrm{A}}$, defined by (20), are obtained from $\boldsymbol{n}$ i.i.d. realizations of $\hat{\lambda}$ and $\hat{\alpha}$. The variance of $\widehat{\Lambda}$ or $\widehat{\mathrm{A}}$ is $\boldsymbol{n}$ times smaller than that of $\hat{\lambda}$ or $\hat{\alpha}$, while the bias of $\widehat{\Lambda}$ or $\widehat{\mathrm{A}}$ is equal to that of $\hat{\lambda}$ or $\hat{\alpha}$.

Estimating $\ln(\lambda)$ and $1/\alpha$ instead of λ and α does not pose a problem, because the parameters λ and α can be approximated with the use of biases of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$, as follows:

$$\alpha \approx \frac{m-1}{m \cdot \widehat{\mathrm{A}}}, \lambda \approx \exp(\widehat{\Lambda}) \cdot \exp[\widehat{\mathrm{A}} \cdot \varphi(m)] \quad (53)$$

where

$$\varphi(m)$$

$$= \frac{m}{m-1} \left[\frac{\ln(m)}{m} - \frac{1}{m} + \Gamma'(1) - \ln(m) + \sum_{j=1}^{m} \frac{1}{j}\right]. \quad (54)$$

The above formulas follow from (20), (41) and (48).

The newly developed method has one important advantage over the standard MLE of the Weibull distribution parameters. Namely, (17), (19), (42) and (48) are explicit and relatively simple formulas expressing the estimators and their biases as functions of $\boldsymbol{m}$ and $t_1, \ldots, t_m$. As shown in Section 3, the standard MLE of the shape parameter, based on a series of i.i.d. TTFs of non-repairable test items, is obtained from an equation that cannot be solved analytically.

As mentioned at the end of Section 4, calculating the variances of $\ln(\hat{\lambda})$ and $1/\hat{\alpha}$ remains an open problem. It seems that explicit formulas for these variances can be found using moment generating functions, which will be attempted in the near future.

References

Alizadeh, M., Rezaei, S. & Bagheri, S.F. 2015. On the estimation for the Weibull distribution. *Annals of Data Science* 2(4), 373–390.

Alkutubi, H.S. & Ali, H.M. 2011. Maximum likelihood estimators with complete and censored data. *European Journal of Scientific Research* 54(3), 407–410.

Almazah, M. & Ismail, M. 2021. Selection of efficient parameter estimation method for two-parameter Weibull distribution. *Hindawi. Mathematical Problems in Engineering* 2021, article ID 5806068.

Chen, M., Zhang, Z. & Cui, C. 2017. On the bias of the maximum likelihood estimators of parameters of the Weibull distribution. *Mathematical and Computational Applications* 22(1), 19.

Chikr el-Mezouar, Z. 2010. Estimation of the shape, location and scale parameters of the Weibull distribution. *Reliability: Theory & Applications (Electronic Journal of International Group on Reliability)* 5(4), 3–40.

Dodson, B. 2006. *The Weibull Analysis Handbook. Second Edition*. American Society for Quality.

Evans, J.W., Kretschmann, D.E. & Green, D.W. 2019. *Procedures for estimation of Weibull Parameters*. United States Department of Agriculture, Forest Service, Forest Products Laboratory. General Technical Report FPL–GTR–264.

Feller, W. 1971. *An Introduction to Probability Theory and its Applications*. John Wiley & Sons, New York.

Lei, Y. 2008. Evaluation of three methods for estimating the Weibull distribution parameters of Chinese pine. *Journal of Forest Science* 54(12), 566–571.

Wu, Y., Xie, H., Chiang, J.-Y., Peng, G. & Qin, Y. 2021. Parameter estimation and applications of the Weibull distribution for strength data of glass fiber. *Hindawi*. *Mathematical Problems in Engineering* 2021, article ID 9175170.