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Extremals of the time optimal control problem for a material point moving along a straight line in the presence of friction and limitation on the velocity*

by

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Abstract: This paper provides an analysis of the time optimal control problem for a material point moving along a straight line in the presence of strength of resistance to movement (friction) and subject to constraint on the velocity. The point is controlled by a limited traction or braking force. The analysis of the problem is based on the maximum principle for state constraints in the Dubovitskii-Milyutin form, see Dubovitskii and Milyutin (1965), and the necessary second-order optimality condition for bang-bang controls, see Milyutin and Osmolovskii (1998).

Keywords: material point, resistance force, acceleration, deceleration, Pontryagin's maximum principle, second–order optimality condition, optimal control, adjoint variable, state constraint, Stieltjes measure, singular arc, boundary arc

1. Introduction

The aim of this article is to attract the attention of transportation and mechanical engineers to such an effective research tool as the Pontryagin maximum principle (MP), see Pontryagin et al. (1964). The MP allows for studying an optimal motion of objects controlled by a limited force. Time optimal control problems for a material point were considered in our previous publications, see Osmolovskii, Figura, Kośka (2013), or Osmolovskii, Wójtowicz, Janiszewski (2013), or Osmolovskii, Figura, Kośka, Wójtowicz (2015). Here we study such

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a problem, taking into account the resistance force and the limitation on the velocity.

More precisely, we investigate the following optimization problem. A train (a tram, a trolley, a bus, etc.), considered as a material point of the mass equal to one (in conventional units), moves along a given segment of a horizontal straight line under the influence of limited force of traction or braking and in the presence of the force of resistance to the motion, depending on the velocity. The initial position and the initial velocity, the terminal position and the terminal velocity are given. The goal is to minimize the time of motion.

We exploit first and second order necessary optimality conditions to evaluate candidates for the optimal solution of this problem. It is very well known that "a solution derived by necessary conditions only is simply no valid solution at all...", see Young (1969), pp. 22-23 (Perron paradox). The necessary conditions may provide the solution of the optimal control problem only in the case where the solution exists.

By the Filippov theorem, see Filippov (1962), the solution to this problem exists. Taking into account this fact, we find extremals satisfying the Dubovitskii-Milyutin maximum principle for problems with state constraints, see Dubovitskii and Milyutin (1965), and then we choose among these extremals the optimal solution using second-order necessary optimality conditions obtained by the first author, see Milyutin and Osmolovskii (1998).

The paper is organized as follows. In Section 2 we give a strict formulation of the problem, together with the necessary assumptions, and provide some explanations and references concerning the existence of a solution. Section 3 contains the formulation of the maximum principle for problems with state constraints in the Dubovitskii–Milyutin form. We recall a general formulation of the MP, and then we write out a full set of conditions of the MP for the case under consideration. Section 4 is dedicated to the direct and detailed analysis of the MP in our problem, as a result of which we obtain five possible cases A)-E) for extremals dependent on the initial and final conditions. In all these cases the control turned out to be piecewise constant. In Section 5 we show that the extremal, obtained in the case E) (in which the state constraint is not active and the control is bang-bang), does not satisfy second-order necessary optimality conditions, see Milyutin and Osmolovskii (1998). Therefore, the possible structure of the optimal solution is exhausted only by cases A)-D). This is the main result of the paper. The numerical example of Section 6 illustrates this result. In Section 7 we mention some general theoretical results related to our problem; their use allows to come a little faster to the results, obtained in Section 4 by the direct analysis of the MP. In Section 8 we show that, due to a simple structure of the problem, another approach is possible, which is not based on application of necessary optimally conditions. However, small complication of the problem eliminates this possibility, while considerations similar to those that were presented in Sections 4 remains in effect. Section 9 summarizes the results of the study.

2. Optimal control problem

According to Newton's second law, the dynamics of the point satisfies the system of equations

$$\dot{x}(t) = y(t) \quad \text{a.e. in} \quad [0,T], \tag{1}$$

$$\dot{y}(t) = -w(y(t)) + u(t) \quad \text{a.e. in} \quad [0,T], \tag{2}$$

$$u(t) \in [a,b] \quad \text{a.e. in} \quad [0,T], \tag{3}$$

where $x(t)$ is the position, $y(t)$ is the velocity of the point at time t, $w(y)$ is the *force of resistance to the motion (friction)*, depending on the velocity y, $a < 0$ and $b > 0$ are given constants.

Assumption 2.1 *The function $w(\cdot) : \mathbb{R} \to \mathbb{R}$ is odd, continuous, twice continuously differentiable on the half-line $(0,\infty)$, and satisfies the conditions*

$$w'(y) > 0 \quad \textit{and} \quad w''(y) \geq 0 \quad \textit{for all} \quad y > 0.$$

Note that Assumption 2.1 implies: $w(0) = 0$ and $w(y) > 0$ for all $y > 0$.

As was said, the initial position, the initial velocity, the terminal position, and the terminal velocity are fixed, that is

$$x(0) = x_0, \quad y(0) = y_0, \quad x(T) = x_T, \quad y(T) = y_T, \tag{4}$$

where x_0, x_T, y_0, y_T are prescribed values, $x_0 < x_T$.

There is a state constraint of the form

$$y(t) \leq V \quad \text{for all} \quad t \in [0,T], \tag{5}$$

where $V > 0$ is the maximal speed possible. To avoid the trivial maximum principle, we make the following assumption.

Assumption 2.2 $0 \leq y_0 < V$ *and* $0 \leq y_T < V$.

We will also need the following important assumption.

Assumption 2.3 $w(V) < b$.

This implies that if $0 \leq y \leq V$, then $0 \leq w(y) \leq w(V)$, and hence $-w(y) + b \geq -w(V)+b > 0$, while $-w(y)+a < 0$. Consequently, the case of $u = b$ corresponds to acceleration, and the case of $u = a$ corresponds to deceleration.

Our goal now is to minimize the time duration of motion:

$$T \to \min\,. \tag{6}$$

For brevity, the problem (1)-(6) is called *Problem A.* Obviously, an optimal process $(x(\cdot), y(\cdot), u(\cdot), T)$ satisfies

Assumption 2.4 $y(t) \geq 0$ *for all* $t \in [0,T]$, *and there is no interval* $[t_1,t_2] \subset [0,T]$ *of positive measure such that* $y(t) = 0$ *for all* $t \in [t_1,t_2]$.

Otherwise, the time T can be reduced. We consider only processes satisfying this assumption.

Since, in this problem, the set of admissible trajectories is obviously nonempty, both state variables are bounded, the system is linear in the control, and the control values lie in a convex compact set, then, by the theorem of Filippov, see Filippov (1962) or Cesari (1983) or Lee and Markus (1986), the optimal trajectory exists. We note that formally, in Filippov's theorem, there is no state constraint. However, this link to Filippov's theorem is appropriate, since the presence of state constraint in the problem leads only to small changes in the proof and formulation of this theorem. Within the framework of this article we would like to confine ourselves to just such a remark about the existence of solution.

3. The Dubovitskii-Milyutin maximum principle

3.1. Maximum principle for a general problem of type A

For the convenience of the reader, we first formulate the maximum principle, obtained by Dubovitskii and Milyutin for a class of problems, containing Problem A, see Dubovitskii and Milyutin (1965) or Milyutin, Dmitruk, and Osmolovskii (2004). Consider the following optimal control problem:

$$T \to \min \tag{7}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U, \quad \text{for a.a.} \quad t \in [0, T], \tag{8}$$

$$x(0) = x_0, \quad x(T) = x_T, \tag{9}$$

$$\varphi(x(t)) \le 0, \quad \text{for all} \quad t \in [0, T]. \tag{10}$$

Here the *state variable* $x(\cdot) : [0, T] \to \mathbb{R}^n$ is a Lipschitz continuous function, the *control variable* $u(\cdot) : [0, T] \to \mathbb{R}^m$ is a measurable and essentially bounded function, the mapping $f : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$ is assumed to be continuous together with its partial derivative f_x, the mapping $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable, $x_0, x_T \in \mathbb{R}^n$ are given vectors, and $U \subset \mathbb{R}^m$ is an arbitrary set.

A pair of functions $(x(\cdot), u(\cdot))$, together with their domain of definition $[0, T]$, is called the *process* of the problem. A process $(x(\cdot), u(\cdot), T)$ is called *admissible* if it satisfies all constraints of the problem. An admissible process $(\hat{x}(\cdot), \hat{u}(\cdot), \hat{T})$ is called a *strong local minimum* if there is an $\varepsilon > 0$ such that $T \ge \hat{T}$ for all admissible processes satisfying $|T - \hat{T}| < \varepsilon$ and $|x(t) - \hat{x}(t)| < \varepsilon$ for all $t \in [0, T] \cap [0, \hat{T}]$.

In order to formulate the necessary conditions for a strong local minimum for a process $(\hat{x}(\cdot), \hat{u}(\cdot), \hat{T})$, we introduce the *Pontryagin function* (or *pre-Hamiltonian*)

$$H(x, u, p) := p\, f(x, u), \tag{11}$$

where p is a row vector of the dimension n. The function

$$\mathcal{H}(x,p) := \sup_{u\in U} H(x,u,p) \tag{12}$$

is called the *Hamiltonian*.

We say that an admissible process $(\hat{x}(\cdot),\hat{u}(\cdot),\hat{T})$ satisfies conditions of the *maximum principle* if there are functions of bounded variation $p(\cdot):[0,T]\to \mathbb{R}^n$ and $\mu(\cdot):[0,T]\to\mathbb{R}$, defining the measures $\mathrm{d}p$ and $\mathrm{d}\mu$, respectively, such that

$$\mathrm{d}\mu \ge 0, \quad \varphi(\hat{x}(\cdot))\,\mathrm{d}\mu = 0, \tag{13}$$

$$\|p\|_\infty + \int_{[0,T]} \mathrm{d}\mu > 0, \tag{14}$$

$$-\,\mathrm{d}p = H_x(\hat{x}(\cdot),\hat{u}(\cdot),p(\cdot))\,\mathrm{d}t - \varphi'(x(\cdot))\,\mathrm{d}\mu, \tag{15}$$

$$\max_{u\in U} H(\hat{x}(t),u,p(t)) = H(\hat{x}(t),\hat{u}(t),p(t)) \quad \text{for a.a. } t\in[0,T], \tag{16}$$

$$H(\hat{x}(t),\hat{u}(t),p(t)) = \text{const} =: \alpha_0 \ge 0 \quad \text{for a.a. } t\in[0,T]. \tag{17}$$

Here, conditions (13) are called the *nonnegativeness of the measure* and *complementarity condition*, respectively, inequality (14) is called the *nontriviality condition*, (15) is the *adjoint equation*, (16) is the *maximum condition for the Pontryagin function*, and (17) is the condition of the *constancy and nonnegativeness of the Hamiltonian*.

Let us emphasize that here and in the sequel we denote by $\mathrm{d}p$ and $\mathrm{d}\mu$ the Lebesgue-Stieltjes measures, generated by the functions of bounded variation $p(\cdot)$ and $\mu(\cdot)$, respectively. (It is convenient and will allow for avoiding new notation.) Recall that, on the real line, each such measure is necessarily a regular Borel measure (and vice versa). The adjoint equation (15) is understood as the equality between measures.

THEOREM 3.1 *If a process $(\hat{x}(\cdot),\hat{u}(\cdot),\hat{T})$ is a strong local minimum in problem (7)-(10), then it satisfies conditions of the maximum principle.*

3.2. Maximum principle for Problem A

Observe, that for Problem A we have:

$$m=1,\ n=2, \quad U=[a,b],$$

$$f_1(x,y)=y,\ f_2(x,y)=-w(y)+u, \quad \varphi(x,y)=y-V,$$

and obviously, all assumptions of problem (7)-(10) are fulfilled. According to (11), the Pontryagin function for Problem A has the form:

$$H = p_1 y + p_2(-w(y)+u). \tag{18}$$

Let a process $(x(\cdot), y(\cdot), u(\cdot), T)$ be *admissible* in Problem A, and $y(t) \geq 0$ for all $t \in [0, T]$. The maximum principle for this process consists in the following: there exist functions of bounded variation $p_1(\cdot) : [0, T] \to \mathbb{R}$, $p_2(\cdot) : [0, T] \to \mathbb{R}$, and $\mu(\cdot) : [0, T] \to \mathbb{R}$ such that

$$\mathrm{d}\mu \geq 0, \quad (y(\cdot) - V)\,\mathrm{d}\mu = 0, \tag{19}$$

$$\|p_1\|_\infty + \|p_2\|_\infty + \int_{[0,T]} \mathrm{d}\mu > 0, \tag{20}$$

$$-\,\mathrm{d}p_1 = 0, \quad -\,\mathrm{d}p_2 = (p_1(\cdot) - p_2(\cdot)w'(y(\cdot)))\,\mathrm{d}t - \,\mathrm{d}\mu, \tag{21}$$

$$\max_{u \in [a,b]} p_2(t)u = p_2(t)u(t) \quad \text{a.e. in} \quad [0, T], \tag{22}$$

$$p_1(t)y(t) + p_2(t)(-w(y(t)) + u(t)) = \text{const} =: \alpha_0 \geq 0 \quad \text{a.e. in} \quad [0, T]. \tag{23}$$

4. Analysis of the maximum principle

Let the triple of multipliers $(p_1(\cdot), p_2(\cdot), \mathrm{d}\mu)$ satisfy conditions (19)-(23) of the maximum principle. According to (21), $\mathrm{d}p_1 = 0$, consequently $p_1 = const$. Set

$$\beta = -p_1, \quad p = p_2.$$

Then, the second equation in (21) takes the form

$$\mathrm{d}p = (p(\cdot)w'(y(\cdot)) + \beta)\,\mathrm{d}t + \,\mathrm{d}\mu, \tag{24}$$

and from the maximum condition (22) it follows that

$$\text{if} \quad p(t) < 0, \quad \text{then} \quad u(t) = a, \tag{25}$$
$$\text{if} \quad p(t) = 0, \quad \text{then} \quad u(t) \in [a, b], \tag{26}$$
$$\text{if} \quad p(t) > 0, \quad \text{then} \quad u(t) = b. \tag{27}$$

As was said, the first case corresponds to *deceleration*, the third one is called *acceleration*. the second one is said to be a *singular regime*.

First of all we show that the measure $\mathrm{d}\mu$ has no atoms and hence the functions of bounded variation $\mu(\cdot)$ and $p(\cdot)$ are continuous. This is not surprising, since the state constraint $y - V \leq 0$ has the order 1, i.e., the control appears after the first differentiation of $y - V$. This property of continuity of $\mu(\cdot)$ and $p(\cdot)$ will considerably simplify the handling of the MP in our considerations.

Lemma 4.1 *If* $y(t') = V$, *then*

$$p(t' - 0) = p(t' + 0) = 0 \quad \textit{and} \quad [\mu](t') := \mu(t' + 0) - \mu(t' - 0) = 0.$$

Proof Let us show that $p(t'+0) \le 0$. Assume the contrary: $p(t'+0) > 0$. Then, condition (27) implies: $u(t) = b$ a.e. in some right half-neighborhood $(t', t'+\varepsilon)$ of the point t' ($\varepsilon > 0$) and hence $\dot{y}(t) = -w(y(t)) + b > 0$ a.e. in $(t', t'+\varepsilon)$. It means that $y(t)$ strictly increases in $(t', t'+\varepsilon)$, that contradicts the conditions $y(t') = V$ and $y(t) \le V$ for all $t \in [0, T]$. Similarly, we show that $p(t'-0) \ge 0$.

Consequently, $[p](t') := p(t'+0) - p(t'-0) \le 0$. But from the adjoint equation (24) and the condition $\mathrm{d}\mu \ge 0$ it follows that $[p](t') = [\mu](t') \ge 0$. Consequently, $[p](t') = [\mu](t') = 0$, and then $p(t'+0) = p(t'-0) = 0$. The lemma is proved. □

Set

$$\mathcal{M}_0(p) := \{t \in [0, T] : \ p(t) = 0\}, \quad \mathcal{M}_V(y) = \{t \in [0, T] : \ y(t) = V\}.$$

Corollary 4.1 *The function $p(\cdot)$ is continuous, and the measure $\mathrm{d}\mu$ has no atoms. Moreover,*

$$\mathcal{M}_V(y) \subset \mathcal{M}_0(p). \tag{28}$$

Denote by $\chi_{\mathcal{M}_0(p)}$ the characteristic function of the set $\mathcal{M}_0(p)$. Then, inclusion (28) and the complementarity condition $(y(\cdot) - V)\,\mathrm{d}\mu = 0$ imply

$$\mathrm{d}\mu\chi_{\mathcal{M}_0(p)} = \mathrm{d}\mu. \tag{29}$$

Lemma 4.2 *There are no points $t_1, t_2 \in [0, T]$ such that*

$$t_1 < t_2, \quad p(t_1) = p(t_2) = 0, \quad p(t) > 0 \quad \forall\, t \in (t_1, t_2). \tag{30}$$

Proof Assume the contrary: let $t_1, t_2 \in [0, T]$ satisfy (30). Then, by Corollary 4.1, $y(t) < V$ for all $t \in (t_1, t_2)$ and hence, in view of (19) and (24),

$$\dot{p}(t) = p(t) w'(y(t)) + \beta \quad \text{a.e. in} \quad (t_1, t_2). \tag{31}$$

Obviously, (30) implies

$$\dot{p}(t_1 + 0) \ge 0 \quad \text{and} \quad \dot{p}(t_2 - 0) \le 0. \tag{32}$$

But, by virtue of (30) and (31), $\dot{p}(t_1 + 0) = \dot{p}(t_2 - 0) = \beta$, whence $\beta = 0$. Consequently,

$$\dot{p}(t) = p(t) w'(y(t)) \quad \text{a.e. in} \quad (t_1, t_2). \tag{33}$$

This and the conditions

$$w'(y(t)) \ge 0 \quad \text{and} \quad p(t) > 0 \quad \forall\, t \in (t_1, t_2)$$

imply

$$\dot{p}(t) \ge 0 \quad \forall\, t \in (t_1, t_2).$$

If $\dot{p}(t) = 0$ for all $t \in (t_1, t_2)$, then $w'(y(t)) = 0$ for all $t \in (t_1, t_2)$, and hence $y(t) = 0$ for all $t \in (t_1, t_2)$, which is impossible for the process $(x(\cdot), y(\cdot), u(\cdot), T)$ by Assumption 2.3. Consequently $\dot{p}(t) > 0$ on a subset of a positive measure of the interval (t_1, t_2), and then $\int_{t_0}^{t_1} \dot{p}(t)\,\mathrm{d}t > 0$. The latter contradicts the conditions $p(t_1) = p(t_2) = 0$. □

Quite similarly, the following lemma can be proved.

Lemma 4.3 *There are no points* $t_1, t_2 \in [0, T]$ *such that*

$$t_1 < t_2, \quad p(t_1) = p(t_2) = 0, \quad p(t) < 0 \quad \forall\, t \in (t_1, t_2). \tag{34}$$

Corollary 4.2 *There are no points* $t_1, t_2 \in [0, T]$ *such that*

$$t_1 < t_2, \quad p(t_1) = p(t_2) = 0, \quad p(t) \neq 0 \quad \forall\, t \in (t_1, t_2). \tag{35}$$

Lemma 4.4 *The set* $\mathcal{M}_0(p)$ *is either empty, or a singleton, or a closed interval* $[t_1, t_2] \subset (0, T)$ $(t_1 < t_2)$ *that coincides with the set* $\mathcal{M}_V(y)$.

Proof Suppose that $\mathcal{M}_0(p) \neq \emptyset$ and $\mathcal{M}_0(p)$ is not a singleton. Let $\tau_1, \tau_2 \in \mathcal{M}_0(p)$, $\tau_1 < \tau_2$. Let us show that then the whole segment $[\tau_1, \tau_2]$ contains in $\mathcal{M}_0(p)$. Assume the contrary. Then, there is a point $\tau \in (\tau_1, \tau_2)$ such that $p(\tau) \neq 0$. In this case, there is an interval $(\tau', \tau'') \subset (\tau_1, \tau_2)$, containing τ, such that $p(\tau') = p(\tau'') = 0$, $\tau' < \tau''$, and $p(t) \neq 0$ for all $t \in (\tau', \tau'')$. The latter is impossible in view of Corollary 4.2. Consequently, $\mathcal{M}_0(p)$ is a connected closed set, i.e., a closed interval. Denote it by $[t_1, t_2]$ $(t_1 < t_2)$.

Let us show that $\beta \neq 0$. Assume the contrary: $\beta = 0$. Then the adjoint equation (24) becomes

$$\mathrm{d}p = p(\cdot) w'(y(\cdot))\,\mathrm{d}t + \mathrm{d}\mu.$$

Multiplying this equation by the characteristic function $\chi_{[t_1,t_2]}$ of the interval $[t_1, t_2]$ and taking into account the definition of this interval and relation (29), we get $\mathrm{d}\mu = 0$. Hence the adjoint equation has the form

$$\dot{p}(\cdot) = p(\cdot) w'(y(\cdot)).$$

Since $p(t_1) = 0$, it follows that $p(t) \equiv 0$, and then the tuple $(\beta, p(\cdot), \mathrm{d}\mu)$ is trivial. Consequently, $\beta \neq 0$.

Upon multiplying equation (24) by $\chi_{[t_1,t_2]}$, we get

$$\chi_{[t_1,t_2]} \beta\,\mathrm{d}t + \mathrm{d}\mu = 0.$$

In turn, by multiplying this equation by $(y(\cdot) - V)$ and taking into account the complementarity condition $(y(\cdot) - V)\,\mathrm{d}\mu = 0$, we obtain

$$(y(\cdot) - V) \chi_{[t_1,t_2]} \beta\,\mathrm{d}t = 0,$$

whence $(y(\cdot) - V)\chi_{[t_1,t_2]} = 0$ (since $\beta \neq 0$). It means that $y(t) = V$ for all $t \in [t_1, t_2]$. Consequently, $\mathcal{M}_0(p) \subset \mathcal{M}_V(y)$. According to (28), the inverse inclusion also holds. Consequently, $\mathcal{M}_0(p) = \mathcal{M}_V(y)$.

Finally, note that $[t_1, t_2] \subset (0, T)$, since $y_0 < V$ and $y_1 < V$. The lemma is proved. □

LEMMA 4.5 *Let $t_1 \in (0, T)$ be such that $p(t) < 0$ for all $t \in (0, t_1)$, $p(t_1) = 0$ and $p(t) > 0$ for all $t \in (t_1, T)$. Then,* $\mathrm{d}\mu = 0$*,* $\beta > 0$*,* $y(t_1) = 0$*, and* $\mathcal{M}_V(y)$ *is an empty set.*

PROOF Let $t_1 \in (0, T)$ be such a point. According to (28), the set $\mathcal{M}_V(y)$ is empty or a singleton $\{t_1\}$. Since the measure $\mathrm{d}\mu$ is concentrated on $\mathcal{M}_V(y)$ and has no atoms, we get: $\mathrm{d}\mu = 0$, and therefore the adjoint equation (24) has the form

$$\dot{p} = p(\cdot)w'(y(\cdot)) + \beta. \tag{36}$$

If $\beta = 0$, then $p(\cdot) = 0$, and hence the tuple $(p, \beta, \mathrm{d}\mu)$ is trivial. Therefore $\beta \neq 0$. Moreover, from equation (36) it follows that the function $p(\cdot)$ is continuously differentiable, and $\dot{p}(t_1) = \beta$, whereas from the conditions of the lemma it follows that $\dot{p}(t_1) \geq 0$, and hence $\beta > 0$.

Further, condition (23) implies

$$-\beta y(t_1) = \alpha_0 \geq 0. \tag{37}$$

Since $y(t_1) \geq 0$ and $\beta > 0$, it follows that $y(t_1) = 0$. Consequently, $\mathcal{M}_V(y)$ is an empty set. The lemma is proved. □

LEMMA 4.6 *There are no points $t_1, t_2 \in (0, T)$, $t_1 < t_2$, such that $p(t) < 0$ for all $t \in (0, t_1)$, $p(t) = 0$ for all $t \in [t_1, t_2]$, and $p(t) > 0$ for all $t \in (t_1, T)$.*

PROOF Let $t_1, t_2 \in (0, T)$ be such points. Then, according to (25), $\dot{y}(t) = -w(y(t)) + a < 0$ a.e. on $(0, t_1)$, that is $y(t)$ decreases on $(0, t_1)$, and by Lemma 4.4, $y(t) = V$ for all $t \in [t_1, t_2]$. The latter is impossible since $y(t) \leq V$ on $[0, T]$. The lemma is proved. □

Since $p(\cdot)$ is a continuous function, it follows from Lemmas 4.4-4.6 that there are only five possible cases.

A) $p(t) > 0$ for all $t \in (0, T)$. In this case

$$u(t) = b \quad \text{a.e. in} \quad [0, T].$$

This corresponds to *acceleration* on the whole segment $[0, T]$.

B) $p(t) < 0$ for all $t \in (0, T)$. In this case

$$u(t) = a \quad \text{a.e. in} \quad [0, T].$$

This corresponds to *deceleration* on the whole segment $[0, T]$.

C) There is a point $t_1 \in (0,T)$ such that $p(t) > 0$ for all $t \in (0,t_1)$ and $p(t) < 0$ for all $t \in (t_1,T)$. In this case

$$u(t) = b \quad \text{a.e. in} \quad (0,t_1) \quad \text{and} \quad u(t) = a \quad \text{a.e. in} \quad (t_1,T).$$

This corresponds to "acceleration – deceleration" mode with one switching at the point $t_1 \in (0,T)$.

D) There are two points $t_1, t_2 \in (0,T)$, $t_1 < t_2$, such that $p(t) > 0$ for all $t \in (0,t_1)$, $p(t) = 0$ for all $t \in [t_1,t_2]$, and $p(t) < 0$ for all $t \in (t_1,T)$. In this case

$$u(t) = b \quad \text{a.e. in} \quad (0,t_1)$$

$$u(t) = a \quad \text{a.e. in} \quad (t_2,T),$$

$$u(t) = w(V) \quad \text{a.e. in} \quad (t_1,t_2), \quad y(t) = V \quad \text{for all} \quad t \in [t_1,t_2].$$

This corresponds to "acceleration – singular – deceleration" mode with two switchings at the points $t_1, t_2 \in (0,T)$. Moreover, the domain $[t_1,t_2]$ of the *singular arc* $p(t) = 0$ coincides with the domain of the *boundary arc* $y(t) = V$.

E) There is a point $t_1 \in (0,T)$ such that $p(t) < 0$ for all $t \in (0,t_1)$, $p(t_1) = 0$, and $p(t) > 0$ for all $t \in (t_1,T)$. In this case

$$u(t) = a \quad \text{a.e. in} \quad (0,t_1) \quad \text{and} \quad u(t) = b \quad \text{a.e. in} \quad (t_1,T).$$

Moreover, $y(t_1) = 0$.
This corresponds to "deceleration – acceleration" mode with one switching at the point $t_1 \in (0,T)$.

The decomposition into cases A)-E) is obviously complete. In the next section we show, using the second-order conditions, that in the case E) the quadruple $(x(\cdot), y(\cdot), u(\cdot), T)$ is not a strong local minimum in the problem.

5. Application of second-order necessary conditions in the case E

5.1. General statement of second-order necessary conditions in the minimum time problem for a system linear in the control

Here we formulate the result, presented in the book by Milyutin and Osmolovskii (1998), Part 2, section 12.4. (A complete proof of this result can be found in the book by Osmolovskii and Maurer, 2012.) Consider the following time-optimal control problem

$$T \to \min, \tag{38}$$

under the constraints:

$$\dot{x}(t) = a(x(t)) + B(x(t))u(t), \quad \text{a.e. in} \quad [0,T], \tag{39}$$

$$x(0) = x_0, \quad x(T) = x_T, \tag{40}$$

$$u(t) \in U, \quad \text{a.e. in} \quad [0, T], \tag{41}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $x_0, x_T \in \mathbb{R}^n$ are given vectors, the mapping $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2, $B(\cdot) = [b_{ij}(\cdot)]$ is an $n \times m$ matrix with coefficients $b_{ij}(\cdot)$ of class C^2, U is a convex polyhedron. So, here the right hand side of the control system has the form

$$f(x, u) = a(x) + B(x)u.$$

Set

$$H(x, u, p) = p(a(x) + B(x)u),$$

where p is an n-dimensional row-vector.

Let the triple $(x(\cdot), u(\cdot), T)$ be a strong local minimum in this problem. Denote by M_0 the set of all absolutely continuous functions $p(\cdot) : [0, T] \to \mathbb{R}^n$ satisfying the conditions of the maximum principle:

$$|p(0)| = 1, \tag{42}$$

$$-\dot{p}(t) = H_x((x(t), u(t), p(t)) \quad \text{for a.a. } t \in [0, T], \tag{43}$$

$$\max_{u \in U} H((x(t), u, p(t)) = H((x(t), u(t), p(t)) \quad \text{for a.a. } t \in [0, T], \tag{44}$$

$$H((x(\cdot), u(\cdot), p(\cdot)) = \text{const} =: \alpha_0 \geq 0 \quad \text{for a.a. } t \in [0, T]. \tag{45}$$

Obviously, these conditions follow from conditions (13)-(17) if in the latter we put $\mathrm{d}\mu = 0$.

Assume that the function $u(\cdot)$ is *piecewise constant,* taking values at the vertex set $\operatorname{ex} U$ of the polyhedron U (in this case we say that $u(\cdot)$ is a *bang-bang control*). Further, assume that $u(\cdot)$ has *only one switching at the point* $t_1 \in (0, T)$. Let u^- and u^+ be the values of $u(\cdot)$ on the intervals $(0, t_1)$ and (t_1, T), respectively. Then $u^-, u^+ \in \operatorname{ex} U$. Denote by

$$[u(\cdot)](t_1) = u(t_1+) - u(t_1-) = u^+ - u^-$$

the jump of $u(\cdot)$ at the switching point t_1. Let $p(\cdot) \in M_0$. In what follows, instead of $[u(\cdot)](t_1)$ we write simply $[u(\cdot)]$, for brevity.

Define the function

$$\Delta H(t) = p(t)B(x(t))[u(\cdot)],$$

and set

$$D^-(H) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta H(t_1 - 0), \quad D^+(H) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta H(t_1 + 0).$$

The following theorem is given in Milyutin and Osmolovskii (1998).

THEOREM 5.1 *For any $p(\cdot) \in M_0$, we have $D^-(H) = D^+(H) \geq 0$.*

For any $p(\cdot) \in M_0$, we denote by $D(H)$ the common value of $D^-(H)$ and $D^+(H)$.

For the triple $(x(\cdot), u(\cdot), T)$, we introduce the *critical cone* $\mathcal{K}$ as the set of all triples $\bar{z} = (\bar{T}, \bar{\xi}, \bar{x}(\cdot))$ such that the following conditions are satisfied

$$\bar{T} \in \mathbb{R}, \quad \bar{\xi} \in \mathbb{R}, \quad \bar{x}(\cdot) \in W^{1,2}([t_1, T], \mathbb{R}^n), \tag{46}$$

$$\bar{x}(t_1) = [\dot{x}]\bar{\xi}, \quad \bar{x}(T) + \dot{x}(T)\bar{T} = 0, \tag{47}$$

$$\dot{\bar{x}}(t) = f_x(x(t), u(t))\bar{x}(t), \tag{48}$$

where $[\dot{x}] := \dot{x}(t_1+) - \dot{x}(t_1-)$ is the jump of the function $\dot{x}(t)$ at the point t_1, and $W^{1,2}([t_1, T], \mathbb{R}^n)$ stands for the Sobolev space of absolutely continuous functions $x : [t_1, T] \to \mathbb{R}^n$ with square integrable derivative.

For $p(\cdot) \in M_0$, let us introduce the quadratic form of $\bar{z}$:

$$\begin{aligned}\Omega(\bar{z}, p) = {} & D(H)\bar{\xi}^2 + [\dot{p}]\bar{x}(t_1)\bar{\xi} - \dot{p}(T)\dot{x}(T)\bar{T}^2 \\ & - \int_{t_1}^{T} \langle H_{xx}((x(t), u(t), p(t))\bar{x}(t), \bar{x}(t)\rangle \, \mathrm{d}t.\end{aligned} \tag{49}$$

Note that for $p(\cdot) \in M_0$ and $\bar{z} = (\bar{T}, \bar{\xi}, \bar{x}(\cdot)) \in \mathcal{K}$ we obviously have

$$D(H)\bar{\xi}^2 + [\dot{p}]\bar{x}(t_1)\bar{\xi} = (D(H) + [\dot{p}][\dot{x}])\bar{\xi}^2.$$

The following theorem was obtained by Osmolovskii, see Milyutin and Osmolovskii (1998).

Theorem 5.2 *If $(x(\cdot), u(\cdot), T)$ is a strong local minimum and $u(\cdot)$ satisfies the above assumptions, then the set M_0 is nonempty, and for any $\bar{z} \in \mathcal{K}$ there exists $p \in M_0$ such that $\Omega(\bar{z}, p) \geq 0$.*

Denote by $\mathcal{K}_1$ the cross section of the cone $\mathcal{K}$ with the hyperplane $\bar{T} = -1$. Obviously, $\mathcal{K}_1$ is defined by the relations

$$\bar{x}(t_1) = [\dot{x}]\bar{\xi}, \quad \bar{x}(T) = \dot{x}(T), \quad \dot{\bar{x}}(t) = f_x(x(t), u(t))\bar{x}(t),$$

while $\Omega(\bar{z}, p)$ on $\mathcal{K}_1$ becomes

$$\begin{aligned}\Omega(\bar{z}, p) &= (D(H) + [\dot{p}][\dot{x}])\bar{\xi}^2 - \dot{p}(T)\dot{x}(T) \\ &- \int_{t_1}^{T} \langle H_{xx}((x(t), u(t), p(t))\bar{x}(t), \bar{x}(t)\rangle \, \mathrm{d}t.\end{aligned}$$

Note that the function $\dot{x}(\cdot)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}(t)) = f_x(x(t), u(t))\dot{x}(t),$$

since the function $u(\cdot)$ is piecewise constant. Further, assume that there is such a number $\bar{\xi}$ that

$$\dot{x}(t_1+) = [\dot{x}]\bar{\xi}. \tag{50}$$

It means that the vectors $\dot{x}(t_1+)$ and $[\dot{x}]$ are collinear (this is true if and only if the vectors $\dot{x}(t_1-)$ and $\dot{x}(t_1+)$ are collinear). Then, obviously, the pair $(\dot{x}(\cdot), \bar{\xi})$ belongs to the set $\mathcal{K}_1$. Thus, the following theorem holds.

THEOREM 5.3 *Suppose that $(T, x(\cdot), u(\cdot))$ is a strong local minimum, $u(\cdot)$ satisfies the above assumptions, M_0 is a singleton, and the vectors $\dot{x}(t_1-)$ and $\dot{x}(t_1+)$ are collinear. Let $\bar{\xi}$ be such that $\dot{x}(t_1+) = [\dot{x}]\bar{\xi}$. Then*

$$\Omega_1 := (D(H) + [\dot{p}][\dot{x}])\bar{\xi}^2 - \dot{p}(T)\dot{x}(T) - \int_{t_1}^{T} \langle H_{xx}((x(t), u(t), p(t))\dot{x}(t), \dot{x}(t)\rangle \,\mathrm{d}t \geq 0. \tag{51}$$

5.2. Second-order necessary conditions in Problem A

Let us apply the second-order necessary conditions, given above, to the extremal $(x(\cdot), y(\cdot), u(\cdot), T)$, described in the case E of the previous section. By Lemma 4.5, in this case $\mathcal{M}_V(y) = \emptyset$, and hence there is an $\varepsilon > 0$ such that $y(t) < V - \varepsilon$ for all $t \in [0, T]$. Hence, the extremal $(x(\cdot), y(\cdot), u(\cdot), T)$ in the problem with the state constraint $y \leq V$ is an extremal in the problem without state constraint, and this extremal has the same multipliers $(\beta, p(\cdot), \mathrm{d}\mu)$, where $\mathrm{d}\mu = 0$. By Lemma 4.5, we have $\beta > 0$, and we can take $\beta = 1$ as the normalization condition. Then the adjoint equation (24) becomes:

$$\dot{p} = p(\cdot)w'(y(\cdot)) + 1. \tag{52}$$

This equation, together with condition $p(t_1) = 0$, determines the function $p(\cdot)$ uniquely. Hence, $M_0 = \{p\}$ is a singleton. Since $p(t_1) = 0$, equation (52) implies $\dot{p}(t_1) = 1$. Further,

$$\Delta H(t) = p(t)[u(\cdot)] = p(t)(b - a)$$

and hence

$$D(H) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta H(t_1) = (b - a) > 0.$$

Recall that $\dot{x}(t) = y(t), \quad \dot{y}(t) = -w(y(t)) + u(t)$, and therefore

$$\dot{x}(t_1) = y(t_1) = 0, \quad \dot{y}(t_1-) = u(t_1-) = a, \quad \dot{y}(t_1+) = u(t_1+) = b.$$

All conditions of Theorem 5.3 are fulfilled. Condition (50) is equivalent to $\dot{y}(t_1+) = [u]\bar{\xi}$, that is: $b = (b - a)\bar{\xi}$, whence $\bar{\xi} = b/(b - a)$.

Further. we have

$$D(H)\bar{\xi}^2 = \frac{b^2}{b - a}, \quad H_{xx} = H_{xy} = 0, \quad H_{yy} = -pw''(y).$$

Therefore,

$$\Omega_1 = \frac{b^2}{b-a} - \dot p(T)\dot y(T) + \int_{t_1}^{T} p(t)w''(y(t))\dot y^2(t)\,\mathrm{d}t.$$

Let us show that if T is close enough to t_1, then $\Omega < 0$. Indeed, equation (52) and the relations

$$\dot y(t) = -w(y(t)) + u(t), \quad y(t_1) = 0, \quad p(t_1) = 0$$

imply

$$\dot p(T) \to \dot p(t_1) = 1, \quad \dot y(T) \to \dot y(t_1+) = b \quad \text{as} \quad T \to t_1 + .$$

Moreover,

$$\int_{t_1}^{T} p(t)w''(y(t))\dot y^2(t)\,\mathrm{d}t \to 0 \quad \text{as} \quad T \to t_1 + .$$

Since

$$\frac{b^2}{b-a} - b = \frac{ab}{b-a} < 0,$$

we get

$$\Omega < 0 \quad \text{if} \quad T - t_1 > 0 \quad \text{is small enough.}$$

This means that, for all $T > t_1$, close enough to t_1, the extremal $(x(\cdot), y(\cdot), u(\cdot), T)$ is not a strong local minimum. Then, this is true for every $T > t_1$, since each part of an optimal solution is an optimal solution.

Thus, we have shown that each solution of the problem A is an extremal of the type A-D, but not E. Now, the final determination of the optimal solution becomes much easier, because it is only required to find the values of at most two parameters: t_1 and t_2. This can be done numerically.

6. Numerical example

We choose the following data:

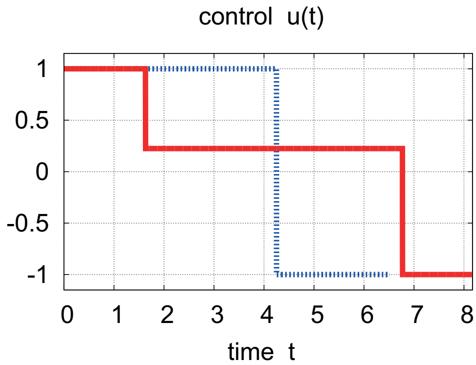
$$w(y) = 0.1\,y^2, \quad x(0) = y(0) = 0,\ x(T) = 10,\ y(T) = 0, \quad a = -1,\ b = 1.$$

Omitting the state constraint $y(t) \le V$ we see that the optimal control is bang-bang, switching from $u(t) = 1$ to $u(t) = -1$ at $t_1 = 4.2507$. The minimal terminal time is $T = 6.51$. We find $\max y(t) = 2.7597$.

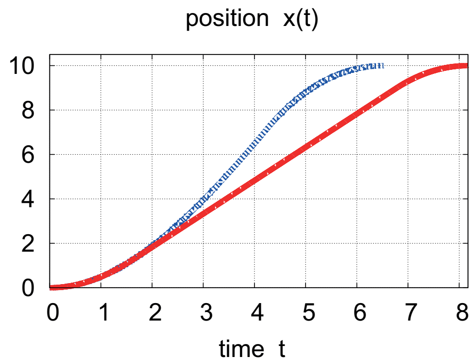
Hence, let us choose the state constraint $y(t) \le V = 1.5$. We get the numerical results

$$T = 8.171755,\ p_1 = 2/3,\ p_2(0) = 1,$$

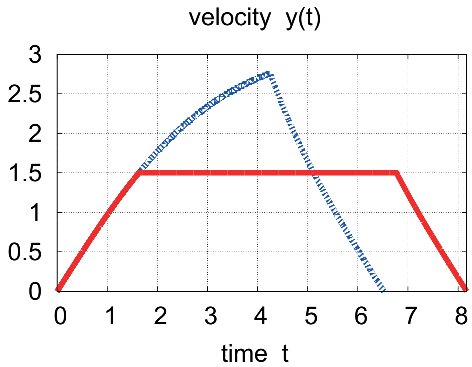
and a boundary arc in $[1.631, 6.771]$ with $\eta(t) = p_1 = 2/3$. A comparison of the unconstrained solution (lighter) and the constrained solution (darker) is shown in Figure 1.



(a) u-comparison



(b) x-comparison



(c) y-comparison

Figure 1: Unconstrained solution (lighter) and state constrained solution with $y(t) \leq V = 1.5$ (darker) for the numerical example

7. Some general theoretical results related to Problem A

Some of the results of this paper can be considered as special cases of the results by Maurer (1977, 1979), see also Hartl (1995). Let us briefly discuss these issues, which also result in an explicit formula for the density of the multiplier μ.

The state constraint $\varphi(y) = y - V \le 0$ has order one, since the the total time derivative

$$\varphi^{(1)}(y(t), u(t)) := \frac{\mathrm{d}}{\mathrm{d}t}\varphi(y(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(y(t) - V) = u(t) - w(y(t))$$

satisfies the regularity condition

$$\frac{\mathrm{d}}{\mathrm{d}u}\varphi^{(1)}(y(t), u(t)) = 1 \ne 0.$$

A boundary arc $y(t) = V$ for $t_1 \le t \le t_2$ with $t_1 < t_2$ then gives the boundary control $u(t) = w(V)$ in view of $\dot{y} = 0$. The boundary control should satisfy the control constraint. Hence, it is reasonable to even require $a < w(V) < b$. Then, it follows from the results in Maurer (1979) on the regularity of the multiplier μ, associated with a state constraint (a measure), that μ has a density η on boundary arcs, i.e., $\mathrm{d}\mu(t) = \eta(t)\,\mathrm{d}t$ with $\eta \ge 0$. This is shown by Maurer (1979) in a much more general context than for the control problem above. We can define the *augmented Hamiltonian* $\bar{H}$ by directly adjoining the state constraint to the standard Hamiltonian,

$$\bar{H}(x, y, p_1, p_2, \eta, u) = p_1 y + p_2(-w(y) + u) - \eta(y - V).$$

(The meaning of the function $\eta(t)$ becomes only clear through the discussion below, where the state constraint is directly adjoined to the standard Hamiltonian by η; as it was said, the existence of $\eta(t)$ follows from the results of Maurer, 1979.) Upon defining the *switching function* $\Phi(t) = p_2(t)$, the control that maximizes the Hamiltonian is given by

$$u(t) = \begin{cases} a & \text{if} \quad \Phi(t) < 0, \\ b & \text{if} \quad \Phi(t) > 0, \end{cases}$$

If the switching function vanishes on an interval we obtain a singular arc.

Now suppose that we have a boundary arc with $y(t) = V$, for $t_1 \le t \le t_2$, for which the reasonable assumption $a < w(V) < b$ holds for the boundary control. Then, the maximum principle implies $\Phi(t) = p_2(t) = 0$ for $t_1 \le t \le t_2$. Hence, the boundary control formally behaves like a singular control. This fact has been extensively used in Maurer (1977) to derive junction theorems for joining interior arcs with boundary arcs. Here, in particular, one obtains the continuity of the adjoint variables, since the control is discontinuous at the entry and exit points of the boundary arc. Since $p_2(t) = 0$, we obtain the multiplier η in the Hamiltonian simply from $0 = \dot{p}_2(t) = -\bar{H}_y = -p_1 + \eta$, which gives

$$\eta(t) = p_1 \quad \text{(constant)}.$$

The "importance" of computing $\eta(t)$ lies in the fact that η_i appears as multiplier of the pointwise state constraint $\varphi(y_i) \le 0$ *in the discretized control problem.* Then, the directly computed discretized values η_i must coincide with p_1.

8. Another approach

In this manner, the problem has been investigated to the end. Although the study required some effort, the result was quite predictable. In the main case (when the distance between the points x_0 and x_T is big enough) the optimal mode is as follows: first acceleration with control $u(t) = b$, then the movement with maximal possible speed $y(t) = V$ and the control $u(t) = w(V)$, and then the deceleration with control $u(t) = a$. If the distance $x_T - x_0$ is not big enough, then one or two of these regimes may be absent.

Let us show that this result can be obtained practically without any theory. Consider again the control system

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = -w(y(t)) + u(t), \quad u(t) \in [a,b], \quad t \in [0,T]. \tag{53}$$

Assuming that $\dot{x} = y > 0$, we can take x as a new independent variable, and consider y and u as the functions depending on x: $y = y(x) \le V$, $u = u(x) \in [a,b]$, $x \in [x_0, x_T]$. Then, we get the following problem with the independent variable x:

$$T(y) = \int_{x_0}^{x_T} \frac{\mathrm{d}x}{y(x)} \to \min, \tag{54}$$

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{-w(y(x)) + u(x)}{y(x)}, \quad y(x_0) = y_0, \quad y(x_T) = y_T \tag{55}$$

$$u(x) \in [a,b], \quad y(x) \le V. \tag{56}$$

Consider two curves, defined as the solutions to the Cauchy problems:

$$a) \quad \frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{-w(y(x)) + b}{y(x)}, \quad x \ge x_0, \quad y(x_0) = y_0,$$

and

$$b) \quad \frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{-w(y(x)) + a}{y(x)}, \quad x \le x_T, \quad y(x_T) = y_T,$$

respectively.

The right hand side of the equation a) is positive and hence the function $y_a(x)$, corresponding to the curve a), is increasing. This curve corresponds to the acceleration with the control $u(x) = b > 0$.

The right hand side of the equation b) is negative, and hence the function $y_b(x)$, corresponding to the curve b), is decreasing. This curve corresponds to the deceleration with the control $u(x) = a < 0$.

Set

$$\mathcal{F} = \{(x,y) \in \mathbb{R}^2 : \ x \in [x_0, x_1], \quad y \in [0,V], \quad y \le y_a(x), \quad y \le y_b(x)\}.$$

Clearly, the graph of any admissible trajectory $y(x)$, $x \in [x_0, x_1]$ belongs to $\mathcal{F}$. Let $\hat{y}$ be the trajectory, which corresponds to the upper bound of the figure $\mathcal{F}$.

Obviously, the functional (54) attains the minimal value (among all admissible curves belonging to $\mathcal{F}$) on the curve $\hat{y}$. This curve corresponds to the optimal solution, which was found above by means of the maximum principle and the second-order necessary condition.

This idea was told to the first author by Andrei V. Dmitruk, who used it in his work with A.K. Vdovina, see Dmitruk and Vdovina (2016). Its implementation turned out to be possible due to the very simple, actually one-dimensional, structure of the problem, and owing to the specific kind of the cost functional (54) which depends only on y and whose integrand decreases on y: if $y(x) \leq \hat{y}(x)$ for all x, then $T(y) \leq T(\hat{y})$ by the comparison theorem for integrals.

In other, more complicated problems, the solution may be not so obvious, and can hardly be found without application of first- and second-order optimality conditions. For example, in a similar problem, but with energy cost functional, the study, carried out using the maximum principle, produced less obvious results, see Asnis, Dmitruk, Osmolovskii (1985).

9. Conclusion

We discussed the time optimal control problem for a nonlinear control system with state constraint, which is a generalization of the classical problem of Pontryagin et al. (1964) of minimization of time T for a material point controlled by a limited force and moving along a straight line without friction. In our generalization, the state constraint has the order one, and therefore the corresponding Lagrange multiplier (the measure $\mathrm{d}\mu$) has no atoms, whereupon the adjoint variable $p(\cdot)$ has no jumps.

Taking into account the existence of the solution, we found extremals of this problem, satisfying the Dubovitskii – Milyutin maximum principle for problems with state constraints. In the case, where the state constraint is not active, we used second-order conditions to eliminate the corresponding extremal.

The maximum principle and the second order necessary conditions allowed for showing that there are only four possible cases regarding optimal control, which is a bang-bang or bang-singular-bang control with at most two switchings. In the last case, the singular arc coincides with the boundary arc, and the switchings of the control coincide with the entry and exit points of the boundary arc. This is the main case, and it was illustrated by a numerical example.

We shortly discussed some theoretical results generalizing the results obtained for our problem. On the other hand, we showed that due to the simple one-dimensional structure of the problem, its solution can be obtained by using rather primitive considerations, without involving the theory of the necessary optimality conditions. This does not diminish the significance of the previous study, where we illustrated an application of optimality conditions in the problem with state constraint. On the contrary, in our opinion, the number of such illustrative examples of state constrained problems, admitting an analytical solution, should be increased.

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