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Free flexural vibrations of an expanded-tapered sandwich beam

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Abstract The paper is devoted to the analytical modelling of a simply supported expanded-tapered sandwich beam. The simplified analytical model of this beam with omitting the shear effect is elaborated. Based on Hamilton's principle, the differential equation of motion of this beam is obtained. This equation is analytically solved with consideration of the deflection line of this beam subjected to its own weight. The fundamental natural frequencies for exemplary beams are derived. Moreover, the FEM model of the beam in the ABAQUS is developed. The calculation results of the fundamental natural frequency of exemplary beams of these two methods are presented in tables and figures.

Keywords: analytical modelling, expanded-tapered sandwich beam, fundamental natural frequency.

1. Introduction

The vibration problems of beams, plates and shells, as main parts of the structures, e.g. rail vehicles, are the subject of a contemporary research.

Rosa and Lippiello [1] studied natural frequencies of tapered beams. Authors used cell discretization method to obtain results. These results were compared with numerical studies. Bayat et al. [2] considered nonlinear free vibrations of tapered beams. Authors used two different approaches to obtain natural frequencies of beams: Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) . Sayyad and Ghugal [3] presented a critical review of literature (515 references) in the last few decades on bending, buckling and free vibration problems of laminated composite and sandwich beams. Meksi et al. [4] developed shear deformation theory of functionally graded sandwich plates. Theory proposed by authors focused on four unknown functions to solve. Authors compared obtained results with numerical studies. Magnucki et al. [5] studied analytically and numerically a three-point bending of an expanded-tapered sandwich beam. Authors developed the analytical model of this beam with consideration of the broken-line theory. Magnucki et al. [6] studied analytically and numerically the free flexural vibrations of homogeneous beams of symmetrically variable depth and bisymmetrical cross sections. Magnucki et al. [7] presented the bending, buckling and free vibration problems of simply-supported sandwich beam. Authors developed three beam models and performed detailed analytical and numerical FEM calculations. Kelly [8] in the monograph presented the basis of vibrations in engineering and showed the mathematical way to model and solved them. The monograph consists of 13 chapters entirely devoted to vibrations.

The subject of the paper is a simply supported expanded-tapered sandwich beam of length *L*. This beam is situated in the Cartesian coordinate system *xyz* (see Fig. 1).

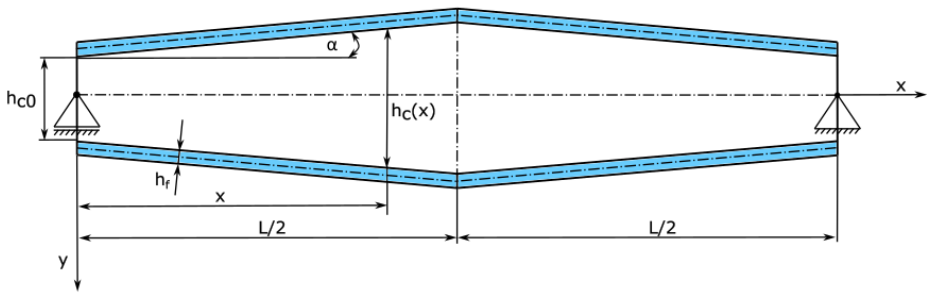

Figure 1. Scheme of the expanded-tapered sandwich beam.

2. Analytical model of the beam – fundamental natural frequency

The plane cross section of the expanded-tapered sandwich beam is shown in Fig. 2.

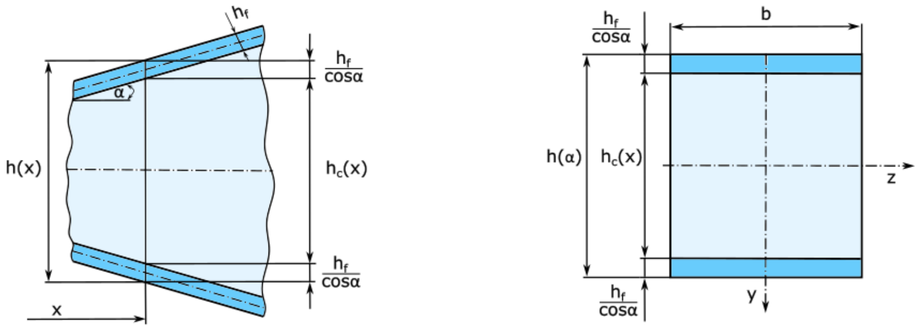

Figure 2. Scheme of the plane cross section of the beam.

The individual dimensions of the beam as shown in Fig.2. are as follows: h_f – thicknesses of the faces, $h_c(x) = h_{c0} + 2x\tan\alpha$ – thickness of the core, $h(x) = h_c(x) + 2\,h_f/\cos\alpha$ – total depth of the beam, for $0 \le x \le L/2$, h_{c0} – initial thickness of the core (Fig. 1) and b – width of the beam.

The simplified analytical model of this beam with consideration of the Bernoulli-Euler beam theory is elaborated. Therefore, a plane cross section of the beam before bending is plane after bending (see Fig. 3).

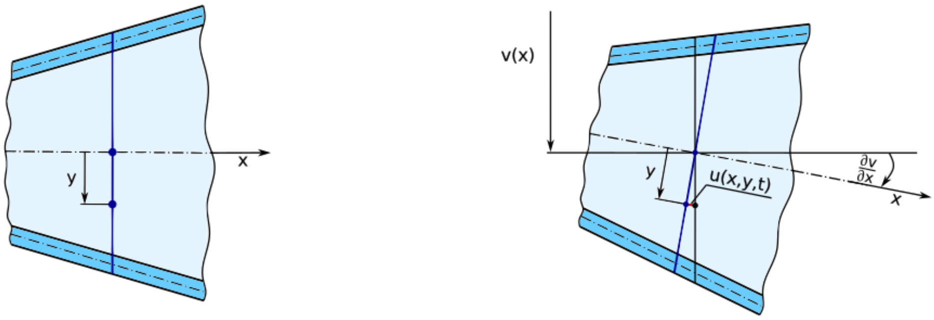

Figure 3. Scheme of the plane cross section of the beam before and after bending.

The longitudinal displacement in accordance with Fig. 3. is as follows

$$u(x,y,t) = -y\frac{\partial v}{\partial x}\ , \tag{1}$$

where $v(x, t)$ – deflection of the beam, t – time.

Consequently, the strains

$$\varepsilon_x(x,y,t) = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2}\ , \qquad \gamma(x,y,t) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{2}$$

and stresses (Hooke's law) for successive layers

$$\sigma_x^{(uf)}(x,y,t) = -E_f y\frac{\partial^2 v}{\partial x^2}, \quad \sigma_x^{(c)}(x,y,t) = -E_c y\frac{\partial^2 v}{\partial x^2}, \quad \sigma_x^{(lf)}(x,y,t) = -E_f y\frac{\partial^2 v}{\partial x^2}, \tag{3}$$

where: E_f – Young's modulus of upper (uf) and lower (lf) faces, E_c – Young's module of the core (c).

The elastic strain energy with consideration of the expressions (3) is as follows

$$U_\varepsilon = b\int_0^{L/2}\left\{E_f\int_{-h(x)/2}^{-h_c(x)/2} y^2dy + E_c\int_{-h_c(x)/2}^{h_c(x)/2} y^2dy + E_f\int_{h_c(x)/2}^{h(x)/2} y^2dy\right\}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx\,. \tag{4}$$

Integrating this expression (4), and after simple transformation, one obtains

$$U_\varepsilon = \frac{E_f b h_{c0}^3}{12L^3} \int_0^{1/2} \left\{ \left[3f_\alpha^2(\xi) + 3f_\alpha(\xi) k_f + k_f^2 \right] k_f + e_c f_\alpha^3(\xi) \right\} \left(\frac{\partial^2 v}{\partial \xi^2} \right)^2 d\xi \,, \tag{5}$$

where: the function

$$f_\alpha(\xi) = 1 + 2\lambda\xi \tan\alpha, \tag{6}$$

and $\chi_f = h_f/h_{c0}$ – relative thickness of the faces, $k_f = 2\chi_f/\cos\alpha$ – coefficient, $\lambda = L/h_{c0}$ – relative length of the beam, $e_c = E_c/E_f$ – relative elasticity module of the core, $\xi = x/L$ – dimensionless coordinate.
The mass intensity of the beam

$$m(x) = \left[\rho_c h_c(x) + 2\rho_f \frac{h_f}{\cos\alpha} \right] b, \tag{7}$$

where ρ_c, ρ_f – mass densities of the core and faces.
However, the mass intensity of the beam with consideration of the dimensionless sizes and dimensionless coordinate is in the form

$$m(\xi) = \left[\sqrt{e_c} f_\alpha(\xi) + k_f \right] \rho_f b h_{c0}, \tag{8}$$

where $\rho c/\rho f = \sqrt{e_c}$ – relative mass density of the core.

The kinetic energy with consideration of the expression (8) is as follows

$$U_k = \rho_f b h_{c0} L \int_0^{1/2} \left[\sqrt{e_c} f_\alpha(\xi) + k_f \right] \left(\frac{\partial v}{\partial t} \right)^2 d\xi. \tag{9}$$

Based on the Hamilton's principle $\delta \int_{t_1}^{t_2} (U_k - U_\varepsilon) dt$, the differential equation of motion of the expanded-tapered sandwich beam is obtained in the following form

$$\left[\sqrt{e_c} f_\alpha(\xi) + k_f \right] \frac{\partial^2 v}{\partial t^2} + \left[f_{v4}(\xi) \frac{\partial^4 v}{\partial \xi^4} + f_{v3}(\xi) \frac{\partial^3 v}{\partial \xi^3} + f_{v2}(\xi) \frac{\partial^2 v}{\partial \xi^2} \right] \frac{E_f}{12\rho_f \lambda^2 L^2} = 0, \tag{10}$$

where occurring functions are as follows:

$$f_{v4}(\xi) = e_c f_\alpha^3(\xi) + 3f_\alpha^2(\xi) k_f + 3f_\alpha(\xi) k_f^2 + k_f^3 \,, \tag{11}$$

$$f_{v3}(\xi) = 12\lambda \left[e_c f_\alpha^2(\xi) + 2f_\alpha(\xi) k_f + k_f^2 \right] \tan\alpha \,, \qquad f_{v2}(\xi) = 24\lambda^2 \left[e_c f_\alpha(\xi) + k_f \right] \tan^2\alpha. \tag{12}$$

The form of the unknown deflection function of the beam is as follows

$$v(\xi, t) = \bar{v}(\xi) v_a(t) \,, \tag{13}$$

where: $\bar{v}(\xi)$ – the dimensionless deflection curve, $v_a(t)$ – the function of the time.

The dimensionless deflection curve of the beam is developed taking into account the paper [6]. The bending moment of this sandwich beam, according to the Bernoulli-Euler beam theory, is formulated as follows

$$M_b(x) = b \left\{ \int_{-h(x)/2}^{-h_c(x)/2} y \sigma_x^{(uf)} dy + E_c \int_{-h_c(x)/2}^{h_c(x)/2} y \sigma_x^{(c)} dy + E_f \int_{h_c(x)/2}^{h(x)/2} y \sigma_x^{(lf)} dy \right\}. \tag{14}$$

Integrating this expression (14), with consideration of the expressions (3), and after transformation, the differential equation of the beam dimensionless deflection curve was obtained in the form

$$\frac{d^2 \bar{v}}{d\xi^2} = -\frac{\bar{M}_b(\xi)}{f_{v4}(\xi)}. \tag{15}$$

Thus

$$\bar{v}(\xi) = C_1\xi - \iint \frac{\bar{M}_b(\xi)}{f_{v4}(\xi)}\,d\xi^2, \tag{16}$$

where the integration constant $C_1 = \int_0^{1/2} \frac{\bar{M}_b(\xi)}{f_{v4}(\xi)} d\xi$. The function (16) satisfies following conditions: $\bar{v}(0) = 0$ and $d\bar{v}/d\xi]_{1/2} = 0$. The beam subjected its own weight is shown in Fig. 4.

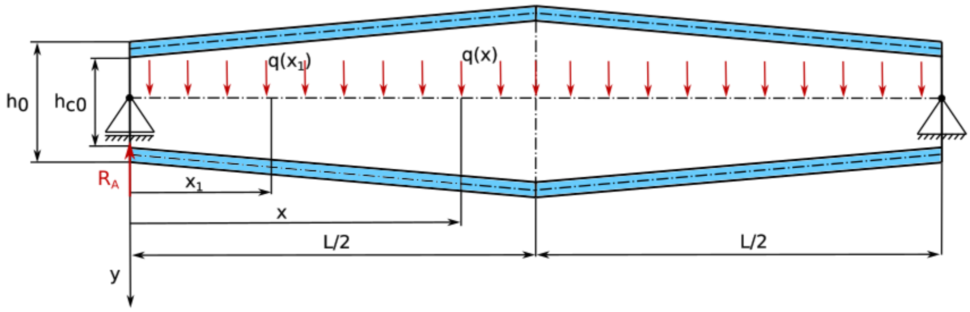

Figure 4. Scheme of the load of the beam under its own weight.

The intensity of this load, with consideration of the expression (6), is as follows

$$q(\xi) = \left[\sqrt{e_c} f_\alpha(\xi) + k_f\right] g\rho_f b h_{c0}, \tag{17}$$

where g = 9.81 m/s^2 – acceleration of gravity. Therefore, after integration the reaction of the supports is in the form:

$$R = g\rho_f b h_{c0} L \int_0^{1/2} \left[\sqrt{e_c} f_\alpha(\xi) + k_f\right] d\xi = \frac{1}{4}\left[\sqrt{e_c}(2 + \lambda \tan\alpha) + 2k_f\right] g\rho_f b h_{c0} L. \tag{18}$$

Consequently, the bending moment:

$$M_b(\xi) = RL\xi - g\rho_f b h_{c0} L^2 \int_0^{\xi} \left[\sqrt{e_c} f_\alpha(\xi_1) + k_f\right](\xi - \xi_1) d\xi_1 \,. \tag{19}$$

Thus, after integration and simple transformation, one obtains

$$M_b(\xi) = \bar{M}_b\,(\xi) q_0 L^2\,, \tag{20}$$

where $q_0 = g\rho_f b h_{c0}$ – substitute intensity of the load, and the dimensionless bending moment

$$\bar{M}_b(\xi) = \frac{1}{12}\left[6\left(\sqrt{e_c} + 2k_f\right)(\xi - \xi^2) + \sqrt{e_c}\lambda(3\xi - 4\xi^3)\tan\alpha\right]. \tag{21}$$

The differential equation of motion of the expanded-tapered sandwich beam Eq. (10), with consideration of the functions (13), (16) and (21), after a simple transformation, one obtains

$$\left[\sqrt{e_c} f_\alpha(\xi) + k_f\right]\bar{v}(\xi)\frac{d^2 v_a}{dt^2} + \left[\sqrt{e_c} f_\alpha(\xi) + k_f\right]\frac{E_f}{12\rho_f \lambda^2 L^2}\, v_a(t) = 0. \tag{22}$$

This equation is approximately solved using the Galerkin method $\int_0^{1/2} \Re(\xi)\bar{v}(\xi)\,d\xi = 0$, where $\Re(\xi)$ – the left part of Eq. (22). Integrating the expression, with consideration of the function (17) and after transformation, one obtains

$$\frac{d^2 v_a}{dt^2} + C_f \frac{E_f}{12\rho_f \lambda^2 L^2} v_a(t) = 0, \tag{23}$$

where: $C_f = J_v/J_k$ – dimensionless coefficient, and $J_k = \int_0^{1/2} f_\alpha(\xi)\bar{v}^2(\xi)\,d\xi$, $J_v = \int_0^{1/2} f_\alpha(\xi)\bar{v}(\xi)\,d\xi$.
The equation (23) is approximately solved with the use of the assumed function

$$v_a(t) = v_a \sin(\omega t)\,, \tag{24}$$

where: v_a – amplitude, ω – fundamental natural frequency.

Substituting this function into Eq. (23), the fundamental natural frequency is obtained

$$\omega = \frac{\sqrt{3}\cdot 10^3}{6\lambda L}\sqrt{C_f\frac{E_f}{\varrho_f}}\ \left[\frac{\text{rad}}{s}\right] \quad \text{or} \quad f_z^{(An)} = \frac{\omega}{2\pi} = \frac{\sqrt{3}\cdot 10^3}{12\pi\lambda L}\sqrt{C_f\frac{E_f}{\varrho_f}}\ \text{[Hz]}. \tag{25}$$

Exemplary calculations are carried out for the following data of beams: h_{c0}=16 mm, h_f=2 mm, E_f=72000 MPa, ρ_f=2710 kg/m^3, ν_c=0.3, e_c=1/32, χ_f=1/8. The results of the calculations for two sample beams are specified in Tabs. 1-2.

Table 1. The calculation results for the beam B-1 of length L=0.6 m and relative length λ=37.5.

α [°]	0	1	2	3	4	5
C_f	225.00	372.50	524.36	679.32	837.12	997.78
$f_z^{(An)}$ [Hz]	157.88	203.14	241.01	274.32	304.52	332.46

Table 2. The calculation results for the beam B-2 of length L=0.8 m and relative length λ=50.0.

α [°]	0	1	2	3	4	5
C_f	225.00	422.67	627.02	836.07	1049.56	1267.68
$f_z^{(An)}$ [Hz]	88.81	121.72	148.25	171.19	191.80	210.79

Moreover, these values of the fundamental natural frequency are graphically presented in Fig. 6.

3. Numerical FEM model of the beam – fundamental natural frequency

The numerical FEM model of the expanded-tapered sandwich beam is developed with the use of the ABAQUS 6.12 (see Fig. 5). The mesh consists of hexahedral quadratic finite elements (C3D8R type).

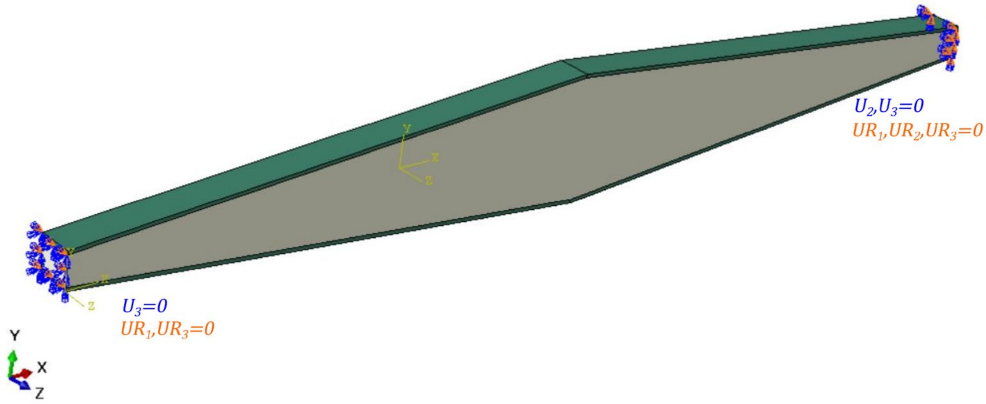

Figure 5. Scheme of the FEM model of the expanded-tapered sandwich beam.

The FEM calculations are performed for the same data as in analytical studies. The results of the calculations for the same two sample beams are specified in Tables. 3-4.

Table 3. The calculation results for the beam B-1 of length L=0.6 m and relative length λ=37.5.

α [°]	0	1	2	3	4	5
$f_z^{(FEM)}$ [Hz]	154.08	196.99	231.76	261.19	286.71	309.19

Table 4. The calculation results for the beam B-2 of length L=0.8 m and relative length λ=50.0.

α [°]	0	1	2	3	4	5
$f_z^{(FEM)}$ [Hz]	87.49	119.41	144.51	165.89	183.90	200.13

Moreover, the values of the fundamental natural frequency are graphically presented in Fig. 6.

4. Conclusions

Comparing the values of the fundamental natural frequency of the beams calculated analytically (An) (see Tabs. 1-2) and numerically (FEM) (see Tabs. 3-4) it is easy to notice, that the differences between them increase with increasing the taper angle α. Moreover, these differences decrease with increasing length L of the beam. These relative differences for two sample beams are specified in Tab. 5.

Table 5. The relative differences $\Delta = \left[f_z^{(An)} - f_z^{(FEM)}\right]/f_z^{(FEM)}$ for two sample beams B-1 and B-2.

α [°]	0	1	2	3	4	5
$\Delta^{(B-1)}\%$	2.5	3.1	4.0	5.0	6.2	7.5
$\Delta^{(B-2)}\%$	1.5	1.9	2.6	3.2	4.3	5.3

The analytical and numerical results are graphically compared in Fig. 6.

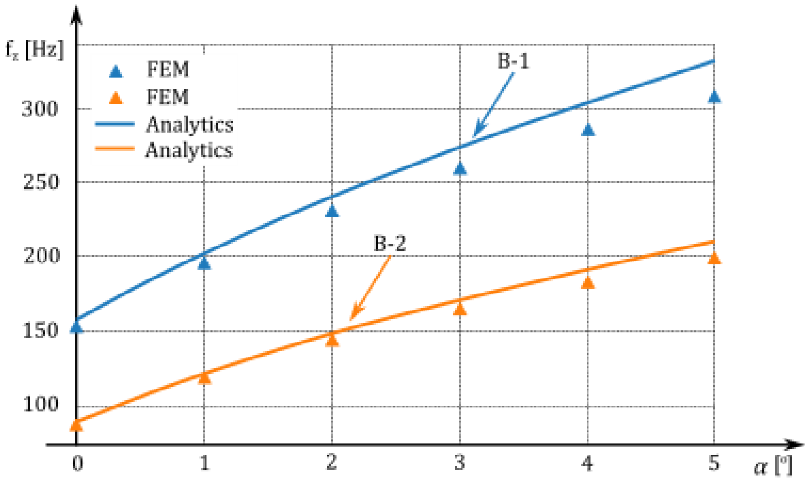

Figure 6. Graphs of the values of the fundamental natural frequency of two sample beams.

These differences between the values of the fundamental natural frequency result primarily from the adopted simplified analytical model of the beam - omitting the shear effect.

To sum up, the novelty of this work is a simplified sandwich tapered beam model that can be successfully used in simple engineering calculations. Differences in fundamental natural frequencies obtained using this model compared to the numerical FEM model are less than 8%.

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Additional information

The authors declare: no competing financial interests and that all material taken from other sources (including their own published works) is clearly cited and that appropriate permits are obtained.

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