10.1515/acsc-2016-0023

Archives of Control Sciences
Volume 26(LXII), 2016
No. 3, *pages 429–435*

Explicit finite-difference scheme for the numerical solution of the model equation of nonlinear hereditary oscillator with variable-order fractional derivatives

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The paper deals with the model of variable-order nonlinear hereditary oscillator based on a numerical finite-difference scheme. Numerical experiments have been carried out to evaluate the stability and convergence of the difference scheme. It is argued that the approximation, stability and convergence are of the first order, while the scheme is stable and converges to the exact solution.

Key words: nonlinear hereditary oscillator, finite-difference scheme, Cauchy problem, fractional derivatives, numerical experiment.

1. Introduction

The development of hereditary processes, i.e. processes with memory, has been reflected in a variety of applications in the last decade. V. V. Uchaikin [9] in his "Method of Fractional Derivatives" devotes a whole chapter to such processes, including a description of hereditary oscillator proposed by Vito Volterra [10]. From the mathematical standpoint, hereditarity, or a memory effect of oscillator, can be demonstrated by inserting integral operator with kernel, which is a memory function, into its model equations. If this kernel is represented by a power series form, the hereditary model equation can be naturally transformed into differential equations with variable-order fractional derivatives [4]. The theory of fractional calculus is quite well developed, and its main provisions can be found in reference books [1, 8].

In this paper we consider the model of nonlinear hereditary oscillator with variable-order derivatives. To do this, we construct an explicit finite-difference scheme for the numerical solution of the corresponding Cauchy problem [2, 5], which will be explored further.

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Received 02.05.2016.

2. Problem

Consider the following Cauchy problem.

$$\begin{gathered}\partial_{0t}^{\beta(t)}x(\tau)+\lambda\partial_{0t}^{\gamma(t)}x(\tau)+\omega^{\beta(t)}\sin(x(t))=f(t),\\ x(0)=x_0,\ \ \dot{x}(0)=y(0),\end{gathered}\tag{1}$$

where

$$\partial_{0t}^{\beta(t)}x(\tau)=\int_0^t\frac{\ddot{x}(\tau)\,d\tau}{\Gamma(2-\beta(\tau))\,(t-\tau)^{\beta(\tau)-1}},\ \partial_{0t}^{\gamma(t)}x(\tau)=\int_0^t\frac{\dot{x}(\tau)\,d\tau}{\Gamma(1-\gamma(\tau))\,(t-\tau)^{\gamma(\tau)}}$$

are the operators of variable-order fractional derivatives $1<\beta(t)<2$ and $0<\gamma(t)<1$, $\Gamma(x)$ is the Euler gamma function, λ, ω, x_0 and y_0 are the given parameters, $f(t)$ is the external stimulus, $t\in[0,T]$ is the process time; the dots over the decision function $x(t)$ mean the classical integer-value derivatives.

Note that problem (1) when $\beta=2$ and $\gamma=1$ transforms into the problem for classical nonlinear oscillator with friction and external force. Note also that the fractional parameters β and γ represent any confined functions.

3. Solution method

The solution to the Cauchy problem (1) in the general case cannot be ob-tained in an explicit form. Therefore, we will seek the solution to this problem using the theory of finite-difference schemes [7]. Let us construct an explicit finite-difference scheme. We divide the segment $[0,T]$ into N equal parts with a constant step τ. Then $x(t_j)=x_j$, $t_j=j\tau$ is the grid solution approximating the solution $x(t)$ of the differential Cauchy problem (1). The operators of the fractional variable-order derivatives are approximated as follows [3].

$$\begin{gathered}\partial_{0t}^{\beta(t)}x(\tau)=\sum_{k=0}^{j-1}\frac{\tau^{-\beta_k}}{\Gamma(3-\beta_k)}\left[(k+1)^{2-\beta_j}-k^{2-\beta_j}\right]\left(x_{j-k+1}-2x_{j-k}+x_{j-k-1}\right)+O\left(\tau^2\right),\\ \partial_{0t}^{\gamma(t)}x(\tau)=\sum_{k=0}^{j-1}\frac{\tau^{-\gamma_k}}{\Gamma(2-\gamma_k)}\left[(k+1)^{1-\gamma_j}-k^{1-\gamma_j}\right]\left(x_{j-k+1}-x_{j-k}\right)+O(\tau).\end{gathered}\tag{2}$$

Substituting relation (2) into equation (1), after some transformations, we come to the following explicit finite-difference scheme.

$$x_1 = \tau y_0 + x_0, \quad j = 0,$$

$$x_{j+1} = A_j x_j - B x_{j-1} - B \sum_{k=1}^{j-1} \frac{\tau^{-\beta_k}}{\Gamma(3-\beta_k)} p_k^j \left(x_{j-k+1} - 2x_{j-k} + x_{j-k-1}\right) -$$

$$-C \sum_{k=1}^{j-1} \frac{\tau^{-\gamma_k}}{\Gamma(2-\gamma_k)} q_k^j \left(x_{j-k+1} - x_{j-k}\right) - \mu \sin(x_j) + \xi f_j,$$

$$A = \frac{2A_0 + B_0}{A_0 + B_0}, \; B = \frac{A_0}{A_0 + B_0}, \; C = \frac{\lambda}{A_0 + B_0}, \tag{3}$$

$$\mu = \frac{\omega^{\beta_j}}{A_0 + B_0}, \; \xi = \frac{1}{A_0 + B_0}, \; A_0 = \frac{\tau^{-\beta_0}}{\Gamma(3-\beta_0)}, \; B_0 = \frac{\lambda \tau^{-\gamma_0}}{\Gamma(2-\gamma_0)},$$

$$p_k^j = (k+1)^{2-\beta_j} - k^{2-\beta_j}, \; q_k^j = (k+1)^{1-\gamma_j} - k^{1-\gamma_j}, \; j = 1, \ldots, N-1.$$

Note that scheme (3) has in its internal points the second order of approximation from the formulas (2); however, due to the approximation in the boundary points, the order is reduced to unity. This can be eliminated by approximating the values in the boundary points in a special way, for example, inserting a dummy node [6]. For the purposes of this paper we do not need to improve scheme (3). We just investigate its stability and convergence by means of a numerical experiment.

Consider the following example. It can be shown that the Cauchy problem with homogeneous initial conditions

$$\partial_{0t}^{\beta(t)} x(\tau) + \lambda \partial_{0t}^{\gamma(t)} x(\tau) = f(t)$$

$$f(t) = \omega^{\beta(t)} \sin\left(t^2\right) + 2 \int_0^t \frac{d\tau}{\Gamma(3-\beta(\tau))(t-\tau)^{\beta(\tau)-1}} + 2 \int_0^t \frac{\tau d\tau}{\Gamma(2-\gamma(\tau))(t-\tau)^{\gamma(\tau)}},$$

$$x(0) = \dot{x}(0) = 0, \tag{4}$$

has an exact solution $x(t) = t^2$. A.A. Samarskii [7] provides definitions of stability on the right side of the equation and with initial data. The essence of these definitions can be summarized as follows. The scheme is stable if a small perturbation introduced to the right side or the initial data leads to a small change in the solution within the accuracy of a constant.

Let us carry out a numerical experiment. To do this we choose the following values of the control parameters of the Cauchy problem (4): $N = 1000$, $\lambda = 1$, $\omega = 2$, $\beta(t) = 2 - 0.006\cos(3\pi t)$, $\gamma(t) = 1 - 0.003\cos(3\pi t)$, $\varepsilon = 10^4$. We find the perturbed and the unperturbed solutions to problem (4) according to scheme (3) and calculate their maximum absolute value error. The results of the experiment are shown in Tabs 1 and 2.

Table 24: Stability with respect to the right side.

N	Maximum error
10	$1.05*10^{-5}$
50	$1.2*10^{-5}$
250	$1.3*10^{-5}$
500	$1.3*10^{-5}$
1000	$1.2*10^{-5}$
2000	$1.2*10^{-5}$
2500	$1.3*10^{-5}$

From Tab. 1 we can conclude that for the chosen values of the control parameters and perturbation ε, explicit finite-difference scheme (3) is stable with respect to the right side, since the maximum error does not exceed perturbation ε.

Table 25: Stability with respect to the initial data.

N	Maximum error
10	$1.633*10^{-4}$
50	$1.634*10^{-4}$
250	$1.633*10^{-4}$
500	$1.633*10^{-4}$
1000	$1.632*10^{-4}$
2000	$1.636*10^{-4}$
2500	$1.635*10^{-4}$

From Tab. 2 it can be concluded that the maximum error values do not practically change with increasing the number of computational grid points N and are commensurate with perturbation ε. Therefore, in this case scheme (3) is stable with respect to the initial data. Let us demonstrate the convergence of scheme (3) for the Cauchy problem through a numerical experiment.

We choose the following values of the control parameters: $N = 1000$, $\lambda = 100$, $\omega = 2$, $t \in (0,1)$ and $\beta(t) = 1.8 - 0.001\cos(3\pi t)$, $\gamma(t) = 0.8 - 0.002\cos(3\pi t)$. We need to find the maximum absolute value error between the numerical and exact solutions depending on step as well as calculate the experimental convergence order of the numerical solution to the exact one. The results of the experiment are shown in Tab. 3.

Table 26: The convergence of scheme (3) to the exact solution.

N	τ	Maximum error	α
10	0.1	0.1172	0.93
20	0.05	0.0573	0.954
40	0.025	0.0219	1.035
80	0.0125	0.00075	1.11

From Tab. 3 it can be concluded that when reducing step τ of the computational grid, the maximum error decreases, while the values of the experimental convergence order $\alpha = \ln(\text{maximum error})/\ln(\text{step})$ are close to unity. Therefore, we can infer that scheme (3) converges to the exact solution with the first order (Fig.1).

4. Conclusion

We have studied the model of variable-order nonlinear hereditary oscillator based on a numerical finite-difference scheme. The stability and convergence of the difference scheme have been evaluated by numerical experiments. The results have shown that the approximation, stability and convergence are of the first order, while the scheme is stable and converges to the exact solution. Certainly, if necessary, scheme (3) can be improved through proper approximation of the initial conditions. Also, using the double counting method we can increase its accuracy. The next step in studying the hereditary nonlinear model of an oscillating system will be the construction and analysis of phase trajectories, as it was carried out in [3] for linear hereditary oscillators.

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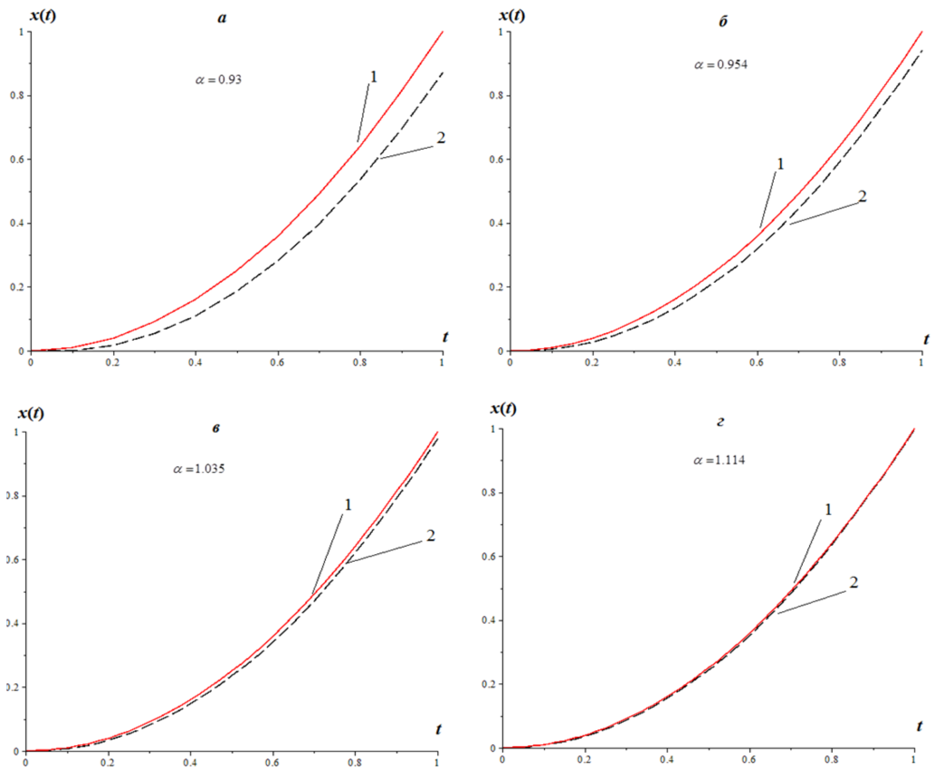

Figure 1: The convergence of scheme (3) to the exact solution.

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