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# Beyond one-point turbulence closures

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**Abstract**

The paper concerns statistical description of turbulence in terms of multipoint velocity moments. A literature survey on possible multipoint turbulence closures and their future perspective is provided. We first consider the transport equations for two-point velocity statistics and their one-point limit. Another form of turbulence description, in terms of multipoint probability density functions is also introduced.

**Keywords:** Turbulence; Turbulence modeling; Multipoint velocity statistics

## 1 Introduction

Although the evolution of turbulent velocity field is governed by the deterministic Navier-Stokes equations, due to its sensitivity to small variations in the initial and boundary conditions the turbulent field may be treated as a stochastic field [15]. Such a field is statistically fully described if, at a given time, all multipoint velocity correlations of arbitrary order are known. Such information can formally be provided by a suitable, infinite hierarchy of equations, namely, the Lundgren-Monin-Novikov (LMN) equations [9] for the multipoint probability density functions (PDF's) or the Friedmann-Keller (FK) hierarchy [5], for the multipoint velocity correlations. The $n$-point velocity correlation is an ensemble average (average over infinitely many realisations) of the product of velocities in

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$n$ different points $\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdot\dots\cdot U_{i_{(n)}}(\boldsymbol{x}_n,t)\rangle$. In the $n$-th equation of the FK hierarchy the unknown correlation of $n+1$-order is present. Similarly, the first $n$ equations of the LMN hierarchy are unclosed due to the presence of an unknown $n+1$-point probability density function.

The $n$-point velocity PDF $f_n = f_n(\boldsymbol{v}_{(1)},\boldsymbol{x}_{(1)};\dots;\boldsymbol{v}_{(n)},\boldsymbol{x}_{(n)},t)$ contains information about all statistics up to $n$-point statistics of infinite order which can be calculated from the PDF by integration over the sample space variables $\boldsymbol{v}_{(1)},\dots,\boldsymbol{v}_{(n)}$, for example

$$\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdot\dots\cdot U_{i_{(n)}}(\boldsymbol{x}_n,t)\rangle = \int v_{i_{(1)}}\dots v_{i_{(n)}} f_n \mathrm{d}\boldsymbol{v}_{(1)}\dots\mathrm{d}\boldsymbol{v}_{(n)}\ , \tag{1}$$

where the integration is performed over the entire sample space, from $-\infty$ to $\infty$ for each $v_{i_{(k)}}$, $k=1,\dots,n$. It should be noted that the FK equations can formally be derived from the LMN hierarchy by a proper multiplication and integration over sample space variables.

Given these multi-point methods of turbulence description, one should notice that, statistical turbulence closures like, e.g. $k$-$\varepsilon$ or the Reynolds-stress models (RSM) [15] provide only very limited information on a turbulent stochastic field. The Reynolds-stresses are one-point second-order moments of turbulent velocity, hence, the effect of all higher-order correlations have to be replaced by a proper closure in the RSM models.

Another class of models is based on the one-point PDF [15], which provides information on one-point statistics of arbitrary order, e.g.,

$$\langle U_i(\boldsymbol{x},t)U_j(\boldsymbol{x},t)U_k(\boldsymbol{x},t)\rangle = \iiint v_i v_j v_k f(\boldsymbol{v},\boldsymbol{x},t)\mathrm{d}\boldsymbol{v}\ .$$

Such closures can be used to model the velocity statistics, see e.g. [2, 10], or the joint velocity-scalar statistics [4].

The one-point statistical models are already well-established and are broadly used in engineering computations. Hence, they will not be discussed in detail here. It has been a common belief that with the increase of available computational powers the statistical Reynolds-averaged models will be replaced by the large-eddy simulations (LES). This latter modeling approach, however, in spite of over 30 years of intensive development, is still deficient in predicting, e.g., the near-wall flows.

Hence, an interesting option for the future study would be to return to the statistical description of turbulence in terms of two-point and/or higher-order correlations. The multipoint closures could also be used as a subgrid models in

LES. The clear advantage of transport equations for two-point velocity moments over the one-point approach is the only one unclosed triple correlation term. The disadvantage is the larger number of independent variables ($3+3$ space variables and time). For this reason the two-point modeling approaches has been put aside, apart from models for homogeneous or weakly inhomogeneous flows. However, the increase of computational powers might be a good reason to reconsider certain alternatives.

In the present work, statistical description of turbulence in terms of multipoint velocity moments and multipoint PDF's is first discussed in Sec. 2. Next, two-point closures for velocity, put forward in the extant literature are presented in Sec. 3 and a perspective for a possible multipoint PDF closure is given in Sec. 4. This is followed by conclusions and perspectives for further study.

# 2 Governing equations

In the present work a turbulent flow of Newtonian, incompressible and isothermal fluid will be considered. With the use of the averaging operator, the instantaneous velocity and pressure is decomposed into the mean and fluctuating parts: $U_i = \langle U_i \rangle + u_i$, $P = \langle P \rangle + p$. After the ensemble averaging of the governing equations we obtain the mean continuity and the Reynolds equation

$$\frac{\partial \langle U_i \rangle}{\partial x_i} = 0 \ , \quad \frac{\partial \langle U_i \rangle}{\partial t} + \langle U_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle u_i u_j \rangle}{\partial x_j} = -\frac{1}{\varrho} \frac{\partial \langle P \rangle}{\partial x_i} + \nu \frac{\partial^2 \langle U_i \rangle}{\partial x_j \partial x_j} \ . \tag{2}$$

## 2.1 Equations for two-point correlation tensor

Before equations for two-point correlation tensor will be written, it is necessary to introduce the following notations, see also [11]. We will consider two points $\boldsymbol{x}$ and $\boldsymbol{x}_{(1)}$ and the distance vector $\boldsymbol{r}$ such that

$$\boldsymbol{r} = \boldsymbol{x}_{(1)} - \boldsymbol{x} \ . \tag{3}$$

The two point correlation tensor will be denoted as

$$R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) = \langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x}_{(1)}, t) \rangle \ . \tag{4}$$

For further purposes the following notations for the two-point triple correlation and two-point velocity-pressure correlations are introduced

$$\begin{aligned} R_{(ik)j}(\boldsymbol{x}, \boldsymbol{r}, t) &= \langle u_i(\boldsymbol{x}, t) u_k(\boldsymbol{x}, t) u_j(\boldsymbol{x}_{(1)}, t) \rangle \ , \\ R_{i(jk)}(\boldsymbol{x}, \boldsymbol{r}, t) &= \langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x}_{(1)}, t) u_k(\boldsymbol{x}_{(1)}, t) \rangle \ , \\ \langle p u_i \rangle(\boldsymbol{x}, \boldsymbol{r}, t) &= \langle p(\boldsymbol{x}, t) u_j(\boldsymbol{x}_{(1)}, t) \rangle \ , \quad \langle u_i p \rangle(\boldsymbol{x}, \boldsymbol{r}, t) = \langle u_i(\boldsymbol{x}, t) p(\boldsymbol{x}_{(1)}, t) \rangle \ . \end{aligned} \tag{5}$$

In order to derive equations for two-point correlation tensor, first, an equation for the evolution of the fluctuating velocity $u_i(\boldsymbol{x},t)$ at point $\boldsymbol{x}$ should be derived by subtracting the Reynolds equation (2) from the momentum equation. Next, this equation should be multiplied by $u_j(\boldsymbol{x}_{(1)},t)$. Similarly, transport equation for fluctuating velocity at point $\boldsymbol{x}_{(1)}$, $u_j(\boldsymbol{x}_{(1)},t)$ should be multiplied by $u_i(\boldsymbol{x},t)$. After adding both equations and changing the system of coordinates from $(\boldsymbol{x}, \boldsymbol{x}_{(1)})$ to $(\boldsymbol{x}, \boldsymbol{r})$ the transport equation for two-point correlation tensor is derived [11]

$$
\begin{aligned}
&\frac{\partial R_{ij}}{\partial t} + \langle U_k\rangle \frac{\partial R_{ij}}{\partial x_k} = \\
&\underbrace{-R_{kj}\frac{\partial \langle U_i(\boldsymbol{x},t)\rangle}{\partial x_k} - R_{ik}\frac{\partial \langle U_j(\boldsymbol{x}+\boldsymbol{r},t)\rangle}{\partial x_k}}_{\mathcal{P}_{ij}} \underbrace{-\left[\langle U_k(\boldsymbol{x}+\boldsymbol{r},t)\rangle - \langle U_k(\boldsymbol{x},t)\rangle\right]\frac{\partial R_{ij}}{\partial r_k}}_{\mathcal{V}_{ij}} \\
&\underbrace{-\frac{1}{\varrho}\left[\frac{\partial\langle pu_j\rangle}{\partial x_i} - \frac{\partial\langle pu_j\rangle}{\partial r_i} + \frac{\partial\langle u_i p\rangle}{\partial r_j}\right]}_{\phi_{ij}} \underbrace{+\nu\left[\frac{\partial^2 R_{ij}}{\partial x_k\partial x_k} - 2\frac{\partial^2 R_{ij}}{\partial x_k\partial r_k} + 2\frac{\partial^2 R_{ij}}{\partial r_k\partial r_k}\right]}_{\varepsilon^{\nu}_{ij}} \\
&\qquad\qquad\qquad\qquad\qquad \underbrace{-\frac{\partial R_{(ik)j}}{\partial x_k} + \frac{\partial}{\partial r_k}\left[R_{(ik)j} - R_{i(jk)}\right]}_{\mathcal{T}_{ij}} .
\end{aligned}
\tag{6}
$$

The above equation is complemented by the relations which follow from the continuity equation

$$
\frac{\partial R_{ij}}{\partial x_i} - \frac{\partial R_{ij}}{\partial r_i} = 0\,, \quad \frac{\partial R_{ij}}{\partial r_j} = 0\,, \quad \frac{\partial R_{(ik)j}}{\partial r_j} = 0\,, \quad \frac{\partial \langle pu_i\rangle}{\partial r_i} = 0\,. \tag{7}
$$

After calculating the divergence of Eq. (6), the Poisson equation for the two-point velocity-pressure correlations $\langle pu_j\rangle$ is obtained. Hence, an interesting observation is, that the only unclosed term in Eq. (6) is the triple correlation term $\mathcal{T}_{ij}$.

In the limit $|\boldsymbol{r}| \to 0$ from the two-point correlation tensor $R_{ij}$ the Reynolds stresses $\langle u_i u_j\rangle$ are obtained

$$
\lim_{|\boldsymbol{r}|\to 0} R_{ij}(\boldsymbol{x},\boldsymbol{r},t) = \langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t)\rangle \tag{8}
$$

and Eq. (6) becomes a transport equation for the Reynolds stresses (RS) in this limit. We obtain, respectively, the production term from $\mathcal{P}_{ij}$

$$
\lim_{|\boldsymbol{r}|\to 0} \mathcal{P}_{ij} = -\left[\langle u_k u_j\rangle \frac{\partial\langle U_i(\boldsymbol{x},t)\rangle}{\partial x_k} + \langle u_i u_k\rangle \frac{\partial\langle U_j(\boldsymbol{x},t)\rangle}{\partial x_k}\right], \tag{9}
$$

the pressure-strain and velocity-pressure correlation from $\phi_{ij}$

$$\lim_{|\boldsymbol{r}|\to 0} \phi_{ij} = \frac{1}{\rho}\left\langle p\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right\rangle - \frac{1}{\rho}\frac{\partial}{\partial x_k}\langle p u_i \delta_{jk} + p u_j \delta_{ik}\rangle \,, \tag{10}$$

dissipation and viscous diffusion from $\mathcal{E}^{\nu}_{ij}$

$$\lim_{|\boldsymbol{r}|\to 0} \mathcal{E}^{\nu}_{ij} = \nu\frac{\partial^2 \langle u_i u_j\rangle}{\partial x_k \partial x_k} - 2\nu\left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right\rangle \,, \tag{11}$$

and turbulent diffusion from $\mathcal{T}_{ij}$

$$\lim_{|\boldsymbol{r}|\to 0} \mathcal{T}_{ij} = -\frac{\partial \langle u_i u_j u_k\rangle}{\partial x_k} \,. \tag{12}$$

The term $\mathcal{V}_{ij}$ in Eq. (6) tends to 0 when $|\boldsymbol{r}| \to 0$. Four terms, i.e., the pressure-strain, velocity-pressure correlation, dissipation and the turbulent diffusion are unclosed in the RS transport equations and are replaced by proper closures in the standard one-point turbulence modelling. The examples are the well-established Launder-Reece-Rodi [8] or Speziale-Sarkar-Gatski [19] models for the pressure-strain correlation. The local isotropy assupmtion is taken as a model for the dissipation term $\epsilon_{ij} = \frac{2}{3}\epsilon\delta_{ij}$ and a separate equation for $\epsilon$ or the turbulence frequency $\omega = \epsilon/k$, where $k$ is the turbulent kinetic energy is solved [21]. The gradient diffusion models are used as a closure for the turbulent diffusion and velocity-pressure correlation terms [15]. The one-point turbulence closure introduces numerous assumptions and modeling constants. Here, the advantage of the two-point equation (6) where only the triple correlation term $\mathcal{T}_{ij}$ is unclosed is clearly seen. Two-point modeling proposals for Eq. (6) are briefly described in Sec. 3.

## 2.2 Multipoint PDF equations

An alternative multipoint description of turbulence can be obtained with the use of multipoint probability density functions. The one-point PDF is related to the probability $f(\boldsymbol{v};\boldsymbol{x},t)\mathrm{d}\boldsymbol{v}$ that the velocity $\boldsymbol{U}(\boldsymbol{x},t)$ at point $\boldsymbol{x}$ and time $t$ is contained within $\boldsymbol{v} \le \boldsymbol{U}(\boldsymbol{x},t) \le \boldsymbol{v}+\mathrm{d}\boldsymbol{v}$. Similarly, $f(\boldsymbol{v}_{(1)},\boldsymbol{x}_{(1)};\boldsymbol{v}_{(2)},\boldsymbol{x}_{(2)},t)\mathrm{d}\boldsymbol{v}_{(1)}\mathrm{d}\boldsymbol{v}_{(2)}$ denotes the joint probability that $\boldsymbol{v}_{(1)} \le \boldsymbol{U}(\boldsymbol{x}_{(1)},t) \le \boldsymbol{v}_{(1)} + \mathrm{d}\boldsymbol{v}_{(1)}$ and $\boldsymbol{v}_{(2)} \le \boldsymbol{U}(\boldsymbol{x}_{(2)},t) \le \boldsymbol{v}_{(2)} + \mathrm{d}\boldsymbol{v}_{(2)}$.

The infinite hierarchy of equations for the multipoint PDF's was derived in [9] in the following form:

$$\frac{\partial f_n}{\partial t}+\sum_{k=1}^{n} v_{i_{(k)}}\frac{\partial f_n}{\partial x_{i_{(k)}}}=-\frac{1}{4\pi}\sum_{k=1}^{n}\frac{\partial}{\partial v_{i_{(k)}}} \tag{13}$$

$$\times \int\int\left(\frac{\partial}{\partial x_{i_{(k)}}}\frac{1}{\left|\boldsymbol{x}_{(k)}-\boldsymbol{x}_{(n+1)}\right|}\right)\left(v_{j_{(n+1)}}\frac{\partial}{\partial x_{j_{(n+1)}}}\right)^2 f_{n+1}\mathrm{d}\boldsymbol{v}_{(n+1)}\mathrm{d}\boldsymbol{x}_{(n+1)}$$

$$-\sum_{k=1}^{n}\frac{\partial}{\partial v_{i_{(k)}}}\left[\lim_{|\boldsymbol{x}_{(n+1)}-\boldsymbol{x}_{(k)}|\to 0}\nu\frac{\partial^2}{\partial x_{j_{(n+1)}}\partial x_{j_{(n+1)}}}\int v_{i_{(n+1)}} f_{n+1}\mathrm{d}\boldsymbol{v}_{(n+1)}\right].$$

In the above equation the pressure was eliminated and instead, a free-space solution of the Poisson equation was used.

The $n+1$-point PDF of velocity can be also related to the PDF of velocity increments $\Delta\boldsymbol{u}(\boldsymbol{x}_{(i)},\boldsymbol{r}_{(j)},t)=\boldsymbol{u}(\boldsymbol{x}_{(i)}+\boldsymbol{r}_{(j)},t)-\boldsymbol{u}(\boldsymbol{x}_{(i)},t)$. The sample space of $\Delta\boldsymbol{u}(\boldsymbol{x}_{(i)},\boldsymbol{r}_{(j)},t)$ will be denoted by $\boldsymbol{\xi}_{(j)}$ and the $n$+1-point PDF of velocity equals

$$f(\boldsymbol{v}_{(1)},\boldsymbol{x}_{(1)};\ldots;\boldsymbol{v}_{(n+1)},\boldsymbol{x}_{(n+1)},t)=f(\boldsymbol{\xi}_{(n)},\boldsymbol{r}_{(n)};\ldots;\boldsymbol{\xi}_{(1)},\boldsymbol{r}_{(1)};\boldsymbol{u}_{(1)},\boldsymbol{x}_{(1)},t)\ .$$

Interestingly, recent studies [17, 18] show that within a certain range of scales $|\boldsymbol{r}|$, $\Delta\boldsymbol{u}(\boldsymbol{x},\boldsymbol{r},t)$ can be treated as a Markov stochastic process in scale $|\boldsymbol{r}|$. This implies that the multipoint statistics of the system could be expressed by a product of three-point statistics and that the system of equations for PDF's (13) could be truncated. This observation could be of particular importance for modeling approaches and will be discussed in more detail in Sec. 4.

# 3 Possible closures for two-point correlations

There exists a class of models for the two-point correlations which are based on the Fourier transform of the transport equation (6). Most of them, however, refer to the homogenoeous or homogeneous and isotropic turbulence. In the homogeneous and isotropic turbulence the two-point correlation tensor is a function of $|\boldsymbol{r}|=r$ and $t$: $R_{ij}(r,t)$. The Fourier transform of the two-point correlation tensor will be denoted by $\Phi_{ij}(\mathbf{k},t)$, where $\mathbf{k}$ is the wavenumber vector. The turbulent kinetic energy spectrum is defined as

$$E(k,t)=\int\frac{1}{2}\Phi_{ii}\mathrm{d}\mathcal{S}(\boldsymbol{k})\ , \tag{14}$$

where the integration is performed over a sphere $\mathcal{S}(\boldsymbol{k})$ in the wavenumber space, centered at the origin, with radius $k$. If $E(k,t)$ is integrated over $k$ from 0 to $\infty$, the kinetic turbulent energy is obtained. From the Fourier transformed Eqs. (6) written for homogeneous, isotropic turbulence, a transport equation for $E(k,t)$ can be derived

$$\frac{\partial E(k,t)}{\partial t} = -2\nu k^2 E(k,t) + T(k,t) \; . \tag{15}$$

The term $T(k,t)$ is the unclosed nonlinear transfer term. Several closures for $T(k,t)$ were proposed in the literature, e.g., the eddy damped quasi-normal Markovian (EDQNM) model [14] or the test field model [6].

The two-point models for homogeneous turbulence can correctly reproduce the cascade of energy from larger to smaller scales. Hence, the two-point models are also used as a closure for subgrid terms in the large eddy simulations (see e.g., [1]) where the subgrid viscosity is expressed as a function of the energy spectrum at the filter cut-off wavenumber.

The use of two-point models in inhomogeneous flow cases was so far limited to relatively simple flows cases. Some authors follow the spectral formulation and aim to extend the EDQNM model towards inhomogeneous turbulence, introducing certain simplifications. In [16] the spectral tensor integrated over the sphere $\mathcal{S}(\boldsymbol{k})$ is considered:

$$\phi_{ij}(k,\boldsymbol{x},t) = \int \Phi_{ij}(\boldsymbol{k},\boldsymbol{x},t)\mathrm{d}\mathcal{S}(\boldsymbol{k}) \; . \tag{16}$$

With this assumption, the pressure-velocity correlation becomes unclosed as in the classical one-point models. Other authors use the weak homogeneity assumption where only the leading terms in an expansion of the triple correlation tensor about homogeneity are retained [7].

In contrast to these latter approaches in [11] Eq. (6) in physical space are considered. To satisfy the continuity equations (7) the following form of the two-point correlation tensor was derived

$$R_{ij} = e_{ikm}e_{jln}\left[\frac{\partial}{\partial x_k} - \frac{\partial}{\partial r_k}\right]\frac{\partial V_{mn}}{\partial x_l} \; , \tag{17}$$

where $V_{mn}$ is the tensor potential and $e_{ijk}$ is the alternating tensor. The product $e_{ijk}e_{lmn}$ equals

$$e_{ijk}e_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \; .$$

In the homogeneous and isotropic turbulence the tensor potential $V_{mn}$ has the following form:

$$V_{mn} = -\frac{1}{2}\langle u_1^2\rangle\delta_{mn}\int_0^r f(r) r \mathrm{d}r \ , \tag{18}$$

where $f(r)$ is the longitudinal velocity autocorrelation function [15], with $\boldsymbol{r} = \mathbf{e}_1 r$, $f(r)\langle u_1^2\rangle = \langle u_1(\boldsymbol{x},t)u_1(\boldsymbol{x}+\mathbf{e}_1 r,t)\rangle$. Equation (17) with (18) introduced into (6) leads finally to the Kármán-Howarth equation for homogeneous and isotropic turbulence.

In order to extend the model towards inhomogeneous flows it was postulated in [11] to present the tensor $V_{mn}$ as a function of a vector $\boldsymbol{r}$ and a new symmetric tensor

$$\int_0^r F_{mn}(\boldsymbol{x},r,t) r \mathrm{d}r \ .$$

It was assumed that $V_{mn}$ is linear in $\int_0^r F_{mn} r \mathrm{d}r$ and has a dimension of $r^2 F_{mn}$. Then, with the use of the tensor invariant theory, a most general form of the tensor $V_{mn}$ satisfying these assumptions was derived.

The assumption that $F_{ij}$ is a function of the magnitude of the correlation distance $|\boldsymbol{r}| = r$ instead of $\boldsymbol{r}$ introduced, of course, limitations to the proposed closure. Especially close to solid boundaries, where the flow is strongly inhomogeneous, the model did not reproduce the statistics correctly. Still, an interesting outcome from the analysis was the equation for nonisotropic length scale $\mathcal{L}_{ij}$, or, alternatively, non-isotropic dissipation $\epsilon_{ij}$. In homogeneous, isotropic turbulence the longitudinal integral lengthscale is defined as

$$L_{11}(t) = \int_0^\infty f(r)\mathrm{d}r \ .$$

In [11] an anisotropic lengthscale tensor was considered

$$\mathcal{L}_{ij} = \frac{1}{4\pi}\int_V F_{ij}\frac{1}{r^2}\mathrm{d}^3 r \ .$$

Next, transport equation for $\mathcal{L}_{ij}$ was derived after integrating the two-point correlation equation (6) taking into account the modelling assumptions. Following the relation for the isotropic turbulent length scale $l \sim \epsilon/k^{2/3}$, it was assumed in [11] that the nonisotropic tensor $\mathcal{L}_{ij}$ is a function of the Reynolds-stress and the dissipation tensor $\epsilon_{ij}$. This led finally to the transport equations for $\epsilon_{ij}$, which provide an interesting alternative to the equation for isotropic dissipation rate $\epsilon$ with $\epsilon_{ij} = 2/3\delta_{ij}\epsilon$ commonly used in turbulence models.

Further works on velocity multipoint correlations [12, 13] show that the FK equations are invariant under additional symmetries which has to be included in order to represent correctly the velocity statistics in inhomogeneous flow cases. Constructing new closures for inhomogeneous flows, based on the extended set of symmetries is a perspective for a further study.

# 4 Perspectives for a possible closure of multipoint PDF's

A new insight into the multipoint closure was given in a series of works on multipoint PDF's [3, 17, 18]. In Ref. [3] the authors considered homogeneous isotropic turbulence and described the turbulent cascade in terms of the longitudinal velocity increments $\Delta u_1(\boldsymbol{x}, r, t) = u_1(\boldsymbol{x} + \mathbf{e}_1 r/2, t) - u_1(\boldsymbol{x} - \mathbf{e}_1 r/2, t)$. The multiscale PDF is denoted as $f(\xi_{(n)}, r_{(n)}; \ldots; \xi_{(1)}, r_{(1)}, t)$ where $\xi_{(i)}$ is the sample space variable of the velocity difference $\Delta u_1(\boldsymbol{x}, r_{(i)}, t)$ and $r_{(i+1)} < r_{(i)}$. It was shown experimentally that in a certain range of $r$ the stochastic cascade process is Markovian, hence its conditional $n$-point PDF can be expressed as

$$f(\xi_{(n)}, r_{(n)} | \xi_{(n-1)}, r_{(n-1)}; \ldots; \xi_{(1)}, r_{(1)}, t) = f(\xi_{(n)}, r_{(n)} | \xi_{(n-1)}, r_{(n-1)}, t) \ . \tag{19}$$

That means that given $\Delta u_1(\boldsymbol{x}, r_{(n-1)}, t) = \xi_{(n-1)}$, knowledge of velocity differences at larger separations $r$ provide no additional information on the velocity increment $\Delta u_1(\boldsymbol{x}, r_{(n)}, t) = \xi_{(n)}$. The authors considered the exact relations for two- and three-point PDF's

$$\begin{aligned} f(\xi_{(2)}, r_{(2)} | \xi_{(1)}, r_{(1)}, t) &= \frac{f(\xi_{(2)}, r_{(2)}; \xi_{(1)}, r_{(1)}, t)}{f(\xi_{(1)}, r_{(1)}, t)} \ , \\ f(\xi_{(3)}, r_{(3)} | \xi_{(2)}, r_{(2)}; \xi_{(1)}, r_{(1)}, t) &= \frac{f(\xi_{(3)}, r_{(3)}; \xi_{(2)}, r_{(2)}; \xi_{(1)}, r_{(1)}, t)}{f(\xi_{(2)}, r_{(2)}; \xi_{(1)}, r_{(1)}, t)} \end{aligned} \tag{20}$$

and investigated experimental data to show that the relation

$$f(\xi_{(3)}, r_{(3)}; \xi_{(2)}, r_{(2)}; \xi_{(1)}, r_{(1)}, t) = f(\xi_{(3)}, r_{(3)} | \xi_{(2)}, r_{(2)}, t) f(\xi_{(2)}, r_{(2)}; \xi_{(1)}, r_{(1)}, t)$$

is satisfied with a good accuracy provided that $r_{(1)} - r_{(2)} > \lambda$, where $\lambda$ is the Taylor microscale.

With the Markov property the $n$-point PDF can be written as a product of conditional probabilities

$$f(\xi_{(n)}, r_{(n)}; \ldots; \xi_{(1)}, r_{(1)}, t) = \\ f(\xi_{(n)}, r_{(n)}|\xi_{(n-1)}, r_{(n-1)}, t) f(\xi_{(n-1)}, r_{(n-1)}|\xi_{(n-2)}, r_{(n-2)}, t) \\ \ldots f(\xi_{(2)}, r_{(2)}|\xi_{(1)}, r_{(1)}, t) f(\xi_{(1)}, r_{(1)}, t) \ , \tag{21}$$

which, once again, is true only if $r_{(i)} - r_{(i+1)} > \lambda$. For such a case the $n$-point information are described by two-scale PDF's, and moreover, the evolution of the conditional PDF in scale $r$ can be described by the Fokker-Planck equation [18]

$$- \quad r\frac{\partial}{\partial r} f(\xi, r|\xi', r', t) = \\ - \quad \frac{\partial}{\partial \xi}\left[D^{(1)}(\xi, r) f(\xi, r|\xi', r', t)\right] + \frac{\partial^2}{\partial \xi^2}\left[D^{(2)}(\xi, r) f(\xi, r|\xi', r', t)\right] , \tag{22}$$

where $r > r'$ which implies the direction of the cascade process from large to small scales.

In [17] a more complete description of the cascade process was performed and the PDF of the from $f(\boldsymbol{\xi}_{(n)}, \boldsymbol{r}_{(n)}; \ldots, \boldsymbol{\xi}_{(1)}, \boldsymbol{r}_{(1)}; \boldsymbol{u}_{(1)}, \boldsymbol{x}_{(1)}) = f(\boldsymbol{v}_{(1)}, \boldsymbol{x}_{(1)}; \ldots; \boldsymbol{v}_{(n+1)}, \boldsymbol{x}_{(n+1)}, t)$ was considered which allowed to calculate multipoint statistics of velocity from the PDF (not only the statistics of velocity increments). Inhomogeneous flow cases were also investigated in [17].

The analysis of DNS and experimental data performed in [3, 17, 18] may deliver new ideas on a possible multipoint closure. It also follows from this analysis that for a certain range of $|\boldsymbol{r}|$, the multipoint hierarchy (13) can be truncated. Lastly, the connections between the described experimental data on PDF's and recent mathematical study on the symmetries of the LMN equations can be investigated [20].

# 5 Conclusions

Although classical statistical turbulence closures are usually restricted to one-point moments of velocity, we could ask a question about other possibilities and approaches which reach beyond the one-point description. In the present work a literature survey on multipoint closures of turbulence was performed. As it was discussed, considering the two-point and higher order statistics may deliver new interesting insight into the phenomenon of turbulence. It is possible that with the constant increase of computational powers, the multipoint closure will become accessible in engineering applications, also as a possible closure for subgrid terms in LES. So far, the large number of independent variables (7 for the two-point statistics) made this approach computationally too expensive. Moreover, the development of multipoint closures for strongly inhomogeneous flows is still an open issue.

Recent experimental studies on multipoint PDF's which show that the turbulence cascade can be described as a Markov process in scale are also an interesting perspective for further investigation. The Markov property was also found for the inhomogeneous flow cases like, e.g., the free jet or cylinder wake. The studies suggest that for a certain range of scales the infinite system of transport equations for multipoint PDF's may be truncated and a model for three-scale (or four-point) statistics can be introduced.

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