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THE GENERAL FORMULA FOR CALCULATION OF FUNDAMENTAL FREQUENCY OF AXISYMMETRIC VIBRATIONS OF CIRCULAR PLATES WITH LINEARLY VARIABLE THICKNESS

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Abstract

This work has derived the general formula, sufficiently precise for engineering calculations of base frequency of axisymmetric free vibrations of uniform, circular diaphragm type plates clamped at the edge with linearly variable thickness. To solve the boundary problem, the Cauchy's function method and characteristic series have been applied. The above formula has been derived on the basis of Dunkerley's formula which is based on the first major term of the characteristic series and results in the simplest, lower bound estimator. The analysis of the formula shows that the base frequency coefficient for diaphragm plates clamped at the edge depends to only a small extent on the Poisson's ratio, and therefore it may be averaged in the case of construction materials. Comparison of the calculations results of the simplest lower bound estimators for the base frequency obtained by using proposed method, with the results known from the literature as precise solutions, including Conway method, confirmed the high accuracy of the proposed method.

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Introduction

The direction of research involving more precise approach to the dynamic model entails considerable complexity of the solution. Often, the initial dynamic calculations are successfully accomplished with the use of simple formulas for estimating natural frequency and mode (VASYLENKO, ALEKSIEJCHUK 2004). This work develops a Cauchy's function method and characteristic series to solve boundary problems of vibrations in uniform, circular plates of variable thickness. It is known that only in a few special cases of power and linear variation of thickness a precise solution is applicable, when the equation takes the form of the Euler equation the solutions are in Bessel's function. Thus, Conway H.D in the works of 1958 a) and b) solved cases of change of flexural rigidity by using the formula: $D = D_0\left(\frac{r}{R}\right)^{\frac{m}{3}}$, where D_0, R – constants, r – coordinate, m – coefficient have the following dependences with Poisson's ratio $\nu = \frac{2m-3}{9}$. JAROSZEWICZ and ZORYJ (2005, 2006), with the assistance of co-authors solved that problem for cases of a much larger range of $0 \le m < 6$ and $\nu = \frac{1}{m}$ applying Cauchy's function, characteristic series and Bernstein--Kieropian estimators (JAROSZEWICZ, ZORYJ 2005).

The work by DOMARADZKI, JAROSZEWICZ, ZORYJ (2006) shows that the Poisson's number significantly affects the base frequency parametres only for certain powers m. This work has also demonstrated that for the diaphragm, when $m = 2$ the effect is quite considerable, and so the simplest estimator of the base frequency is as follows: $7.8 \le \gamma_{-}(\nu) \le 9.3$, the difference relating to the average value is 15.7%. On the other hand, it is also known that in the case of plates of uniform thickness, clamped at the edge ($m = 0$) natural frequency coefficients do not depend on the ν value. Therefore, the examination of influence of and ν on roots of characteristic equation solution for $L[u] = 0$ has a significant consequence in the study of boundary problem of flexural vibrations.

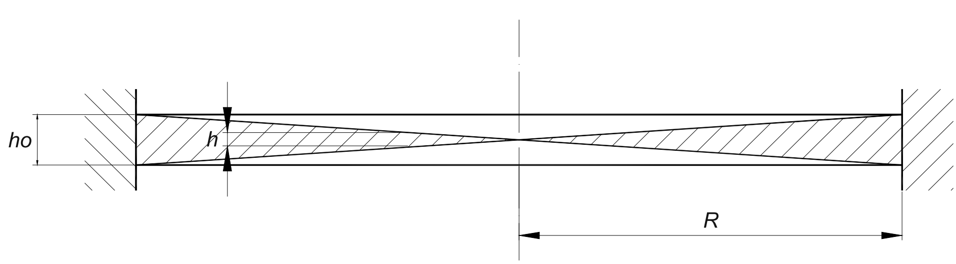

Fig. 1. A model of studied diaphragm

Formulation of the problem

The object of study is a circular plate, rigidly clamped at the edge with radius R, thickness h and flexural rigidity D as definite functions of radial coordinate r (Fig. 1):

$$h = h_0 \cdot \frac{r}{R}, \quad D = D_0\left(\frac{r}{R}\right)^3, \quad D_0 = \frac{Eh_0}{12(1 - \nu^2} \tag{1}$$

In this case $0 < r \leq R, h_0, D_0$ are constants. The following nomenclature was used: – Young's modulus ν – Poisson's ratio.

It should be explained at this point that the tested plate is a thin diaphragm with thickness linearly growing towards the centre of symmetry, where for $r \Rightarrow 0$ this value is close to $(h \to 0)$.

Axisymmmetrical free vibrations of such diaphragm for $m = 3$ are characterized by a following boundary problem (CONWAY 1958a, JAROSZEWICZ, ZORYJ 2005).

$$L_0[u] - pr^{-2m}u = 0, \quad u(R) = 0, \quad u'(R) = 0 \tag{2}$$

where:

$$\mathrm{L}_0[u] \equiv u^{IV} + \frac{8}{r}u^{III} + (11 + 3\nu)\frac{u^{II}}{r^2} + 2(3\nu - 1)\frac{u^I}{r^3}, \; p = \frac{\rho \cdot h}{D} \cdot R^2 \cdot \omega^2 \tag{3}$$

ρh – mass – density of plate, $u = u(r)$ – amplitude of deflection, ω – frequency parameter. Besides of boundary conditions (2) existing requirements concern limiting of deflection and angle of deflection for $r = 0$, which consider limited strength of plate material.

$$|u(0)| < \infty, \quad |u'(0)| < \infty \tag{4}$$

We will further use a no dimensional form $r = \frac{r}{R}$.

Solution of the boundary problem

Linearly independent solutions of equation (5) which correspond to static problem of deflection ($\omega = 0$), and are created by the following four roots (JAROSZEWICZ, ZORYJ 2005, 2006):

$$L[u] = 0 \tag{5}$$

$$u_1 = 1, \quad u_2 = r^{\mu_1}, \quad u_3 = r^{\mu_2}, \quad u_4 = r^{-1}$$
$$\mu_1 = -\frac{1}{2} + \sqrt{\Delta}, \quad \mu_2 = -\frac{1}{2} + \sqrt{\Delta}, \quad \Delta = \frac{1}{4} + q, \quad q = 3(1 - \nu) \tag{6}$$

The solution of boundary problem (2.2), which is necessary for further consideration, is constructed in the form of the following power series (JAROSZEWICZ, ZORYJ 2005, 2006):

$$S_j = s_{j0} + ps_{j1} + p^2 s_{j2} + ..., j = 1,2 \tag{7}$$

$$s_{jk} = \int_0^r K_0(r,\tau) \cdot \tau^{-2} \cdot s_{j,k-1}(\tau)dt \tag{8}$$

$$s_{10} = 1, s_{20} = r^{\mu_1} \tag{9}$$

$K_0(r,\tau)$ – Cauchy's function corresponds to operator $L_0[u]$.

The function $K_0(r,\tau)$, as presented in literature (JAROSZEWICZ, ZORYJ 2005), can be determined by means of a formula:

$$K_0(r,\tau) = \frac{1}{2q\sqrt{\Delta}} (r^{-\mu_1}\tau^{\mu_2+4} - r^{-\mu_2}\tau^{\mu_1+4}) + \frac{1}{q} (r^{-1}\tau^4 - \tau^3) \tag{10}$$

After considering recurrent model of formulas (3.3) the parametre α was introduced in integrals τ^α and S_{j1} ($j = 1,2$) values were calculated on the base of formulas (7, 8, and 9):

$$S_{j1} = \frac{1}{2q\sqrt{\Delta}} \left[(J_1 - J_2) + \frac{1}{q} (J_3 - J_4)\right] \tag{11}$$

$$J_1 = r^{\mu_1}\int_0^r \tau^{2+\alpha+\mu_1} d\tau = \frac{r^{2+\alpha}}{3 + \mu_2 + \alpha}$$
$$J_2 = r^{\mu_2}\int_0^r \tau^{2+\alpha+\mu_1} d\tau = \frac{r^{2+\alpha}}{3 + \mu_1 + \alpha} \tag{12}$$

$$J_3 = r^{-1}\int_0^r \tau^{2+\alpha} d\tau = \frac{1}{3 + \alpha} r^{2+\alpha}, \quad (\alpha \neq -3)$$

$$J_4 = \int_0^r \tau^{(1+\alpha)} d\tau = \frac{1}{3 + \alpha} r^{2+\alpha}, \quad (\alpha \neq -2) \tag{12}$$

Flowing limitations ($\alpha \neq -3$) and ($\alpha \neq -2$) result from indeterminacy of expressions (12), received by mean of integration.

Integration (12) is conducted with respect to τ; r denotes variable limit of integration, therefore we receive $K_0(r, \tau)$.

The following formula has been obtained on the base of (11) and with consideration of (12):

$$S_{j1} = \frac{1 \cdot r^{2+\alpha}}{(3\nu + 3 + 5\alpha + \alpha^2)(6 + 5\alpha + \alpha^2)} \tag{13}$$

However, when $j = 1$ it is necessary to assume $\alpha = 0$, because ($s_{10} = u_1 = r^0$), and for $j = 2$ it is $\alpha = \mu_1$, because $s_{20} = u_2 = r^{\mu_1}$ which yields the following:

$$s_{11} = \frac{r^2}{(1 + \nu) \cdot 18}, \quad s_{21} = \frac{r^{2+\mu_1}}{(3\nu + A_{21})(3 + A_{21}}, \quad A_{21} = 3 + 5\mu_1 + \mu_1^2 \tag{14}$$

Analogically, when determining s_{12} and s_{22} and also applying (3.8) $\alpha = 2$ and $\alpha = 2 + \mu_1$ and so on, are introduced respectively. As a result the following formulas can be obtained:

$$S_1 = 1 + pr^2 a_{11} + p^2 r^4 a_{12} + ...,$$

$$S_2 = r^{\mu_1} + pr^2 + \mu_{1\alpha_{21}} + p^2 r^4 + \mu_{1\alpha_{22}} + ... \tag{15}$$

And solution limited in measure zero:

$$u = C_1 S_1(r) + C_2 S_2(r) \tag{16}$$

$C_1 S_2$ – arbitrary constants.

In formulas above:

$$a_{11} = (3\nu + 3)^{-1}6^{-1}, \quad a_{21} = [(3\nu + A_{21})(3\nu + A_{21})]^{-1} \tag{17}$$

The characteristic equation and calculation results of simplest estimator of fundamental frequency

Substitution of (13) to boundary conditions (2) resulted in a system of two homogenous algebraic equations in relation to constants C_1 and C_2. By equalling system determinant to zero, the frequency equation, also known as a characteristic equation is derived. The ultimate form of this equation is presented as follows:

$$\begin{vmatrix} S_1 & S_2 \\ S_1' & S_2' \end{vmatrix}_{|r=R} = R^{\mu_1-1}\{\mu_1 + pR^2[\mu_1 a_{11}] - 2a_{11} + (\mu_1 + 2)\, a_{21}\} = 0 \tag{18}$$

$$S_1' = 2pra_{11}, \quad S_2' = \mu_1 r^{\mu_1-1} + (\mu_1 + 2)pr^{\mu_1+1}a_{21}$$

In this way, reduced multiplication R^{μ_1-1} gave an equation which allows calculating the simplest lower estimator of base frequency:

$$\mu_1 - pR^2[(2 - \mu_1)a_{11} - (\mu + 2)a_{21}] = 0 \tag{19}$$

Table 1 summarises forms and calculation results of the simplest estimator of base frequency coefficient for a few selected values of Poisson's number.

Table 1

Derivation of the formula for fundamental frequency of characteristic equation (19)

Material	ν	μ_1	a_1	a_{21}	Equation (17)	γ_A
Glass	0	1.3	0.056	0.0063	$1.3 - 0.0184pR^2 = 0$	8.41
Steel	$^1/_3$	1	0.042	0.0083	$1 - 0.01(6)pR^2 = 0$	7.75
Rubber	$^1/_2$	0.8	0.037	0.0103	$0.8 - 0.0156pR^2 = 0$	7.16

In Table 1 a well known Dunkerley's formula was applied to estimate fundamental frequency:

$$\gamma_A = \frac{1}{\sqrt{a_1}} \tag{20}$$

Where a_1 is the first term of characteristic series (21)

$$1 - a_1\lambda^2 + a_2\lambda^4 - ... = 0 \tag{21}$$

Hence values of base frequency (equation 17) with deficiency in respect to appropriate values ν can be determined by applying the following formulas:

$$\mu_1 - pR^2[(2 - \mu_1)a_{11} - (\mu + 2)a_{21}] = 0 \tag{22}$$

On the base of above formulas (22) a general formula for any given values of Poisson's ratio $\nu \in (0;0;5)$ was proposed. Incomplete fundamental frequency of membrane plate clamped at the edge with linearly variable thickness can be calculated by using a simple formula:

$$\omega_1 \approx 2.4 \cdot \frac{h_0}{R^2} \cdot \sqrt{\frac{E}{\rho}} \tag{23}$$

where:

$$\mu_1 = 1, (\nu = {}^1/_3)$$

Considering the formulas (16–19) the first approximation (accurate to p, hence to the square of frequency) has been gained, which takes the following form:

$$1 - a_1 pR^a + a_2 p^2 R^{2a} - ... = 0 \tag{24}$$

where

$$a_1 = -[A_1(1 - a) + B_1(1 + a)] \tag{25}$$

$$a_2 = [A_1B_1\ A_2(1 - 2a) + B_2(1 + 2a)] \tag{26}$$

Taking into consideration the equation:

$$pR^a = \frac{\rho h_0}{D_0} R^4\omega^2 \tag{27}$$

Neglecting second terms of series a2 it can be transformed (3.16) to the following form:

$$1 - a_1 \frac{\rho h_0}{D_0} R^4 \omega^2 = 0 \tag{28}$$

Remarks:
For $m = 3$, $\nu = {}^1/_3$ we have obtained $A_1 = (24)^{-1}$, $A_2 = (24 \cdot 360)^{-1}$, $B_1 = (120)^{-1}$, B_2 $(120 \cdot 840)^{-1}$ (JAROSZEWICZ, ZORYJ 2005)

Results of calculations of fundamental natural frequency by mean described in this paper methods

Take into account the series (24) and known Bernstein-Kieropian's estimators with the following form (BERSTEIN, KIEROPIAN 1960) can be applied.

$$(a_1^2 - 2a_2)^{-\frac{1}{4}} < \gamma < \sqrt{2} \cdot (a_1 + \sqrt{a_1^2 - 4a_2})^{-\frac{1}{2}} \tag{29}$$

Coefficients of the series a_1, dependent on can be constructed on the basis of (25). Coefficient $a1$ scrutinized in previous work, where exact formula has been constructed.

$$a_1 = \frac{9^4 \cdot 3 \cdot 10}{[(6 - m) \cdot (6 + m) \cdot (9 + m) \cdot (12 + m)]} \tag{30}$$

Formulas (30), (31) are in accordance with the values quoted in the above remarks.

Obviously first of solution (25) and second derivative of solutions (26) for $a = 0$ $(m = 6)$ are independent of r, in this case coefficients lose sense for $m \to 6$, we have $B_1 \to \infty$, $B_2 \to \infty$, $A_1 \to \infty$ for we have $m \to 4.5$ $(a \to 1)$ and $A_2 \to \infty$ for $m \to 5.25$ $(a \to 0.5)$.

To develop formula (26) for in a similar form formula in following form should be present:

$$a_2 = \frac{3^9 \cdot 5}{2 \cdot (12+m) \cdot (6+m) \cdot (18-m) \cdot (21-m) \cdot (24-m) \cdot (6-m)^2 \cdot (9+m)^2} \tag{31}$$

Example results of calculation for changed cases $2 < m < 6$ and $\frac{1}{\nu} = m$ obtain on the basis of formulas (21), (29) present on the Table 2, where mean arithmetic value of basic frequency coefficient for $m = 3$, $\nu = {}^1/_3$ is:

$$\gamma_0 = \frac{1}{2}(\gamma_- + \gamma_+) = \frac{1}{2}(8.65.05 + 8.8429) = 8.7467$$

the basic frequency can be found (JAROSZEWICZ et al. 2004)

$$\omega = \gamma \cdot \frac{1}{R^2}\sqrt{\frac{D_0}{\rho h_0}} \equiv \gamma \frac{h_0}{R^2}\sqrt{E[12\rho(1 - \nu^2)]^{-1}} \tag{32}$$

Table 2

The fundamental frequency of the plate with nonlinearly variable thickness

No.	Material	ν	Coefficient m	D_0 [Nm]	Masses [kg]	Average of bilateral estimator γ_0	Frequency of the bilateral estimator f[Hz]	Frequency in the FEM analysis f[Hz]	Difference Δ %
1.	titanium	0.36	2.78	9778	6.093	8.697682	322.09 Hz	330.19 Hz	2.45
2.	steel	0.27	3.7	18560	9.834	8.90738	340.23 Hz	347.15 Hz	1.95
3.	zinc	0.25	4.0	10531	8.410	9.356813	286.75 Hz	267.35 Hz	-7.26
4.	concrete	0.17	5.9	1716	1.983	unknown	unknown	211.76 Hz	unknown

Source: JAROSZEWICZ et al. (2008).

It should be noticed that in the case of a constant thickness plate ($m = 0$), the multiplier $\gamma = \gamma_0 = (3.196)^2 = 10.2122$ is independent from ν (VASYLENKO, OLEKSIEJCUK 2004, KOVALENKO 1959). So the ratio of coefficients:

Summary

Deriving of the above mentioned formulas for the Cauchy's functions, as well as fundamental systems of function operator $L_o[u]$ allows to study the convergence problem (velocity of convergence) of solutions of equation (2) in form of power series in respect to parameters of frequencies, depending on values of parameters m and ν.

With the present influence functions of operator $L_o[u]$ corresponding solutions and the use them for any given physically justified values of parameters m and ν ($m \in -\infty, + \infty$); $\nu \in (0;0.5)$) can be consequently determined, when the exact solutions are unknown on the base of general form of Cauchy's function (10). On the basis of quoted solutions, simple engineering formulas for frequencies estimators of circular plates which are characterized by variable parameters distribution can be derived and limits of their application can be identified. The bilateral estimators calculated using four first

elements of the series, allow to credibly observe an influence material constants: Young modulus – E, Poisson ratio – ν density – ρ on the frequencies on axisymmetrical vibrations of circular plates whose thickness or rigidity changes along the radius according to the power function.

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