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The Use of Idea of Time Scaling to Reconstruction of Measuring System Input Signal

Abstract

The way of reconstruction of measuring system input signal has been presented. The measuring system output signal is followed by classic feedback system containing "associated model" of measuring system. This model is represented with structure defined by state variable scheme. If error of follow-up action is small, then output of associated model and its accessible derivatives can be used to reconstruction of measuring system input. Due to virtual operation of slowing down the measuring system output signal and multi-option possibility of choice of form of "associated model" one can substantially improve the accuracy of follow-up action and input signal recovery. The examples confirming advantageous properties of proposed algorithm are attached.

Keywords: measuring systems, reconstruction of measuring system input signal, time-scaling of signals.

1. Introduction

There are numerous concepts for processing of measuring system output in order to reconstruct its input - see [1, 2, 3, 4, 5]. These concepts are based on algorithms for solution of inverse problem. The widely known algorithm uses the properties of classic feedback system where measuring system output is treated as its reference signal and measuring system accurate model plays the role of "plant" in feedback loop. If controller in feedback loop reduces the error of follow-up action to small values then input signal of perfect model of measuring system ("plant") can be treated as approximation of measuring system input signal. This is difficult task to find the author of this concept. This idea was already mentioned in "historical" book [6]. The currently proposed approach simplifies the procedure of controller design and yields excellent performance of follow-up action. This advantages were obtained due to introduction of time-scaling operation to algorithm processing the measuring system output signal. Some additional benefits were obtained due to application of structure of "associated model" of measuring system in form of state variables scheme. Those benefits are result of multi-option possibility of choice of form of "associated model". That is why "easy to control" form can be chosen. To apply the proposed algorithm one has to know mathematical model, i.e. linear or nonlinear ordinary differential equation (ODE) describing the measuring system input-output relation. This condition does not seem to be restrictive one because parameters of contemporary sensors, including those influencing dynamic properties (time constants, gains, delays, etc.) are presented in technical data or can be identified. It is worth emphasizing that measuring sensors and systems described by ODE can be treated as members of important for measurement science class of devices [7, 8, 9]. For example, temperature sensors are often described by ODE: $aY'+Y=kU$, $aY''+bY'+cY=kU$, $aY''+'bY'+cY=dU'+kU$, where a,b,c,d,k - coefficients, U - input signal, Y - output signal.

2. Idea of method

Let us assume, that measuring system containing sensor/transducer and auxiliary equipment (amplifier, pre-filter, etc.) can be described by state equations (1), where: $x_1,\dots,x_n$ – components of state vector $\boldsymbol{x}$, $f_1,\dots,f_n$, f – static linear or nonlinear functions, u - model input, y - model output. If $f(..)=F(..)$, $f_l(..)=F_l(..)$ for $l=1,..,n$, then model (2) is identical to measuring system representation (1).

$$
\begin{aligned}
X_1' &= F_1(X_1, X_2,\dots,X_n, U)\\
X_2' &= F_2(X_1, X_2,\dots,X_n, U)\\
&\dots\dots\\
X_n' &= F_n(X_1, X_2,\dots,X_n, U)\\
Y &= F(X_1, X_2,\dots,X_n)
\end{aligned}
\tag{1}
$$

where: $X_1,\dots,X_n$ – components of state vector $\boldsymbol{X}$, $F,F_1,..,F_n$ - static linear or nonlinear functions, U – input signal, Y - output signal. Let us introduce the "associated model" of system (1):

$$
\begin{aligned}
x_1' &= f_1(x_1, x_2,\dots,x_n, u)\\
x_2' &= f_2(x_1, x_2,\dots,x_n, u)\\
&\dots\dots\\
x_n' &= f_n(x_1, x_2,\dots,x_n, u)\\
y &= f(x_1, x_2,\dots,x_n)
\end{aligned}
\tag{2}
$$

Let us assume that form of (1) allows to express the input-output relation of measuring system (1) by differential equation:

$$
\frac{d^nY}{dt^n} + g\left(Y,\frac{dY}{dt},\dots,\frac{d^{n-1}Y}{dt^{n-1}}\right) = h\left(U,\frac{dU}{dt},\dots,\frac{d^{k-1}U}{dt^{k-1}}\right) \tag{3}
$$

where $g(.)$,$h(.)$ – static linear or nonlinear functions, $k \le n$. Equations (1),(3) illustrate known property: the measurement system output signal differs from its input, i.e. $U(t)\neq Y(t)$. In order to restore $U(t)$ one has to execute the inversion of operation (3). This may occur as quite complicated task. The idea of well-known method of inversion [6] was based on properties of feedback system – see Fig. 1. To obtain high accuracy of reconstruction of $U(t)$ according to classic approach $u(t)\approx U(t)$ we should fulfil the obvious conditions: "associated model" in Fig.1 should represent highly accurate model of measuring system (1),(3) and controller should hold $e(t)\approx 0$. The first condition will be fulfilled if $f(..)=F(..)$, $f_l(..)=F_l(..)$ for $l=1,..,n$. The second condition can be fulfilled by implementation of advanced control algorithm.

During design of controller realizing the highly accurate "follow-up" action one can encounter serious difficulties, especially in case of "speedy" signals $Y(t)$. On the other hand, if $Y(t)$ is slowly altering then even simple control algorithms can make that $e(t)\approx 0$ [10]. Hence, we can propose to "slow down" A-times the feedback system reference signal putting the signal $Y(At)$ instead of original measuring system output $Y(t)$, where the time-scale coefficient $A>1$. We can organize calculations involved in process of modelling of feedback system operations using the "associated model" (see Fig. 1) in form of structure representing the state variable scheme defined by (2). For example, it can be most popular structure based on "chain of integrators". The use of "associated model" in form of structure imitating the state variable scheme enables the direct access to derivatives of $y(t)$ or possibility of forming them on the basis of components of state vector $\boldsymbol{x}(t)$. Thus, if response of feedback system $y(t)$ to slowed reference input $Y(At)$ fulfils condition $y(t)\approx Y(At)$, then

$$y(A^{-1}t) \cong Y(t), \quad A\frac{dy(A^{-1}t)}{dt} \cong \frac{dY(t)}{dt}$$
$$A^2\frac{d^2y(A^{-1}t)}{dt^2} \cong \frac{d^2Y(t)}{dt^2}, \quad A^3\frac{d^3y(A^{-1}t)}{dt^3} \cong \frac{d^3Y(t)}{dt^3} \qquad (4)$$

and so on. Owing to the access to derivatives of $y(t)$ the sum $S(t)$ (compare to formula (3)) can be created:

$$A^n\frac{d^ny}{dt^n} + g(y, A\frac{dy}{dt},..., A^{n-1}\frac{d^{n-1}y}{dt^{n-1}}) = S(t) \qquad (5)$$

Now we can „speed up" A-times signal $S(t)$ recalculating it to the form $S(t/A)$. If $y(t) \approx Y(At)$, then the left member of (3) can be approximated by $S(t/A)$, i.e.

$$\frac{d^nY}{dt^n} + g(Y, \frac{dY}{dt},..., \frac{d^{n-1}Y}{dt^{n-1}}) \cong S(\frac{1}{A}t) \qquad (6)$$

In order to obtain approximation $\tilde{U}(t)$ of input signal $U(t)$ we should solve the differential equation (see (3)):

$$h(\tilde{U}, \frac{d\tilde{U}}{dt},..., \frac{d^{k-1}\tilde{U}}{dt^{k-1}}) = S(\frac{1}{A}t) \qquad (7)$$

If the right hand member of (3) does not depend on derivatives of $U(t)$ then necessity of solving of differential equation (7) disappears since $\tilde{U}(t)$ can be obtained directly as result of static operation. Of course, the stability of equation (7) has to be treated as necessary condition deciding on possibility of application of proposed method. Considering dynamical properties of real sensors/transducers [8,9] we can state that in most cases the above stability condition will not be obstruction.

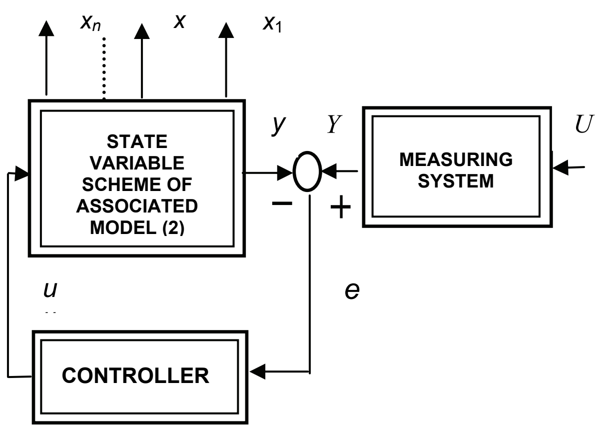

Fig. 1. Idea of recovery of U: if $e(t)\approx 0$ and "associated model" represents measuring system with high accuracy then $u(t) \approx U(t)$. If "associated model" does not represent measuring system then state variables $x_1, x_2, \ldots, x_n$ are used to generation of $S(t)$ defined by (5)

The accuracy of follow up action organized according to scheme of feedback system in Fig. 1 influences directly the accuracy of input reconstruction. Due to operation of slowing the measuring system output $Y(t)$ we can achieve high accuracy of follow up action, i.e. $e(t) \approx 0$. If accuracy of follow up action yielded by classic control algorithms, such as these based on PID approach, is not sufficient then more advanced control algorithms, those processing state variables of "associated model can be used. The extensive overview of basic and advanced control algorithms for linear and non-linear, continuous-time and discrete-time systems can be found in [11]. The "associated model" (2) can be chosen in "easy to control" form which can be different than measuring system representations (1),(3). Furthermore, the structure in Fig. 1 has to be treated as scheme for design of calculation procedures. That is why limitations imposed on design of control laws for real, "physical" plants (like values of acceptable amplitudes of input $u(t)$ and its derivatives, presence of saturations, etc.) can be neglected. The mentioned properties can be efficiently exploited in order to increase the final accuracy of reconstruction of $U(t)$. The *Examples* 1,2 present the consecutive steps leading to reconstruction of measuring system input signal.

Example 1. Let the hypothetical measuring system (Fig. 1) be represented by transmittance:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \qquad (8)$$

Hence n=3, $F_1(.)=X_2$, $F_2(.)=X_3$, $F_3(.)=-X_1-X_2-3X_3+U$, $F(.)=X_1$ (compare to (1)). Let us define the "associated model" (2) by relations: n=3, $f_1(.)=x_2$, $f_2(.)=x_3$, $f_3(.)=-x_1-2x_2-3x_3+u$, $y=x_1$. Putting those expressions to (1),(2) respectively we obtain identical representations of measuring system and its model (in general this condition is not necessary). The measuring system representation of type (3) is given by:

$$g(.) = Y + 2\frac{dY}{dt} + 3\frac{d^2Y}{dt^2}; \qquad h(.) = U \qquad (9)$$

The derivatives of y can be expressed by formulae:

$$y = x_1, \quad \frac{dy}{dt} = x_2, \quad \frac{d^2y}{dt^2} = x_3$$
$$\frac{d^3y}{dt^3} = -(x_1 + 2x_2 + 3x_3 - u) \qquad (10)$$

The slowed down, approximate representation of left–side member of (3) obtains the form (compare to (5)):

$$S(t) = y + 2A\frac{dy}{dt} + 3A^2\frac{d^2y}{dt^2} + A^3\frac{d^3y}{dt^3} \qquad (11)$$

Finally, the approximation of input signal $\tilde{U}(t)$ can be obtained as result of speeding the $S(t)$ up:

$$h(.) = \tilde{U}(t) = S(t/A) \qquad (12)$$

The exemplary results of simulations carried out in accordance with proposed reconstruction algorithm in case of "difficult" to reconstruction input signal of rectangular form are shown in Fig. 2. All experiments were carried out for identical parameters of feedback loop elements, i.e. PID controller was tuned to the transfer function $(3+0.8s^{-1}+5s)$ and exact state space model of (8) was used as "associated model". In Fig. 2 (upper figure) one can observe input signal U, output of measuring system Y, output of "associated model" y and its input $u(t)$ denoted by U_1. In this case $U_1=u(t)=\tilde{U}(t)$ because A=1 (the classic approach has been applied). Beneath, (figure placed in the middle) we can observe the slowed down (10-times, A=10) output Y of measuring system, output of "associated model" y and result of forming of signal $U_2=S(t)$ according to (11). The signal $U_2=S(t)$ represents the slowed down approximation of input U. In the bottom figure the input signal U and its approximations, i.e. result of classic approach U_1 and result of improved approach U_2 are shown. Comparing those results we observe, that accuracy of follow-up action is much better in case of proposed way of signal processing. That is why the "improved" reconstruction of input signal U_2 is much better than "classic" reconstruction U_1.

Example 2. Let the measuring system (Fig. 1) be represented by transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{s^3 + 3s^2 + 2s + 1} \qquad (13)$$

Hence $n=3$, $F_1(.)=X_2$, $F_2(.)=X_3$, $F_3(.)=-X_1-2X_2-3X_3+U$, $F(.)=X_1+X_2$ (see (1)). Let "associated model" be identical to this in Example 1. For given assumptions the representations of measuring system and "associated model" are different ($F(.)\neq f(.)$). The measuring system model of type (3) is given by:

$$g(.) = Y + 2\frac{dY}{dt} + 3\frac{d^2 Y}{dt^2}; \qquad h(.) = U + \frac{dU}{dt} \tag{14}$$

Hence, for slowed down reference signal $Y(At)$ the equation (5) determining the slowed down, approximate representation of left – side member of (3) obtains identical to (11) form:

$$S(t) = y + 2A\frac{dy}{dt} + 3A^2\frac{d^2 y}{dt^2} + A^3\frac{d^3 y}{dt^3} \tag{15}$$

Finally, the approximation of input signal $\tilde{U}(t)$ can be obtained as solution of first order differential equation:

$$h(.) = \tilde{U} + \frac{d\tilde{U}}{dt} = S\left(\frac{t}{A}\right) \tag{16}$$

where $S(t/A)$ represents speeded signal $S(t)$. The solution of (16) can be obtained as response of first order inertia system to excitation $S(t/A)$. The exemplary results of simulations carried out in accordance with proposed reconstruction algorithm for input signal $U(t)$ of sine wave form are shown in Fig. 3.

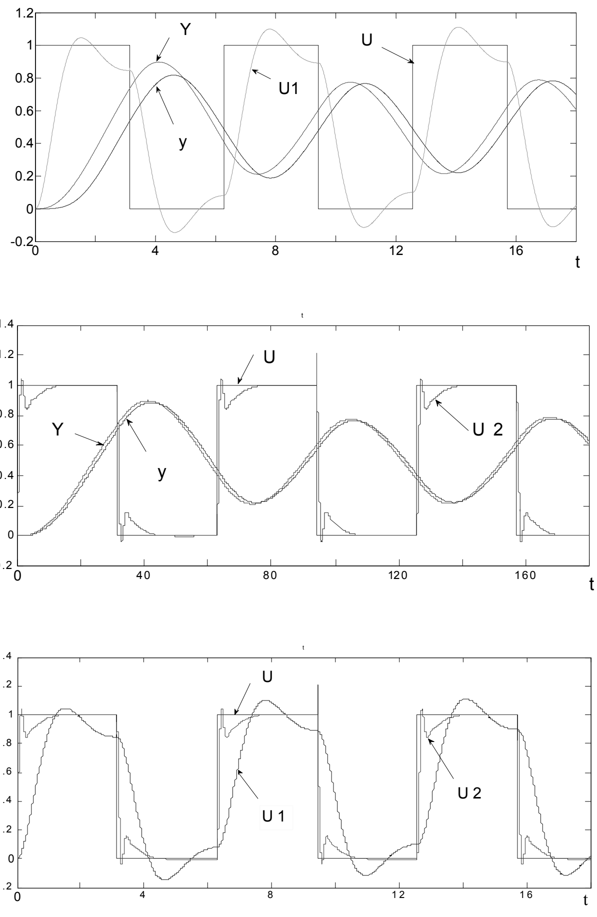

Fig. 2. The reconstruction of measuring system input signal: U-input signal, Y-output of measuring system (8), y – output of "associated model" (see Fig. 1), U_1- result of "classic" reconstruction of input signal, U_2–result of improved method of reconstruction. Note, that slowed down representations of signals U, U_2, Y, y are presented in figure placed in the middle

The given above presentation of idea and exemplary simulation results "fits" conceptually to "off-line" way of calculation. The use of slowed down output of measuring system $Y(At)$ allows to design feedback loop controller with minimal effort. This "primary" solution is sufficient for off-line reconstruction of input signal but it can be easily utilized for real-time mode. If accuracy like that attained for slowed down signal $Y(At)$ is needed for real-time reconstruction, then elements of feedback system have to be modified. If we put real-time representation of signal $Y(t)$ to input of feedback system previously calculated for $Y(At)$ then accuracy of follow up action will be poor. To return the previous good accuracy we must speed up A-times operation of feedback system. It can be easily done, namely, we have to substitute all operations of integration ($1/s$) in feedback loop with operations (A/s) and shorten A-times all time-constants of controller. Due to introduction of multipliers (gains) A to structure of feedback system (see Fig. 1) its operation is speeded up A-times and form of output $y(t)$ is conserved (because $y(t)$ can be treated as speeded up response of "primary" model of feedback system to reference signal $Y(At)$). Since feedback system is treated as computational structure, then we do not encounter problems connected with feasibility of this operation, like in case of real control tasks. Note, that typical limitations encountered during design of real control systems, such as saturations, the maximum values of amplitudes of signals and their derivatives, can be neglected in numerical reality created on the basis of state space model of feedback system in Fig. 1. Under the above conditions the "associated model" yields $y(t)$ and its derivatives with $A=1$. Nevertheless, the multipliers A have to be taken into consideration when excitation $S(t)$ for equation (7) is created (see Example 3).

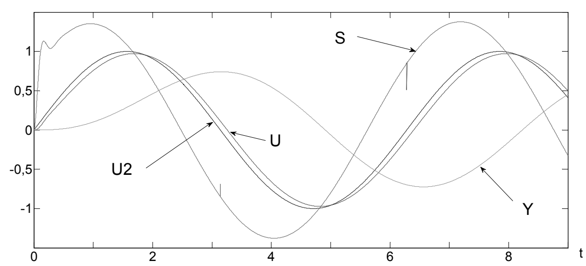

Fig. 3. The reconstruction of input signal in form of sine wave U: Y - output of measuring system, S - speeded signal $S(t)$, $U_2 = \tilde{U}(t)$ - result of reconstruction of U

Example 3. Let us adjust results presented in Example 2 to real-time way of reconstruction of input. To speed up operation of "associated model" described in Examples 1,2 one should introduce multipliers $A>1$ to its state variable scheme. The obtained structure is shown in Fig. 4. For $A=1$ structure in Fig.4 represents state variable scheme of "associated model" for assumptions in Examples 1,2. To speed up A-times operation of controller its primary time-constants have to be decreased A–times. If we put $A=10$ then both operations, those modifying state variable scheme of "associated model" and controller, allow to conserve the form of result of reconstruction $\tilde{U}(t)$, this shown in Fig. 3, however now $\tilde{U}(t)$ is generated in real-time mode, i.e. on the basis of response of feedback system to real-time reference signal $Y(t)$. To obtain real-time expressions for derivatives of y one has to take into consideration the multipliers A. For state variable scheme shown in Fig. 4 the following dependences hold:

$$y = \underline{x}_1, \quad \frac{dy}{dt} = A\,\underline{x}_2, \quad \frac{d^2 y}{dt^2} = A^2\,\underline{x}_3, \quad \frac{d^3 y}{dt^3} = A^3\,\underline{x}'_3 \tag{17}$$

where $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ - state variables of structure in Fig. 4. Hence:

$$S=\underline{x}_1+2A\underline{x}_2+3A^2\underline{x}_3+A^3\underline{x}_3'=$$
$$= y + 2A\frac{dy}{dt} + 3A^2\frac{d^2y}{dt^2} + A^3\frac{d^3y}{dt^3} \quad (18)$$

which is apparently identical to (15). Note, that S in (15) is created on the basis of slowed down response of feedback system while S in (18) is formed in real-time mode. Finally, the approximation of input signal $\tilde{U}(t)$ can be obtained as solution of differential equation (19) excited with signal (18):

$$\tilde{U} + \frac{d\tilde{U}}{dt} = S \quad (19)$$

The final result of approach discussed in Example 3 can be illustrated with identical forms of curves U, U_2, Y to those in Fig. 3.

To apply considered concept one will use undoubtedly the discrete-time representation of differential equations given above, i.e. the samples of signal $Y(t)$ will be processed and differential equations describing feedback system in Fig.1 will be substituted with respective difference equations. Note, that necessary digital calculations can be carried out on the basis of recursive formulae which process current and previous samples of $Y(t)$. This means, that in practice we do not need to slow down the calculation process waiting for consecutive samples of slowed down signal $Y(At)$. The discrete–time representations of $y(t)$ and its derivatives can be calculated on the basis of formulae created as if $Y(t)$ was slowed down. So, the consecutive samples of $\tilde{U}(t)$ will be generated with small delays in relation to referring values of input signal $U(t)$. The delays of samples of $\tilde{U}(t)$ in relation to referring samples of $Y(t)$ caused by time necessary for execution of calculations basing on recursive formulae will depend on computing performance of involved "hardware", sample time, order of difference equation (the higher order the bigger number of samples of $Y(t)$ necessary to calculation of high-order differences) and software solutions. Let us note, that delays between "delivery" of input data and access to results of calculations can be treated as common feature for all types of digital signal processing.

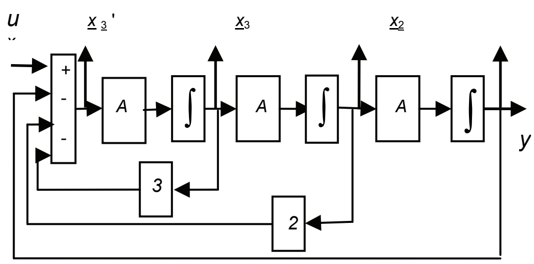

Fig. 4. The way of speeding the operation of "associated model" by means of multipliers (gains) $A>1$

3. Conclusions

1. The alternative way of use of classic method for reconstruction of measuring system input signal has been presented. The novelty lies in fact that introductory calculations of "associated model" response $y(t)$ and its derivatives proceed on the basis of formulae as if measuring system output signal $Y(t)$ was slowed down.

2. The use of feedback system containing structure of "associated model" in form defined by state variables scheme causes that referring calculations yield "accessible" $Y(t)$ and its derivatives. They can be utilized as components of $S(t)$ which can be treated as virtually slowed right-side member of equation (3). Thus, recalculating $S(t)$ as if was speeded up A-times one can obtain excitation of equation (7). Its solution approximates input $U(t)$.

3. The accuracy of $\tilde{U}(t)$ depends on accuracy of follow-up action realized in feedback system. The introductory control algorithms are synthesized for slowed down reference signal Y. That is why relatively simple algorithms for controller can guarantee sufficient accuracy. Additionally, to fulfil the condition $y(t)\approx Y(At)$ one can choose the "associated model" in "easy to control" form. It can be different than measuring system accurate model (Examples 2,3).

4. During choice of input-output relation for "associated model" and its state variables scheme one should take into account the required quality of follow-up action as well as possibility of generating of derivatives of $y(t)$. The mentioned properties constitute competitive advantages of proposed method in relation to classic one where the accurate model of measuring system has to be put as "associated model".

5. The feedback system in Fig. 1 should be treated as structure for creating of calculation procedures. That is why typical limitations referring to controller design for real control systems can be neglected. The continuous-time algorithm for input recovery can be transformed to discrete-time version. The introduction of time-scale A to reconstruction concept can yield possibility of operation in real-time mode, where samples of $\tilde{U}(t)$ will be slightly delayed in relation to referring values of $U(t)$.

The considerations in current paper were concentrated on presentation of general idea of method. Basing on this idea one can obtain many particular solutions depending on required accuracy of input reconstruction, class of input signals, presence of disturbances as well as invention of measuring system designer. That is why important problems for particular implementations (accuracy of identification of measuring system model, optimization of control law, choice of sample time and method for discretization, pre-filtering of disturbances, etc.) have not been discussed. All those agents influence the final accuracy of input signal reconstruction. Nevertheless, the usefulness of presented concept has been undoubtedly confirmed by results of attached "non-optimized" examples.

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