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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

A NOTE ON THE GENERAL MOMENT PROBLEM

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Abstract. In this note we show that given an indeterminate Hamburger moment sequence, it is possible to perturb the first moment in such way that the obtained sequence remains an indeterminate Hamburger moment sequence. As a consequence we prove that every sequence of real numbers is a moment sequence for a signed discrete measure supported in $\mathbb{R}_+$.

Keywords: general moment problem, charge sequences, atomic measure.

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1. INTRODUCTION

The moment problem is related to numerous fields of mathematics and others areas of applied science. It is widely investigated in the literature and its formulation can be started as follows. Let K be a closed subset of $\mathbb{R}$ and let $\{\gamma_n\}_{0\leq n\leq p}$ $(p \leq +\infty)$ be a sequence of $\mathbb{R}$. The associated K-moment problem consists of finding a positive Borel measure μ such that

$$\gamma_n = \int\limits_K t^n d\mu(t) \quad \text{for every } \ n\,(0\leq n\leq p) \quad \text{and} \quad \operatorname{supp}(\mu)\subset K. \tag{1.1}$$

For $K = [a, b]$, $\mathbb{R}$ or $[0, +\infty)$ respectively, the problem (1.1) is known in the literature as the *Hausdorff*, the *Hamburger* and the *Stieltjes* moment problem respectively. On the other hand the K-moment problem is called *full* when $p = +\infty$, and is *truncated* in the case where $p < +\infty$. A positive Borel measure μ solution of the problem (1.1) is called a *representing measure* of $\{\gamma_n\}_{0\leq n\leq p}$. In the case where the problem owns a solution, the sequence is said to be a moment sequence. The methods for solving the full K-moment problem vary depending on the geometric nature of K and also vary from the full to the truncated case.

For example, if we consider the Hamburger moment problem, one approach consists in writing positive polynomials as sum of squares and extending the Riesz functional

$L(X^n) = \gamma_n$ to a nonnegative functional on some adequate Hilbert space of functions. This leads, thanks to representation theorems, to the necessary and sufficient condition on γ to own a solution, that the Hankel matrix $(\gamma_{i+j})_{i,j}$ is positive semi-definite.

In the truncated case, the positive condition is still necessary, but here the strategy of solving is totally algebraic and there is no need of extension theorems. The existence of solutions in this case depends essentially on initial data and is equivalent to the existence of a solutions with finite support.

Historically, the moment problem was defined by T.J. Stieltjes in [12], for $K = \mathbb{R}_+$. His main result is

Theorem 1.1 (Stieltjes, 1894). *A sequence $\gamma = (\gamma_i)_{i\geq 0}$ is a $\mathbb{R}_+$-moment sequence, if and only if both Hankel matrices $(\gamma_{i+j})_{i,j}$ and $(\gamma_{i+j+1})_{i,j}$ are positives semi-definite.*

In the same paper, Stieltjes introduced the notion of determinacy for the moment problem. More precisely, the problem is said to be K-determinate if there is a unique measure solution for the K-moment problem, and K-indeterminate if there are many solutions. An example of $\mathbb{R}_+$-indeterminate moment sequence was provided by Stieltjes himself with the sequence $(e^{n^2/2})_n$, associated to the log-normal distribution. For further information about the classical moment problem we refer to [1, 11].

Little attention, however, has been given to the possibility of finding a signed measure of finite variation, which satisfies equation (1.1), called usually the general moment problem and the associated sequences are called charge moment sequences. To avoid confusion, such measure will be called general in the sequel and the associated moment problem will be called the general moment problem.

The first known result is this direction goes back to 1938 when G. Pólya [10], proved that every sequence is a Hamburger charge moment sequence. In 1939 R.P. Boas [3] improved this result by proving that every sequence is a Stieltjes charge moment sequence. Also, J. Duran proved in [4] a more precise result. The measure solution of equation (1.1) can be chosen with density in the Schwartz's space $\mathcal{S}$.

Recently, it has been observed that the charge moment problem has many useful applications in physics. Particularly, the mathematical results and techniques of the charge moment problem can be applied successfully in the theory of renormalized or effective iteration in quantum many-body physics, in nuclear physics, and in solid state physics [7]. It is also applicable in optimization problems and is closely related to the maximal entropy, see [8].

The main objective of this note is to prove that for every sequence of real numbers, we can find an atomic solution for the charge moment problem, supported in $\mathbb{R}_+$. In particular, given any sequence of real numbers (γ_n), there exists two indeterminate Stieltjes moment sequences, (γ_n^+) and (γ_n^-), such that $\gamma_n = \gamma_n^+ - \gamma_n^-$, for every $n \geq 0$. Such solutions are more interesting for applications in various branches in science.

This note is organized as follows. The first section is started by a quick review of classical notion associated to the theory of orthogonal polynomials up to Hamburger indeterminacy criterion [1], we prove then that we can perturb the first moment of every indeterminate Hamburger moment sequence and remains indeterminate. The proof of this result is slightly different of [5] where the question of perturbation of finite number of moments is treated. In the last section we recall some ideas from the

Stieltjes monograph [12] on indeterminate moment sequence, and use them to get the main result about general moment problem.

In the whole text δ_a designs the Dirac measure on a.

2. INDETERMINATE HAMBURGER MOMENT SEQUENCE

The main ingredient in our paper is Hamburger's theorem that is stated as follows:

Theorem 2.1 (Hamburger, 1920)**.** *A sequence of real numbers γ is a $\mathbb{R}$-moment sequence, if and only if the Hankel matrix $(\gamma_{i+j})_{i,j}$ is positive semi-definite.*

In particular we have $\det((\gamma_{i+j})_{0\le i,j\le p}) \ge 0$, for every $p \ge 0$ and since the sequence $\gamma'' := (\gamma_{n+2})_n$ is also a Hamburger moment sequence (associated to the positive measure $(d\mu' = x^2 d\mu)$) we get $\det((\gamma_{i+j+2})_{0\le i,j\le p}) \ge 0$, and then $\det((\gamma_{i+j+2n})_{0\le i,j\le p}) \ge 0$ for any $n, p \ge 0$.

For a non-negative integer k, we adopt the next notations in the sequel:

$$\Delta_k(\gamma) = \det((\gamma_{i+j})_{i,j\le k}),$$

and

$$\Delta'_k(\gamma) = \det((\gamma_{4+i+j})_{i,j\le k-2}).$$

For a nondegenerate moment sequence $(\det(\gamma_{i+j})_{0\le i,j\le p}) > 0$ for every $p \ge 0$ associated to a measure μ, we introduce the next two families of important polynomials:

– orthogonal polynomials $(p_k)_k$ given by

$$p_k(x) = \frac{1}{\sqrt{\Delta_k(\gamma)\Delta_{k+1}(\gamma)}} \begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \ldots & \vdots \\ \gamma_{k-1} & \gamma_k & \cdots & \gamma_{2k-1} \\ 1 & x & \ldots & x^k \end{vmatrix};$$

in particular, this family verifies the orthogonality conditions:

$$\int_{\mathbb{R}} p_n(t)p_k(t)d\mu(t) = \delta_{n,k}, \quad n, k \in \mathbb{N};$$

– polynomials of the second kind $(q_k)_k$:

$$q_k(x) = \int_{\mathbb{R}} \frac{p_k(x) - p_k(t)}{x - t} d\mu(t).$$

We also consider the reproducing kernel of degree k given by

$$K_k(x, y) = \sum_{j\le k} p_j(x)p_j(y) = -\frac{1}{\Delta_k(\gamma)} \begin{vmatrix} 0 & 1 & x & \ldots & x^k \\ 1 & \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ y & \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ y^k & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{vmatrix}. \tag{2.1}$$

For $x = y$, we put $\rho_k(x) = (K_k(x,x))^{-1}$. The sequence $(\rho_k(x))_k$ is clearly decreasing. Moreover, $\rho_k(x)$ is the minimum of L^2-norm on the family of polynomials of degree less than k which equal to 1 on x (see [1, 11]). Setting $\rho(x) = \lim_k \rho_k(x) \geq 0$, we state an important criterion for indeterminacy given by Hamburger [1, 9]:

Theorem 2.2 (Hamburger, 1921)**.** *Let γ be a Hamburger moment sequence. The following are equivalent:*

1. *The Hamburger moment problem is indeterminate,*
2. *$\rho(x) \neq 0$ for all $x \in \mathbb{R}$,*
3. *$\rho(0)\rho''(0) \neq 0$, where $\rho''(x)$ is associated to the sequence γ'',*
4. *$\rho(0) \neq 0$ and $\sum_{k\geq 0} q_k^2(0) < \infty$.*

We begin by the next technical observation:

Lemma 2.3. *Let γ be a Hamburger moment sequence, then the sequence*

$$\left(\frac{\Delta_k(\gamma)}{\det((\gamma_{2+i+j})_{i,j\leq k-1})}\right)_k$$

decreases to $\rho(0)$. In particular, for any moment sequence γ the sequence

$$\left(\frac{\Delta_k(\gamma)}{\Delta'_k(\gamma)}\right)_k$$

decreases to $\rho(0)\rho''(0)$.

Proof. We obtain our assumption by considering the determinant formula for the reproducing kernel $K_k(z,w)$ associated to our sequence (see [1]), as follows:

$$\sum_{j=0}^{k} p_j^2(0) = K_k(0,0) = -\frac{1}{\Delta_k(\gamma)} \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & \gamma_0 & \gamma_1 & \dots & \gamma_k \\ 0 & \gamma_1 & \gamma_2 & \dots & \gamma_{k+1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \gamma_k & \gamma_{k+1} & \dots & \gamma_{2k} \end{vmatrix} = \frac{\det((\gamma_{2+i+j})_{i,j\leq k-1})}{\Delta_k(\gamma)}.$$

□

We deduce the next corollary:

Corollary 2.4. *Let γ be a sequence of real number. Then γ is an indeterminate Hamburger moment sequence if and only if for every $n \in \mathbb{N}$ $\Delta_n(\gamma) > 0$, and* $\lim_{n\to\infty} \frac{\Delta_n(\gamma)}{\Delta'_n(\gamma)} \neq 0$.

3. PERTURBATION OF HAMBURGER MOMENT SEQUENCES

It is not difficult to see that if γ is an indeterminate Hamburger moment sequence γ, then there exist t, such that the next first order perturbed moment sequence

$$\gamma_0 + t, \gamma_1, \gamma_2, \gamma_3, \dots$$

is also an indeterminate Hamburger moment sequence. Indeed, let μ^1 and μ^2 be two different positive Borel measures such that $\gamma_k(\mu^1) = \gamma_k(\mu^2) = \gamma$. Then,

$$\nu^i = \mu^i + t\delta_0, \quad i = 1, 2,$$

are two different solutions to perturbed moment problem $(\gamma_n + t\delta_{n,0})_n$.

The representing measures above, ν^1 and ν^2 are positive for $t \in (-\rho(0), 0)$ thanks to the next classical result:

Theorem 3.1. *For any solution μ of the Hamburger moment problem we have*

$$\mu(\{0\}) \leq \rho(0).$$

Furthermore, there is a unique solution μ_0 such that $\mu_0(\{0\}) = \rho(0)$.

Notice also that, if $t = \rho(0)$, then the moment problem $\gamma_0 - \rho(0), \gamma_1, \gamma_2, \gamma_3, \ldots$, is a determinate Hamburger moment problem by the same result.

Finally if $t < -\rho(0)$ the sequence $\gamma_0 + t, \gamma_1, \gamma_2, \gamma_3, \ldots$ is not a moment sequence, in fact if there is any measure solution ν for this moment problem, then $\nu - t\delta_0$ is a solution for the moment problem associated to γ, then:

$$[\nu - t\delta_0](\{0\}) = \nu(\{0\}) - t \geq -t > \rho(0)$$

which contradicts Theorem 3.1. In conclusion, we have,

Proposition 3.2. Let γ be an indeterminate Hamburger moment sequence, then

$$\gamma_0 + x, \gamma_1, \gamma_2, \gamma_3, \ldots$$

is an indeterminate Hamburger moment sequence if and only if $x \in (-\rho(0), +\infty)$.

Remark 3.3. The result remains true for any sequence such that $\rho(0) \neq 0$, in particular for any measure charging 0.

The problem of perturbing the second moment term is much more difficult. We have the following result:

Theorem 3.4. *Let γ be an indeterminate Hamburger moment sequence, then there is a constant $\delta > 0$ such that the sequence:*

$$\gamma_0, \gamma_1 + x, \gamma_2, \gamma_3, \ldots$$

remains an indeterminate Hamburger moment sequence, for every $|x| < \delta$.

Proof. We will prove that, there is a positive constant τ such that the moment sequence $s(t)$ is an indeterminate Hamburger moment sequence, where

$$s = s(t) = \begin{cases} s_n = \gamma_n & \text{if } n \neq 1, \\ s_1 = \gamma_1 + t. \end{cases}$$

For $k \geq 2$, let $f_k(t)$ be the function defined on $\mathbb{R}_+$ by

$$f_k(t) := \frac{\Delta_k(s)}{\Delta'_k(s)} = \frac{\Delta_k(s)}{\Delta'_k(\gamma)}$$

$$= \frac{1}{\Delta'_k(\gamma)} \begin{vmatrix} \gamma_0 & \gamma_1 + t & \gamma_2 & \dots & \gamma_k \\ \gamma_1 + t & \gamma_2 & \dots & \dots & \vdots \\ \gamma_2 & \gamma_3 & \dots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_k & \gamma_{k+1} & \ddots & \ddots & \gamma_{2k} \end{vmatrix} = -t^2 + \alpha_k t + \frac{\Delta_k(\gamma)}{\Delta'_k(\gamma)},$$

where

$$\alpha_k = f'_k(0) = -\frac{2}{\Delta'_k(\gamma)} \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \dots & \gamma_k \\ \gamma_3 & \gamma_4 & \dots & \dots & \gamma_{k+2} \\ \gamma_4 & \gamma_5 & \dots & \dots & \gamma_{k+3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_{k+1} & \gamma_{k+2} & \ddots & \ddots & \gamma_{2k} \end{vmatrix}.$$

Lemma 3.5.

$$\alpha_k = 2\frac{\Delta_k(\gamma)}{\Delta'_k(\gamma)} \sum_{j \leq k} p'_j(0) p_j(0).$$

In particular, the sequence $(\alpha_k)_k$ is bounded.

Proof. We recall the expression of the reproducing kernel $K_k(x, y)$:

$$K_k(x,y) = \sum_{j=0}^{k} p_j(x) p_j(y) = -\frac{1}{\Delta_k(\gamma)} \begin{vmatrix} 0 & 1 & x & \dots & x^k \\ 1 & \gamma_0 & \gamma_1 & \dots & \gamma_k \\ y & \gamma_1 & \gamma_2 & \dots & \gamma_{k+1} \\ \dots & \dots & \dots & \dots & \dots \\ y^k & \gamma_k & \gamma_{k+1} & \dots & \gamma_{2k} \end{vmatrix}.$$

Then

$$\partial_x K_k(x,y) = \sum_{j=0}^{k} p'_j(x) p_j(y) = -\frac{1}{\Delta_k(\gamma)} \begin{vmatrix} 0 & 0 & 1 & \dots & kx^{k-1} \\ 1 & \gamma_0 & \gamma_1 & \dots & \gamma_k \\ y & \gamma_1 & \gamma_2 & \dots & \gamma_{k+1} \\ \dots & \dots & \dots & \dots & \dots \\ y^k & \gamma_k & \gamma_{k+1} & \dots & \gamma_{2k} \end{vmatrix}.$$

In particular, we have

$$\sum_{j=0}^{k} p'_j(0) p_j(0) = -\frac{1}{\Delta_k(\gamma)} \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \dots & \gamma_k \\ \gamma_3 & \gamma_4 & \dots & \dots & \gamma_{k+2} \\ \gamma_4 & \gamma_5 & \dots & \dots & \gamma_{k+3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_{k+1} & \gamma_{k+2} & \ddots & \ddots & \gamma_{2k} \end{vmatrix} = \alpha_k \frac{\Delta'_k(\gamma)}{2\Delta_k(\gamma)}.$$

For the second part, we note that

$$|\alpha_k| \leq 2\left|\frac{\Delta_k(\gamma)}{\Delta_k'(\gamma)}\right| \sum_{j\leq k} |p_j'(0)p_j(0)| \leq 2\left|\frac{\Delta_2(\gamma)}{\Delta_2'(\gamma)}\right| \sum_{j\leq k} |p_j'(0)p_j(0)|.$$

Since γ is indeterminate, we have $\sum_n |p_n(0)|^2 < \infty$ (and also $\sum_n |q_n(0)|^2 < \infty$). So it suffices to show that $\sum_n |p_n'(0)|^2 < \infty$. We first observe that

$$p_n(x) = p_n(0) + x \sum_{k\leq n-1} \alpha_{n,k} p_k(z), \quad \text{where} \quad \alpha_{n,k} = p_k(0)q_n(0) - p_n(0)q_k(0)$$

Indeed, since

$$\frac{p_n(x)-p_n(0)}{x} \in \mathbb{R}_{n-1}[X],$$

we get

$$\frac{p_n(x)-p_n(0)}{x} = \sum_{k\leq n-1} \alpha_{n,k} p_k(x)$$

with

$$\begin{aligned}
\alpha_{n,k} &= \varphi\left(\frac{p_n(x)-p_n(0)}{x} p_k(x)\right) = \varphi\left(\frac{p_n(x)-p_n(0)}{x}(p_k(x)-p_k(0)+p_k(0))\right) \\
&= p_k(0)\varphi\left(\frac{p_n(x)-p_n(0)}{x}\right) - p_n(0)\varphi\left(\frac{p_k(x)-p_k(0)}{x}\right) \\
&= p_k(0)q_n(0) - p_n(0)q_k(0),
\end{aligned}$$

where $\varphi(p) = \int_{\mathbb{R}} p(x)d\mu(x)$ for every polynomial p. This implies that

$$p_n'(0) = \sum_{k\leq n-1} \alpha_{n,k} p_k(0),$$

and using the Cauchy–Schwarz inequality we get

$$|p_n'(0)|^2 \leq \sum_{k\leq n-1} |\alpha_{n,k}|^2 \sum_{k\leq n-1} |p_k(0)|^2 \leq \sum_{k\geq 0} |p_k(0)|^2 \sum_{k\leq n-1} |\alpha_{n,k}|^2.$$

It follows that

$$\begin{aligned}
\sum_{n\geq 0} |p_n'(0)|^2 &\leq \sum_{n\geq 0} |p_n(0)|^2 \left(\sum_{n\geq 0}\sum_{k\leq n-1} |\alpha_{n,k}|^2\right) \\
&\leq \sum_{n\geq 0} |p_n(0)|^2 \left(\sum_{n\geq 0}\sum_{k\geq 0} |p_k(0)q_n(0) - p_n(0)q_k(0)|^2\right) \\
&\leq 4\sum_{n\geq 0} |p_n(0)|^2 \left(\sum_{n\geq 0} |q_n(0)|^2 \sum_{k\geq 0} |p_k(0)|^2\right) \\
&= 4\frac{\sum_{n\geq 0} |q_n(0)|^2}{\rho^2(0)} < \infty.
\end{aligned}$$

□

In particular,

$$F(t) := -t^2 + \inf_{k\geq 2} \alpha_k t + \rho(0)\rho'(0) \leq \inf_{k\geq 2} f_k(t), \quad t \geq 0,$$

and

$$G(t) := -t^2 + \sup_{k\geq 2} \alpha_k t + \rho(0)\rho'(0) \leq \inf_{k\geq 2} f_k(t), \quad t \leq 0.$$

Since $F(0) = G(0) = \rho(0)\rho'(0) > 0$ and F, G are continuous there exist $\delta > 0$ such that $F(x) > \epsilon > 0$ and $G(x) > \epsilon$ for $|x| < \delta$.

In particular, $\lim_k \frac{\Delta_k(s(x))}{\Delta'_k(s(x))} \neq 0$ and $\Delta_k(s(x)) = f_k(x)\Delta'_k(\gamma) > 0$ for all k. The claimed is proved by Corollary 2.4. □

Remark 3.6. Unlike the first order perturbation, the interval in question is never unbounded, since $\lim_{|t|\to+\infty} f_k(t) = -\infty$.

As a direct consequence, we get the following result:

Theorem 3.7. *Let γ be a Stieltjes moment sequence, which is Hamburger indeterminate. Then $(\gamma_{2n})_n$ is an indeterminate Stieltjes moment sequence.*

Proof. It is clear that $(\gamma_{2n})_n$ is a Stieltjes moment sequence. To prove that $(\gamma_{2n})_n$ is indeterminate we will use Theorem 3.4, which provides a Hamburger moment sequence $s = (s_n)_n$, such that

$$s = \begin{cases} s_n = \gamma_n & \text{if } n \neq 1, \\ s_1 = \gamma_1 + \tau \end{cases}$$

with $\tau > 0$.

Let us show that s is a Stieltjes moment sequence. By Stieltjes's theorem, it remains to check that $(s_{i+j+1})_{i,j}$ is positive definite. To this aim, let $k \geq 1$ be given,

$$\begin{aligned}
\det((s_{i+j+1})_{i,j\leq k}) &= \begin{vmatrix} \gamma_1+\tau & \gamma_2 & \cdots & \gamma_{k+1} \\ \gamma_2 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{k+1} & \ddots & \ddots & \gamma_{2k+1} \end{vmatrix} \\
&= \tau \begin{vmatrix} \gamma_3 & \gamma_4 & \cdots & \gamma_{k+2} \\ \gamma_4 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{k+2} & \ddots & \ddots & \gamma_{2k+1} \end{vmatrix} + \begin{vmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \gamma_2 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{k+1} & \ddots & \ddots & \gamma_{2k+1} \end{vmatrix} \\
&= \tau \det((\gamma_{i+j+3})_{i,j\leq k-1}) + \det((\gamma_{i+j+1})_{i,j\leq k}) > 0,
\end{aligned}$$

where to get that $\det((\gamma_{i+j+3})_{i,j\leq k-1})$ and $\det((\gamma_{i+j+1})_{i,j\leq k}) > 0$, we use a similar idea to [2, 6]. In fact, if $\det((\gamma_{i+j+k_0})_{i,j\leq k}) = 0$ for some $k_0, k \in \mathbb{N}$, we get that

$\det((\gamma_{i+j})_{i,j\leq n}) = 0$ for all $n \geq k_0 + 1$, in particular the moment sequence γ will be Hamburger determinate, which is impossible.

Since γ and s are Stieltjes moment sequences, there exist two positive measures μ and ν such that

$$\gamma_{2n} = \int_0^\infty t^{2n} d\mu(t) = \int_0^\infty u^n d\mu(\sqrt{u}) = \int_0^\infty u^n d\mu'(u).$$

and

$$\gamma_{2n} = s_{2n} = \int_0^\infty u^n d\nu'(u).$$

Moreover,

$$s_1 = \int_0^\infty \sqrt{u} d\nu'(u) = \tau + \gamma_1 = \tau + \int_0^\infty \sqrt{u} d\mu'(u)$$

Finally $\mu' \neq \nu'$, and hence $(\gamma_{2n})_n$ is indeterminate. □

Remark 3.8. We mention here that our conditions does not imply that the sequence γ is a determinate Stieltjes moment sequence. It is known that there exists a determinate Stieltjes moment sequence which it Hamburger indeterminate; see, for example, [11, Example 8.11].

Corollary 3.9. *Let γ be a sequence of positive numbers, and γ' the shifted sequence of γ. We assume that*

$$\inf_{k\geq 0} \Delta_k(\gamma) > 0, \quad \inf_{k\geq 0} \Delta_k(\gamma') > 0 \tag{3.1}$$

and

$$\lim_{k\to\infty} \frac{\Delta_k(\gamma)}{\Delta'_k(\gamma)} > 0. \tag{3.2}$$

Then $(\gamma_{2n})_n$ is an indeterminate Stieltjes moment sequence.

4. EVERY SEQUENCE IS A DIFFERENCE OF INDETERMINATE STIELTJES MOMENT SEQUENCE

The starting point of Stieltjes is the study of the continued fractions:

$$\cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \ddots + \cfrac{1}{a_{2n} + \cfrac{1}{a_{2n+1} z + \cdots}}}}} \tag{4.1}$$

with coefficients $a_i > 0$. He proved in particular the following: If $(a_n)_n \in \ell^1(\mathbb{N})$, then the odd convergent $\frac{p_{2n}(z)}{q_{2n}(z)}$ and the even convergent $\frac{p_{2n+1}(z)}{q_{2n+1}(z)}$ have different limits in $\mathbb{C} \setminus \mathbb{R}_-$. Furthermore, there exist $\alpha_i, \beta_i \geq 0$ and $\lambda_i, \zeta_i > 0$ such that:

$$\lim_{n\to\infty} \frac{p_{2n}(z)}{q_{2n}(z)} = \sum_{n\geq 0} \frac{\alpha_n}{z+\lambda_n},$$
$$\lim_{n\to\infty} \frac{p_{2n+1}(z)}{q_{2n+1}(z)} = \sum_{n\geq 0} \frac{\beta_n}{z+\zeta_n},$$

where $\lambda_n \neq \zeta_n$ for every n. If we put $\nu = \sum_{n\geq 0} \beta_n \delta_{\zeta_n}$ and $\mu = \sum_{n\geq 0} \alpha_n \delta_{\lambda_n}$, then we can write

$$\lim_{n\to\infty} \frac{p_{2n}(z)}{q_{2n}(z)} = \int_0^\infty \frac{d\mu(t)}{z+t},$$
$$\lim_{n\to\infty} \frac{p_{2n+1}(z)}{q_{2n+1}(z)} = \int_0^\infty \frac{d\nu(t)}{z+t}.$$

Furthermore, he proved that for every $n \in \mathbb{N}$, $\gamma_n := \int_0^\infty t^n d\mu(t) = \int_0^\infty t^n d\nu(t)$. This case is in particular indeterminate, since the continuous fraction can not converge to a unique function.

If $(a_n)_n \notin \ell^1(\mathbb{N})$, then $\frac{p_n(z)}{q_n(z)}$ converges in $\mathbb{C} \setminus \mathbb{R}_-$ to $\int_0^\infty \frac{d\mu(t)}{z+t}$ for some measure μ supported on $\mathbb{R}_+$, which is the unique solution of a moment problem $\gamma_n := \int_0^\infty t^n d\mu(t)$, this is the determinate case.

As a consequence, we have the following theorem.

Theorem 4.1 (Stieltjes)**.** *Every indeterminate Stieltjes moment sequence γ has a discrete solution μ.*

We are ready to show our main result.

Theorem 4.2. *Let γ be a sequence of real numbers, then there exists a sequence of real numbers $(c_n)_n$ and a sequence of positive numbers $(\xi_n)_n$ such that*

$$\gamma_k = \sum_{n\geq 0} c_n \xi_n^k \quad \text{for every } k \geq 0.$$

Proof. Let $u = (u_n)_n$ be the sequence defined by

$$\begin{cases} u_{2n} = \gamma_n, \\ u_{2n+1} = 0. \end{cases}$$

We will write u as a difference of two Stieltjes moment sequences v and w verifying (3.1) and (3.2) in Corollary 3.9.

We proceed by induction in our construction.

1. We choose $v_0 \geq 1$, $w_0 \geq 1$, $v_1 = w_1 \geq 1$, such that $v_0 - w_0 = u_0$,
2. We take v_2, w_2 and $v_3 = w_3$ in $[1, +\infty[$ great enough such that

$$\Delta_1(v) = \begin{vmatrix} v_0 & v_1 \\ v_1 & v_2 \end{vmatrix} = v_0 v_2 - v_1^2 \geq 1 \quad \text{and} \quad \Delta_1(w) = \begin{vmatrix} w_0 & w_1 \\ w_1 & w_2 \end{vmatrix} = w_0 w_2 - w_1^2 \geq 1$$

and

$$\Delta_1(v') = \begin{vmatrix} v_1 & v_2 \\ v_2 & v_3 \end{vmatrix} = v_3 v_1 - v_2^2 \geq 1 \quad \text{and} \quad \Delta_1(w') = \begin{vmatrix} w_1 & w_2 \\ w_2 & w_3 \end{vmatrix} = w_3 w_1 - w_2^2 \geq 1.$$

3. Now, since $\Delta_2(v) = \begin{vmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \\ v_2 & v_3 & v_4 \end{vmatrix} = \Delta_1(v) v_4 + c$, and $\Delta_1(v) > 0$ and

$$\frac{\Delta_2(v)}{\Delta_2'(v)} = \frac{\Delta_1(v) v_4 + c}{v_4} = \Delta_1(v) + \frac{c}{v_4},$$

for some real number c, we can pick out $v_4, w_4 > 0$ sufficiently large such that

$$\Delta_2(v) \geq 1, \quad \Delta_2(w) \geq 1, \quad \frac{\Delta_2(v)}{\Delta_2'(v)} \geq \frac{\Delta_1(v)}{2} \quad \text{and} \quad \frac{\Delta_2(w)}{\Delta_2'(w)} \geq \frac{\Delta_1(w)}{2}.$$

A similar reasoning can be applied with regard to w_4, so we can choose them such that $v_4 - w_4 = u_4$.

Suppose that we have constructed

$$v_k - w_k = u_k, \quad k \leq 2n - 1.$$

such that, for any $k \leq n - 1$

$$\begin{aligned} &\Delta_k(v) \geq 1, \quad \Delta_k(w) \geq 1, && \Delta_k(v') \geq 1, \quad \Delta_k(w') \geq 1, \\ &\frac{\Delta_k(v)}{\Delta_k'(v)} \geq \frac{\Delta_{k-1}(v)}{\Delta_{k-1}'(v)} - a_k, && \frac{\Delta_k(w)}{\Delta_k'(w)} \geq \frac{\Delta_{k-1}(w)}{\Delta_{k-1}'(w)} - b_k, \end{aligned}$$

where a_n and b_n will be fixed later.

For a real number x, we have

$$F(x) \coloneqq \frac{\Delta_n(w)}{\Delta_n'(w)} = \frac{\begin{vmatrix} w_0 & w_1 & \dots & w_n \\ w_1 & w_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ w_n & \dots & w_{2n-1} & x \end{vmatrix}}{\begin{vmatrix} w_4 & w_5 & \dots & w_{n+2} \\ w_5 & w_6 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ w_{n+2} & \dots & \dots & x \end{vmatrix}} = \frac{x\Delta_{n-1}(w) + f(w_0, w_1, \dots, w_{2n-1})}{x\Delta_{n-1}'(w) + g(w_4, w_5, \dots, w_{2n-1})}.$$

Since $\lim_{x\to\infty} F(x) = \frac{\Delta_{n-1}(w)}{\Delta'_{n-1}(w)}$, for any $0 < b_n < \frac{\Delta_{n-1}(w)}{\Delta'_{n-1}(w)}$, we can take $x = w_{2n}$ great enough such that $F(w_{2n}) > \frac{\Delta_{n-1}(w)}{\Delta'_{n-1}(w)} - b_n$, and $\Delta_n(w) \geq 1$.

Using the same method, we have

$$G(x) \coloneqq \frac{\Delta_n(v)}{\Delta'_n(v)} = \frac{\begin{vmatrix} v_0 & v_1 & \dots & v_n \\ v_1 & v_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ v_n & \dots & v_{2n-1} & x \end{vmatrix}}{\begin{vmatrix} v_4 & v_5 & \dots & v_{n+2} \\ v_5 & v_6 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ v_{n+2} & \dots & \dots & x \end{vmatrix}} = \frac{x\Delta_{n-1}(v) + f(v_0, v_1, \dots, v_{2n-1})}{x\Delta'_{n-1}(v) + g(v_4, v_5, \dots, v_{2n-1})}.$$

Since $\lim\limits_{x\to\infty} G(x) = \frac{\Delta_{n-1}(v)}{\Delta'_{n-1}(v)}$, for any $0 < a_n < \frac{\Delta_{n-1}(w)}{\Delta'_{n-1}(v)}$, we can take $x = v_{2n}$ large enough such that $G(x) > \frac{\Delta_{n-1}(v)}{\Delta'_{n-1}(v)} - a_n$ and $\Delta_n(v) \geq 1$.

Since the choice is arbitrary, we can take w_{2n} and v_{2n} such that $u_{2n} = v_{2n} - w_{2n}$. Similarly, we can find $v_{2n+1} = w_{2n+1}$ such that

$$\Delta_n(v') \geq 1, \quad \Delta_n(w') \geq 1$$

This completes the induction.

We choose a_n and b_n arbitrary to satisfy

$$\tfrac{\Delta_1(v)}{4} \geq \textstyle\sum_{k\geq 2} a_k \qquad \tfrac{\Delta_1(w)}{4} \geq \textstyle\sum_{k\geq 2} b_k \ .$$

Such v and w verify (3.1) and (3.2), then by Lemma 3.9 $(v_{2n})_n$ and $(w_{2n})_n$ are indeterminate. Using Theorem 4.1, there exists two atomic measures $\mu = \sum_{n\geq 0} \beta_n \delta_{\zeta_n}$ and $\nu = \sum_{n\geq 0} \alpha_n \delta_{\lambda_n}$ supported on $\mathbb{R}_+$ solution to the moment problem $(v_{2n})_n$ and $(w_{2n})_n$ respectively. Hence, for $k \geq 0$, we get

$$\gamma_k = u_{2k} = v_{2k} - w_{2k} = \sum_{n\geq 0} \beta_n \zeta_n^k - \sum_{m\geq 0} \alpha_m \lambda_m^k = \sum_{n\geq 0} c_n \xi_n^k. \qquad \square$$

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