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A NEW MODIFICATION OF THE REDUCED DIFFERENTIAL TRANSFORM METHOD FOR NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The objective of this study is to present a new modification of the reduced differential transform method (MRDTM) to find an approximate analytical solution of a certain class of nonlinear fractional partial differential equations in particular, nonlinear time-fractional wave-like equations with variable coefficients. This method is a combination of two different methods: the Shehu transform method and the reduced differential transform method. The advantage of the MRDTM is to find the solution without discretization, linearization or restrictive assumptions. Three different examples are presented to demonstrate the applicability and effectiveness of the MRDTM. The numerical results show that the proposed modification is very effective and simple for solving nonlinear fractional partial differential equations.

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1. Introduction

The exact solutions and numerical solutions of the nonlinear fractional partial differential equations play an important role in physical science and in engineering fields such as viscoelasticity, fluid mechanics, acoustics, electromagnetism, diffusion, analytical chemistry, control theory, biology, and so on [1–14]. Consequently, there have been attempts to develop new methods to obtain approximate analytical solutions which converge to exact solutions. Among these methods are: the natural decomposition method (NDM) [15], homotopy perturbation transform method (HPTM) [16], homotopy analysis transform method (HATM) [17], optimal homotopy asymptotic method (OHAM) [18], fractional variational iteration method (FVIM) [19], residual power series method (RPSM) [20]. In this paper, we present a new modification of the reduced differential transform method (MRDTM) which is a combination of the Shehu transform method and the reduced differential transform method for solving a certain class of nonlinear fractional differential equations. The advantage of the MRDTM is to solve nonlinear fractional differential equations without using any complicated polynomials like as the Adomian polynomials that are used in the Adomian decomposition method (ADM) and He's polynomials that are used in the homotopy perturbation method (HPM).

Consider the following nonlinear time-fractional wave-like equations with variable coefficients

$$D_t^{\alpha} u = \sum_{i,j=1}^{n} F_{1ij}(X,t,u) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(u_{x_i}, u_{x_j}) + \sum_{i=1}^{n} G_{1i}(X,t,u) \frac{\partial^p}{\partial x_i^p} G_{2i}(u_{x_i}) + H(X,t,u) + S(X,t), \tag{1}$$

subject to the initial conditions

$$u(X,0) = a_0(X), u_t(X,0) = a_1(X), \tag{2}$$

where D_t^{α} is the Caputo fractional derivative operator of order α, $1 < \alpha \leq 2$, u is a function of $(X,t) \in \mathbb{R}^n \times \mathbb{R}^+$, F_{1ij}, G_{1i} $i,j \in \{1,2,...,n\}$ are nonlinear functions of X,t and u, F_{2ij}, G_{2i} $i,j \in \{1,2,...,n\}$, are nonlinear functions of derivatives of u with respect to x_i and x_j $i,j \in \{1,2,...,n\}$, respectively. Also H,S are nonlinear functions and k,m,p are integers.

These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, and velocity distributions of fluid particles in turbulent flows.

2. Definition and preliminaries

In this section, we define some basic definitions and properties of the fractional calculus theory and the Shehu transform which shall be used in this paper.

Definition 1 [21] A real function $f(t), t > 0$, is considered to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h(t) \in C([0,\infty[)$, and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$. □

Definition 2 [21] The Riemann-Liouville fractional integral operator I^{α} of order $\alpha \geq 0$ for a function $f \in C_{\mu}, \mu \geq -1$ is defined as follows

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\xi)^{\alpha-1} f(\xi) d\xi, t > 0. \tag{3}$$

where $\Gamma(.)$ is the well-known Gamma function. □

Definition 3 [21] The Caputo fractional derivative operator of order $n-1 < \alpha \leq n$ for a function $f \in C_{-1}^{n}$ is defined as follows

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, t > 0. \tag{4}$$

Definition 4 [22] The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions

$$A = \left\{ f(t)/\exists N, \eta_1, \eta_2 > 0, |f(t)| < N \exp\left(\frac{|t|}{\eta_j}\right), \text{ if } t \in (-1)^j \times [0, \infty) \right\}, \tag{5}$$

by the following integral

$$\mathbb{S}[f(t)] = F(s,v) = \int_{0}^{\infty} \exp\left(-\frac{st}{v}\right) f(t) dt, t > 0. \tag{6}$$

Theorem 1 *[23] Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $F(s,v)$ be the Shehu transform of the function $f(t)$, then the Shehu transform denoted by $F_{\alpha}(s,v)$ of the Caputo fractional derivative of $f(t)$ of order α, is given by*

$$\mathbb{S}[D^{\alpha} f(t)] = F_{\alpha}(s,v) = \frac{s^{\alpha}}{v^{\alpha}} F(s,v) - \sum_{k=0}^{n-1} \left(\frac{s}{v}\right)^{\alpha-(k+1)} \left[D^{k} f(t)\right]_{t=0}. \tag{7}$$

3. Reduced differential transform method (RDTM)

In this section, we apply the reduced differential transform method (RDTM) for $(n+1)-$variables function $u(x_1, x_2, ..., x_n, t)$ which has been developed in [24].

Consider a function $u(x_1, x_2, ..., x_n, t)$ of $(n+1)-$variables and assume that it can be represented as a product of $(n+1)$ single-variable function, i.e.

$$u(x_1, x_2, ..., x_n, t) = F_1(x_1) F_2(x_2) ... F_n(x_n) F_m(t). \tag{8}$$

On the basis of the properties of the one-dimensional differential transform, the function $u(x_1,x_2,...,x_n,t)$ can be represented as

$$\begin{aligned} u(x_1,x_2,...,x_n,t) &= \left(\sum_{k_1=0}^{\infty} F_1(k_1)x_1^{k_1}\right)\left(\sum_{k_2=0}^{\infty} F_2(k_2)x_2^{k_2}\right)\times ... \\ &\quad \times\left(\sum_{k_n=0}^{\infty} F_n(k_n)x_n^{k_n}\right)\times\left(\sum_{k_m=0}^{\infty} F_m(k_m)t^{k_m}\right) \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}\sum_{k_m=0}^{\infty} U(k_1,k_2,...,k_n,k_m)x_1^{k_1}x_2^{k_2}...x_n^{k_n}t^{k_m}, \quad (9) \end{aligned}$$

where $U(k_1,k_2,...,k_n,k_m) = F_1(k_1)\times F_2(k_2)\times ...\times F_n(k_n)\times F_m(k_m)$ is called the spectrum of $u(x_1,x_2,...,x_n,t)$.

Next, we assume that $u(X,t), X = (x_1,x_2,...,x_n)$ is a continuously differentiable function with respect to space variable and time in the domain of interest.

Definition 5 Let $u(X,t)$, $X = (x_1,x_2,...,x_n)$ be an analytic function, then the RDT of u is given by

$$U_k(X) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(X,t)\right]_{t=t_0}. \quad (10)$$

Here the lowercase $u(X,t)$ represents the original function while the uppercase $U_k(X)$ stands for the reduced transformed function. □

Definition 6 The inverse RDT of $U_k(X)$ is defined by

$$u(X,t) = \sum_{k=0}^{\infty} U_k(X)(t-t_0)^k. \quad (11)$$

Combining Eqs. (10) and (11), we have

$$u(X,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(X,t)\right]_{t=t_0}(t-t_0)^k. \quad (12)$$

In particular, for $t_0 = 0$, Eq. (12) becomes

$$u(X,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(X,t)\right]_{t=0} t^k. \quad (13)$$

From the above definitions, the fundamental operations of the RDTM are given by the following theorems.

Theorem 2 *Let $U_k(X), V_k(X)$ and $W_k(X)$ be the fractional reduced differential transform of the functions $u(X,t), v(X,t)$ and $w(X,t)$ respectively, then*

(1) if $w(X,t) = \lambda u(X,t) + \mu v(X,t)$, then $W_k(X) = \lambda U_k(X) + \mu V_k(X)$, $\lambda,\mu \in \mathbb{R}$.

(2) if $w(X,t)=u(X,t)v(X,t)$*, then* $W_k(X)=\sum_{r=0}^{k}U_r(X)V_{k-r}(X)$.

(3) if $w(X,t)=u^1(X,t)u^2(X,t)...u^n(X,t)$*, then*

$$W_k(X)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U^1_{k_1}(X)U^2_{k_2-k_1}(X)\cdot\ldots\cdot U^{n-1}_{k_{n-1}-k_{n-2}}(X)U^n_{k-k_{n-1}}(X).$$

(4) if $w(X,t)=\frac{\partial^n}{\partial t^n}u(X,t)$*, then* $W_k(X)=\frac{(k+n)!}{k!}U_{k+n}, n=1,2,\ldots.$ □

4. MRDTM for nonlinear time-fractional wave-like equations with variable coefficients

Theorem 3 *Consider the following nonlinear time-fractional wave-like equations with variable coefficients (1) subject to the initial conditions (2). Then, by MRDTM, the approximate analytical solution of Eqs. (1) and (2) is given in the form of infinite series which converges rapidly to the exact solution as follows*

$$u(X,t)=\sum_{k=0}^{\infty}U_k(X), \tag{14}$$

where $U_k(X)$ *is the reduced differential transformed function of* $u(X,t)$. □

Proof In order to achieve our goal,we consider the following nonlinear time-fractional wave-like equations with variable coefficients (1) subject to the initial conditions (2).

Taking the Shehu transform on both sides of Eq. (1) subject to the initial conditions (2) and using the Theorem 1, we get

$$\begin{aligned}\mathbb{S}[u(X,t)] &= \frac{v}{s}a_0(X)+\left(\frac{v}{s}\right)^2a_1(X)+\frac{v^\alpha}{s^\alpha}\mathbb{S}[S(X,t)]\\ &+\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\sum_{i,j=1}^{n}F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k\partial x_j^m}F_{2ij}(u_{x_i},u_{x_j})\right.\\ &\left.+\sum_{i=1}^{n}G_{1i}(X,t,u)\frac{\partial^p}{\partial x_i^p}G_{2i}(v_{x_i})+H(X,t,u)\right].\end{aligned} \tag{15}$$

Applying the inverse Shehu transform on both sides of Eq. (15), we have

$$\begin{aligned}u(X,t) &= L(X,t)+\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[\sum_{i,j=1}^{n}F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k\partial x_j^m}F_{2ij}(v_{x_i},v_{x_j})\right.\right.\\ &\left.\left.+\sum_{i=1}^{n}G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p}G_{2i}(v_{x_i})+H(X,t,v)\right]\right),\end{aligned} \tag{16}$$

where $L(X,t)$ is a term arising from the source term and the prescribed initial conditions.

We now apply the reduced differential transform method to Eq. (16), and get

$$U_0(X) = L(X,t), \tag{17}$$

$$U_{k+1}(X) = \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_k(X)+B_k(X)+C_k(X)\right]\right), k \geq 0, \tag{18}$$

where $A_k(X), B_k(X)$ and $C_k(X)$ are a transformed forms of the nonlinear terms,

$\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(u_{x_i},u_{x_j})$, $\sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p}{\partial x_i^p}G_{2i}(u_{x_i})$ and $H(X,t,u)$.

From Eqs. (17) and (18), we have

$$\begin{aligned}
U_0(X) &= L(X,t),\\
U_1(X) &= \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_0(X)+B_0(X)+C_0(X)\right]\right),\\
U_2(X) &= \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_1(X)+B_1(X)+C_1(X)\right]\right),\\
&\vdots
\end{aligned}$$

■

Hence, the approximate analytical solution of Eqs. (1) and (2) is given as

$$u(X,t) = \sum_{k=0}^{\infty} U_k(X). \tag{19}$$

5. Numerical examples

In this section, we consider three different examples of nonlinear time-fractional wave-like equations with variable coefficients to demonstrate the applicability and effectiveness of the MRDTM.

Example 1 Consider the following two-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha u = \frac{\partial^2}{\partial x \partial y}(u_{xx}u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyu_xu_y) - u, 1 < \alpha \leq 2, \tag{20}$$

subject to the initial conditions

$$u(x,y,0) = e^{xy}, u_t(x,y,0) = e^{xy}, \tag{21}$$

where u is a function of $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

By applying the steps involved in the MRDTM, as presented in Section 4, to Eqs.

(20) and (21), we have the following iteration formula

$$U_0(x,y) = e^{xy} + te^{xy}, \tag{22}$$

$$U_{k+1}(x,y) = \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left(\frac{\partial^2}{\partial x\partial y}A_k(x,y) - \frac{\partial^2}{\partial x\partial y}B_k(x,y) - U_k(x,y)\right)\right), \tag{23}$$

where $A_k(x,y)$ and $B_k(x,y)$ are a transformed forms of the nonlinear terms, $u_{xx}u_{yy}$ and xyu_xu_y. For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A_0 &= U_{0xx}U_{0yy}, \\ A_1 &= U_{0xx}U_{1yy} + U_{1xx}U_{0yy}, \\ A_2 &= U_{0xx}U_{2yy} + U_{1xx}U_{1yy} + U_{2xx}U_{0yy}, \end{aligned}$$

$$\begin{aligned} B_0 &= xyU_{0x}U_{0y}, \\ B_1 &= xyU_{0x}U_{1y} + xyU_{1x}U_{0y}, \\ B_2 &= xyU_{0x}U_{2y} + xyU_{1x}U_{1y} + xyU_{2x}U_{0y}. \end{aligned}$$

From Eqs. (22) and (23), we obtain

$$\begin{aligned} U_0(x,y) &= (1+t)e^{xy}, \\ U_1(x,y) &= -\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy} \\ U_2(x,y) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}, \\ &\vdots \end{aligned}$$

Hence, the approximate analytical solution of Eqs. (20) and (21) is given as

$$u(x,y,t) = \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \ldots\right)e^{xy}. \tag{24}$$

Taking $\alpha = 2$ in Eq. (24), we have

$$\begin{aligned} u(x,y,t) &= \left(1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \ldots\right)e^{xy} \\ &= (\cos t + \sin t)\, e^{xy}, \end{aligned} \tag{25}$$

which is the same solution as obtained by using the FRPSM [25].

Example 2 Consider the following one dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha u = u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3) - 18u^5 + u, 1 < \alpha \leq 2, \tag{26}$$

subject to the initial conditions

$$u(x,0)=e^{x}, u_t(x,0)=e^{x}, \tag{27}$$

where u is a function of $(x,t)\in\,]0,1[\,\times\mathbb{R}^{+}$.

By applying the steps involved in the MRDTM, as presented in Section 4, to Eqs. (26) and (27), we have the following iteration formula

$$U_0(x)=e^{x}+te^{x}, \tag{28}$$

$$U_{k+1}(x)=\mathbb{S}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{S}\left(A_k(x)+B_k(x)-18C_k(x)+U_k(x)\right)\right), \tag{29}$$

where $A_k(x)$, $B_k(x)$ and $C_k(x)$ are a transformed forms of the nonlinear terms, $u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}), u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3)$, and u^5. For the convenience of the reader, the first few nonlinear terms are as follows:

$$\begin{aligned}
A_0 &= U_0^2\frac{\partial^2}{\partial x^2}\left[U_{0x}U_{0xx}U_{0xxx}\right],\\
A_1 &= 2U_0U_1\frac{\partial^2}{\partial x^2}\left[U_{0x}U_{0xx}U_{0xxx}\right]+U_0^2\frac{\partial^2}{\partial x^2}\left[U_{1x}U_{0xx}U_{0xxx}\right.\\
&\quad\left.+U_{0x}U_{1xx}U_{0xxx}+U_{0x}U_{0xx}U_{1xxx}\right],
\end{aligned}$$

$$\begin{aligned}
B_0 &= U_{0x}^2\frac{\partial^2}{\partial x^2}U_{0xx}^3,\\
B_1 &= 2U_{0x}U_{1x}\frac{\partial^2}{\partial x^2}U_{0xx}^3+3U_{0x}^2\frac{\partial^2}{\partial x^2}\left[U_{0xx}^2U_{1xx}\right],
\end{aligned}$$

$$C_0=U_0^5, C_1=5U_0^4U_1.$$

From Eqs. (28) and (29), we obtain

$$\begin{aligned}
U_0(x) &= (1+t)e^{x},\\
U_1(x) &= \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{x},\\
U_2(x) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{x},\\
&\vdots
\end{aligned}$$

Hence, the approximate analytical solution of Eqs. (26) and (27) is given as

$$u(x,t)=\left(1+t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+...\right)e^{x}. \tag{30}$$

Taking $\alpha = 2$ in Eq. (30), we have

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right) e^x = e^{x+t}, \tag{31}$$

which is the same solution as obtained by using the FRPSM [25].

Example 3 Consider the following one dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} u = x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2(u_{xx}^2) - u, 1 < \alpha \leq 2, \tag{32}$$

subject to the initial conditions

$$u(x,0) = 0, u_t(x,0) = x^2, \tag{33}$$

where u is a function of $(x,t) \in]0,1[\times \mathbb{R}^+$.

By applying the steps involved in the MRDTM, as presented in Section 4, to Eqs. (32) and (33), we have the following iteration formula

$$U_0(x) = tx^2, \tag{34}$$

$$U_{k+1}(x) = \mathbb{S}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}} \mathbb{S}\left(x^2 \frac{\partial}{\partial x} A_k(x) - x^2 B_k(x) - U_k(x)\right)\right), \tag{35}$$

where $A_k(x)$ and $B_k(x)$ are a transformed forms of the nonlinear terms, $u_x u_{xx}$ and u_{xx}^2. For the convenience of the reader, the first few nonlinear terms are as follows:

$$\begin{aligned} A_0 &= U_{0x}U_{0xx}, \\ A_1 &= U_{0x}U_{1xx} + U_{1x}U_{0xx}, \\ A_2 &= U_{0x}U_{2xx} + U_{1x}U_{1xx} + U_{2x}U_{0xx}, \end{aligned}$$

$$\begin{aligned} B_0 &= U_{0xx}^2, \\ B_1 &= 2U_{0xx}U_{1xx}, \\ B_2 &= 2U_{0xx}U_{2xx} + U_{1xx}^2. \end{aligned}$$

From Eqs. (34) and (35), we obtain

$$\begin{aligned} U_0(x) &= tx^2, \\ U_1(x) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2, \\ U_2(x) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2, \\ &\vdots \end{aligned}$$

Hence, the approximate analytical solution of Eqs. (32) and (33) is given as

$$u(x,t) = x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \ldots \right). \tag{36}$$

Taking $\alpha = 2$ in Eq. (36), we have

$$u(x,t) = x^2 \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \right) = x^2 \sin t, \tag{37}$$

which is the same solution as obtained by using the FRPSM [25].

6. Numerical results and discussion

Figures 1, 3 and 5 show the surface graph of the exact solution and 3-term approximate solutions by MRDTM at $\alpha = 1.7, 1.8, 2$.

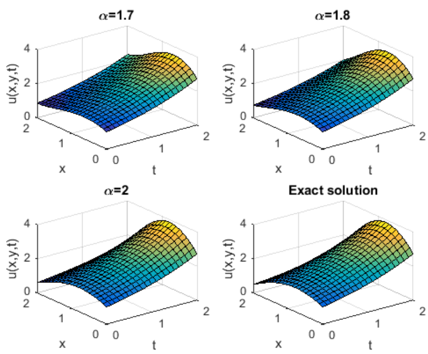

Fig. 1. 3D plots of the approximate solutions and exact solution for Eq. (20) when $y = 0.5$

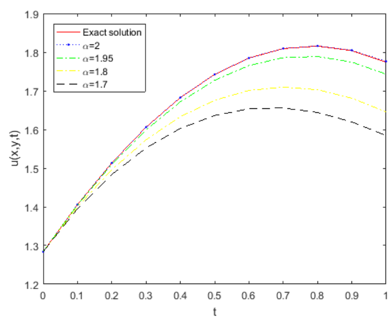

Fig. 2. 2D plots of the approximate solutions and exact solution for Eq. (20) when $x = y = 0.5$

Figures 2, 4 and 6 show the behavior of the exact solution and 3-term approximate solutions by MRDTM at $\alpha = 1.7, 1.8, 1.95, 2$. From these figures, we can confirm that when α approaches to 2, the approximate solution obtained by MRDTM converges towards the exact solution. Tables 1, 2 and 3 show the comparison between the FRPSM-approximate solutions (see [25]) and the obtained results by the MRDTM. From these tables, we can see that the solution obtained by the MRDTM match well with the FRPSM and coincide with the exact solution.

Table 1. Comparison of the FRPSM-approximate solution and the obtained results by the MRDTM and the exact solution for Eq. (20) when $x = y = 0.5$ and $\alpha = 2$

t	u_{FRPSM}	u_{MRDTM}	u_{exact}	$\lvert u_{exact} - u_{MRDTM} \rvert$
0.1	1.4058	1.4058	1.4058	3.2196×10^{-13}
0.3	1.6061	1.6061	1.6061	2.1569×10^{-9}
0.5	1.7424	1.7424	1.7424	1.3095×10^{-7}
0.7	1.8093	1.8093	1.8093	1.9680×10^{-6}
0.9	1.8040	1.8040	1.8040	1.4947×10^{-5}

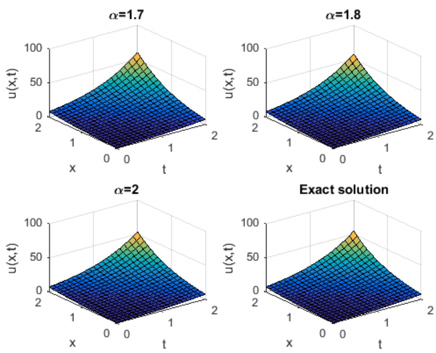

Fig. 3. 3D plots of the approximate solutions and exact solution for Eq. (26)

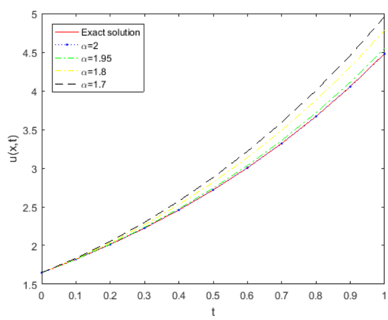

Fig. 4. 2D plots of the approximate solutions and exact solution for Eq. (26) when $x = 0.5$

Table 2. Comparison of the FRPSM-approximate solution and the obtained results by the MRDTM and the exact solution for Eq. (26) when $x = y = 0.5$ and $\alpha = 2$

t	u_{FRPSM}	u_{MRDTM}	u_{exact}	$\lvert u_{exact} - u_{MRDTM} \rvert$
0.1	1.8221	1.8221	1.8221	4.1350×10^{-13}
0.3	2.2255	2.2255	2.2255	2.7750×10^{-9}
0.5	2.7183	2.7183	2.7183	1.6907×10^{-7}
0.7	3.3201	3.3201	3.3201	2.5543×10^{-6}
0.9	4.0552	4.0552	4.0552	1.9535×10^{-5}

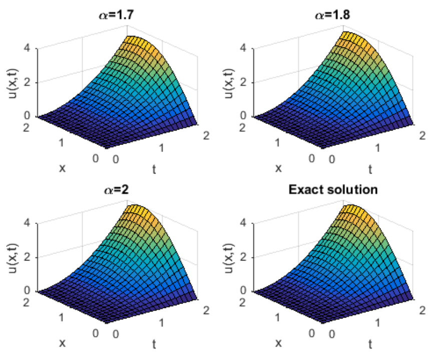

Fig. 5. 3D plots of the approximate solutions and exact solution for Eq. (32)

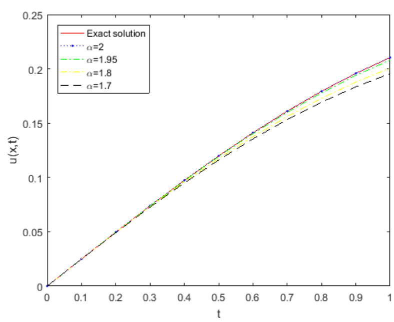

Fig. 6. 2D plots of the approximate solutions and exact solution for Eq. (32) when $x = 0.5$

Table 3. Comparison of the FRPSM-approximate solution and the obtained results by the MRDTM and the exact solution for Eq. (32) when $x = y = 0.5$ and $\alpha = 2$

t	u_{FRPSM}	u_{MRDTM}	u_{exact}	$\lvert u_{exact} - u_{MRDTM} \rvert$
0.1	0.02496	0.02496	0.02496	6.8887×10^{-16}
0.3	0.07388	0.07388	0.07388	1.3549×10^{-11}
0.5	0.11986	0.11986	0.11986	1.3425×10^{-9}
0.7	0.16105	0.16105	0.16105	2.7677×10^{-8}
0.9	0.19583	0.19583	0.19583	2.6495×10^{-7}

7. Conclusions

In this paper, a new modification of the reduced differential transform method (MRDTM) has been successfully applied to find approximate analytical solutions for nonlinear time-fractional wave-like equations. The results shows that the MRDTM is an efficient and easy to use technique for solving these types of equations. The obtained approximate solution using the suggested method is in excellent agreement with the exact solution. This confirms our belief that the efficiency of our method gives it much wider applicability for general classes of nonlinear fractional partial differential equations.

References

[1] Pinar, Z. (2019). On the explicit solutions of fractional Bagley-Torvik equation arises in engineering. *An International Journal of Optimization and Control: Theories & Applications*, *9*(3), 52-58.

[2] Ashyralyev, A., Dal, F., & Pinar, Z. (2011). A note on the fractional hyperbolic differential and difference equations. *Applied Mathematics and Computation*, *217*(9), 4654-4664.

[3] Ashyralyev, A., Dal, F., & Pinar, Z. (2009). On the numerical solution of fractional hyperbolic partial differential equations. *Mathematical Problems in Engineering*, DOI: 10.1155/2009/730465.

[4] Vinagr, B.M., Podlubny, I., Hernandez, A., & Feliu, V. (2000). Some approximations of fractional order operators used in control theory and applications. *Fractional Calculus and Applied Analysis*, *3*(3), 231-248.

[5] Fitt, A.D., Goodwin, A.R., Ronaldson, K.A., & Wakeham, W.A. (2009). A fractional differential equation for a MEMS viscometer used in the oil industry. *Journal of Computational and Applied Mathematics*, *229*(2), 373-381.

[6] Zhou, Y., & Peng, L. (2017). Weak solution of the time-fractional Navier-Stokes equations and optimal control. *Computers & Mathematics with Applications*, *73*(6), 1016-1027.

[7] Mainardi, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity*. London: Imperial College Press.

[8] Herzallah, M.A.E., Muslih, S.I., Baleanu, D., & Rabei, E.M. (2011). Hamilton-Jacobi and fractional like action with time scaling. *Nonlinear Dynamics*, *66*(4), 549-555.

[9] Doungmo Goufo, E.F., Kumar, S., & Mugisha, S.B. (2020). Similarities in a fifth-order evolution equation with and with no singular kernel. *Chaos, Solitons & Fractals*, *130*, 10946.

[10] Kumar, S., Nisar, K.S., Kumar, R., Cattani, C., & Samet, B. (2020). A new Rabotnov fractional-exponential function based fractional derivative for diffusion equation under external force. *Mathematical Methods in Applied Science*, DOI: 10.1002/mma.6208.

[11] Kumar, S., Kumar, R., Agarwal, R.P., & Samet, B. (2020). A study on fractional Lotka Volterra population model by using Haar wavelet and Adams Bashforth-Moulton methods. *Mathematical Methods in Applied Science*, DOI: 10.1002/mma.6297.

[12] Kumar, S., Kumar, R., Singh, J., Nisar, K.S., & Kumar, D. (2020). An efficient numerical scheme for fractional model of HIV-1 infection of $CD4^{+}$ T-cells with the effect of antiviral drug therapy. *Alexandria Engineering Journal*, DOI: 10.1016/j.aej.2019.12.046.

[13] Ghanbari, B., Kumar, S., & Kumar, R. (2020). A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. *Chaos, Solitons & Fractals*, *133*, 109619.

[14] Khalouta, A., & Kadem, A. (2019). A new numerical technique for solving Caputo time--fractional biological population equation. *AIMS Mathematics*, *4*(5), 1307-1319.

[15] Khalouta, A., & Kadem, A. (2020). A new numerical technique for solving fractional Bratu's initial value problems in the Caputo and Caputo-Fabrizio sense. *Journal of Applied Mathematics and Computational Mechanics*, *19*(1), 43-56.

[16] Jleli, M., Kumar, S., Kumar, R., & Samet, B. (2019). Analytical approach for time fractional wave equations in the sense of Yang-Abdel-Aty-Cattani via the homotopy perturbation transform method. *Alexandria Engineering Journal*, DOI: 10.1016/j.aej.2019.12.022.

[17] Kumar, S., Kumar, A., Abbas, S., Al Qurashi, M., & Baleanu, D. (2020). A modified analytical approach with existence and uniqueness for fractional Cauchy reaction-diffusion equations. *Advances in Difference Equations*, *28*, DOI: 10.1186/s13662-019-2488-3.

[18] Hamarsheh, M., Ismail, A.I., & Odibat, Z. (2016). An analytic solution for fractional order Riccati equations by using optimal homotopy asymptotic method. *Applied Mathematical Sciences*, *10*(23), 1131-1150.

[19] Singh, B.K., & Kumar, P. (2017). Fractional variational iteration method for solving fractional partial differential equations with proportional delay. *International Journal of Differential Equations*, Article ID 5206380, 1-11.

[20] Kumar, S., Kumar, A., Momani, S., Aldhaifallah, M., & Nisar, K.S. (2019). Numerical solutions of nonlinear fractional model arising in the appearance of the strip patterns in two-dimensional systems. *Advances in Difference Equations*, *413*.

[21] Kilbas, A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and Application of Fractional Differential Equations*. Amsterdam: Elsevier.

[22] Maitama, S., & Zhao, W. (2019). New integral transform: Shehu transform a genaralization of Sumudu and laplace transform for solving differential equations. *International Journal of Analysis and Applications*, *17*(2), 167-190.

[23] Khalouta, A., & Kadem, A. (2019). A new method to solve fractional differential equations: Inverse fractional Shehu transform method. *Applications and Applied Mathematics*, *14*(2), 926-941.

[24] Keskin, Y. & Oturanç, G. (2011). Reduced differential transform method for partial differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, *10*(6), 741-750.

[25] Khalouta, A., & Kadem, A. (2019). An efficient method for solving nonlinear time-fractional wave-like equations with variable coefficients. *Tbilisi Mathematical Journal*, *12*(4), 131-147.