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A NOVEL ANALYTICAL METHOD FOR THE EXACT SOLUTION OF THE FRACTIONAL-ORDER BIOLOGICAL POPULATION MODEL

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Abstract: In this research, we develop a new analytical technique based on the Elzaki transform (ET) to solve the fractional-order biological population model (FBPM) with initial and boundary conditions (ICs and BCs). This approach can be used to locate both the closed approximate solution and the exact solution of a differential equation. The usefulness and validity of this strategy for managing the solution of FBPM are demonstrated using a few real-world scenarios. The dependability of the suggested strategy is also shown using a table and a few graphs. The approximate solutions that were achieved and the convergence analysis are shown in numerical simulations in a range of fractional orders. From the numerical simulations, it can be seen that the population density increases with increasing fractional order, whereas the population density drops with decreasing fractional order.

Key words: fractional biological population model, novel analytical method, Elzaki transform, Mittag-Leffler

1. INTRODUCTION

Although fractional derivatives have a long mathematical history, science did not use them frequently for a very long time. One possible explanation for the unpopularity of fractional derivatives is the prevalence of various non-equivalent definitions of them [1–3]. Furthermore, due to their non-locality, fractional derivatives lack a precise geometrical interpretation [1]. Over the past 10 years, however, fractional calculus has begun to attract the attention of mathematicians and engineers much more. It was discovered that fractional derivatives can effectively imitate a variety of applications, notably interdisciplinary ones. Fractional derivatives can be used to explain a variety of phenomena, such as the non-linear oscillation of earthquakes [3], Kilbas et al. [1] provide an overview of a few fractional derivative applications in continuous mechanics and statistical mechanics.

Many authors have researched the analytical findings on the existence and distinctiveness of fractional differential equation [FDE] solutions. Various techniques, including Adomian decomposition (ADM), Homotopy analysis [5], and many more, have been used in recent years to solve FDEs, FPDEs, and dynamic systems incorporating fractional derivatives. Fractional operators can be used to effectively represent phenomena with the memory effect since they are non-local. We stress that a particular fractional operator can change a PDE from a local to a non-local one by substituting it for the classical derivative with respect to time.

In this essay, the FBPM will be resolved using a novel approach called the Elzaki transform (ET) approach. ET and its variations are used to tackle boundary value problems. The recommended approach [10, 11, 21, 22, 24] presents the solution in a finite series form that is straightforward to compute, but the real strategy offers greater precision because different starting approximations are employed in any iterations. This class of equations including linear fractional differential equations did not have an analytical solution method prior to the 17th century. Linear and non-linear population problems were handled in [8, 9] using the VIM and HPM. Akinfe and Loyinmi [13] have examined other earlier research attempts on the current fractional biological population model and its applications in quantum physics, optics, fluid modelling employing the ET and a solitary wave solution to the generalised Burgers-Fisher's equation using an improved differential transform method in [14–17, 28]. Additionally, the fractional order model is examined in [18–20] with regard to Esmehan Ucar et al.

The Caputo fractional derivative was used for this study because it enables the formulation of the physical problems to include conventional initial and boundary conditions (BCs). Provide a few other crucial properties of fractional derivatives. A few instances of the identified difficulties are addressed using the general description of the suggested solution. Finding analytical solutions to FBPM using initial conditions (ICs) and BCs is fairly difficult. The current study uses a relatively simple and straightforward methodology to obtain closed-form analytical answers for the FBPM. We'll talk about the fractional biological population model in this article, and this strategy is a potent method for resolving the functional equations that arise from modelling various systems analytically.

The plan of our paper is as follows: Brief definitions of fractional calculus are given in Section 1. Some theorems of the ET are given in Section 2. The novel analytical method is presented in Section 3. The convergence analysis is presented in Section 4. In Section 5, three numerical examples are given to illustrate the applicability of the considered method. Numeric results are presented in Section 6.Section 7 is devoted to the conclusions of the work.

The generalised time-fractional non-linear biological population equation we suggest in this paper is as follows:

$$D_t^\alpha \Phi(x,y,t) = \frac{\partial^2}{\partial x^2}(\Phi^2) + \frac{\partial^2}{\partial y^2}(\Phi^2) + f(\Phi), \quad (1)$$
$$0 < \alpha \le 1, \quad t > 0,$$

Given ICs and BCs, and based on Verhulst and Malthusian law, we explore a more generic version of,

$$f(\Phi) = h\,\Phi^a(1 - r\Phi^b), \quad h, a, b, r \in \mathbb{R},$$

They switch to Verhulst and Malthusian laws when choosing exceptional values.

Definition 1: The following definition for the Riemann–Liouville (R–L) [23], fractional integral (FI) operator, of the order

$\alpha > 0$,of $f \in C_\mu$, $\mu \ge -1$, is given:

$J^\alpha f(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta - v)^{\alpha-1} f(v)\,dv, \ \alpha > 0\,.J^0 f(\eta) = f(\eta).$

Some properties of J^α, for, $f^n \in C_\mu, n \in N, \alpha, \beta \ge 0$ and $\gamma \ge -1$:

(a) $J^\alpha J^\beta f(\eta) = J^{\alpha+\beta} f(\eta)$

(b) $J^\alpha \eta^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\,\eta^{\alpha+\gamma}$

Definition 2: According to Caputo, the fractional derivative of $f(\eta)$, is: $D^\alpha f(\eta) = J^{m-\alpha} D^m f(\eta)$. For $m - 1 < \alpha \le m$, $m \in N, \eta > 0$,and$f \in C_{-1}^m$.

Caputo's fractional derivative (CFD) computes an ordinary derivative first, then a FI to determine the right order of a fractional derivative. The FI operator of RL and the integer order integration are both linear operations:

$J^\alpha \left(\sum_{i=1}^n c_i f_i(\eta)\right) = \sum_{i=1}^n c_i J^\alpha f_i(\eta),$

where,$\{c_i\}_{i=1}^n$are constants.

Fractional derivatives are interpreted as having a Caputo, meaning, in the current investigation, provides the reason for using the Caputo definition.

2. ELZAKI TRANSFORM

Here is a short explanation of the modified Sumudu transform, also known as the ET of the function $\Phi(\eta)$,

$$E[\Phi(\eta)] = p\int_0^\infty \Phi(\eta)\, e^{-\frac{\eta}{p}} d\eta = T(p)\ , \qquad \eta > 0,$$

where, p is a complex value.

Tarig M. Elzaki has shown in [10, 11, 5] that PDEs, ODEs, systems of PDEs, and Euler-Bernoulli Beam's can all be solved using the modified Sumudu transform, or ET. When Sumudu and Laplace transforms are unsuccessful in solving DEs with variable coefficients, ET can be effectively used [12].

Theorem 1: [2] If$\Phi = \Phi(x,y,t)$, then the partial derivatives are transformed by ET as follows:

$$E\left[\frac{\partial \Phi}{\partial t}\right] = \frac{1}{p}T(x,y,p) - p\,\Phi(x,y,0)\,,$$
$$E\left[\frac{\partial^2 \Phi}{\partial t^2}\right] = \frac{1}{p^2}T(x,y,p) - \Phi(x,y,0) - p\frac{\partial \Phi(x,y,0)}{\partial t}\,,$$
$$E\left[\frac{\partial \Phi}{\partial x}\right] = \frac{d}{dx}[T(x,y,p)], \qquad E\left[\frac{\partial^2 \Phi}{\partial x^2}\right] = \frac{d^2}{dx^2}[T(x,y,p)],$$
$$E\left[\frac{\partial \Phi}{\partial y}\right] = \frac{d}{dy}[T(x,y,p)], \qquad E\left[\frac{\partial^2 \Phi}{\partial y^2}\right] = \frac{d^2}{dy^2}[T(x,y,p)].$$

ET of some functions:

$\Phi(\eta)$	$E[\Phi(\eta)] = T(p)$
1	p^2
η	p^3
η^n	$n!\ \ p^{n+2}$
$e^{a\eta}$	$\frac{p^2}{1 - ap}$
$\sin a\eta$	$\frac{ap^3}{1 + a^2p^2}$
$\cos a\eta$	$\frac{p^2}{1 + a^2p^2}$

Here, we show some lemmas that can be applied to extract the function $\Phi(\eta)$, from its ET.

Lemma 1: ET of R–L FI operator of order $\alpha > 0$, is represented as:

$E[J^\alpha \Phi(\eta)] = p^\alpha\, T(p).$

Proof: We begin by:

$$E[J^\alpha \Phi(\eta)] = E\left[\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta - \alpha)^{\alpha-1}\Phi(\eta)\ d\eta\right],$$
$$= \frac{1}{\Gamma(\alpha)}\ \frac{1}{p}\ T(p)\, G(p) = p^\alpha\, T(p)$$

where

$G(p) = E[\eta^{\alpha-1}] = p^{\alpha+1}\Gamma(\alpha).$

Lemma 2: ET of CFD for $\alpha > 0$, $m - 1 < \alpha \le m$, $m \in N$, is;

$$E[D_t^\alpha \Phi] = p^{m-\alpha}\begin{bmatrix} \frac{T(x,y,p)}{p^m} - \frac{\Phi(x,y,0)}{p^{m-2}} - \frac{\frac{\partial \Phi(x,y,0)}{\partial t}}{p^{m-3}} \\ - \cdots - p\,\frac{\partial^{m-1}\Phi(x,y,0)}{\partial t^{m-1}} \end{bmatrix},$$

or

$E[D_t^\alpha \Phi] = \frac{1}{p^\alpha}E[\Phi] - \sum_{k=0}^{m-1}\frac{\partial^k \Phi(x,y,0)}{\partial t^k}p^{2-\alpha+k}, m - 1 < \alpha \le m,$

The following is the definition of the normal and generalized Mittag-Leffler functions:

$\in_\alpha(\eta) = \sum_{n=0}^\infty \frac{\eta^n}{\Gamma(n\alpha+1)}, \in_{\alpha,\beta}(\eta) = \sum_{n=0}^\infty \frac{\eta^n}{\Gamma(n\alpha+\beta)}.$

Lemma 3: If, $\alpha, \beta > 0, a \in C$ and $\frac{1}{p^\alpha} > |a|$, the formula for inverse ET is as follows:

$$E^{-1}\left[\frac{p^{\beta+1}}{1 + a\,p^\alpha}\right] = \eta^{\beta-1} \in_{\alpha,\beta}(-a\,\eta^\alpha).$$

Proof:

$$\frac{p^{\beta+1}}{1 + a\,p^\alpha} = p^{\beta+1}\frac{1}{1 + a\,p^\alpha} = p^{\beta+1}\sum_{n=0}^\infty (-a)^n (p^\alpha)^n$$
$$= \sum_{n=0}^\infty (-a)^n\ p^{n\alpha+\beta+1},$$

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Then:

$$E^{-1}\left[\frac{p^{\beta+1}}{1+a\,p^{\alpha}}\right] = E^{-1}\left[\sum_{n=0}^{\infty}(-a)^n\, p^{n\alpha+\beta+1}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-a)^n\eta^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = A^{\beta-1}\sum_{n=0}^{\infty}\frac{(-a\eta^{\alpha})^n}{\Gamma(n\alpha+\beta)} = \eta^{\beta-1}\in_{\alpha,\beta}(-a\eta^{\alpha}).$$

3. THE NOVEL ANALYTICAL METHOD

We explain the basic tenets of the suggested approach in this section. Let's look at the fractional non-linear non-homogeneous PDE,

$$D_t^{\alpha}\Phi + R[\Phi] + N[\Phi] = g(x,y,t)\,,\quad 0<\alpha\le 2,\ 0\le x\le d\,,\ 0\le y\le j\,,\ t\ge 0,\ d,j\in\mathbb{R}, \tag{2}$$

with the ICs, and BCs,

$$\begin{aligned}&\Phi(x,y,0)=h_1(x,y)\,,\ \ \Phi_t(x,y,0)=h_2(x,y)\,,\\ &\Phi(0,y,t)=k_1(y,t)\,,\ \ \Phi(d,y,t)=k_2(y,t)\,,\\ &\Phi(x,0,t)=k_3(x,t)\,,\ \ \Phi(x,j,t)=k_4(x,t)\,,\end{aligned} \tag{3}$$

R, N are linear and non-linear operators, and g, inhomogeneous terms.

Considering that, Φ_n^* the new method was employed to calculate the new solution,

$$\Phi_n^* = \Phi_n + (d-x)(k_1-\Phi_n(0,y,t)) + x(k_2-\Phi_n(d,y,t)) + (j-y)(k_3-\Phi_n(x,0,t)) + y(k_4-\Phi_n(x,j,t)), \tag{4}$$

where *n* = 0, 1, ⋯.

It is obvious that Φ_n^* will meet both the ICs' and BCs' standards. We can resolve Eq. (2) by using the ET to deduce Eq. (3).

$$\begin{aligned}&E[D_t^{\alpha}\Phi] + E\{R[\Phi]+N[\Phi]\} = E[g]\,,\\ &E(\Phi) = p^2h_1(x,t) + p^3h_2(x,t) - p^{\alpha}E\{R[\Phi]+N[\Phi]-g\}.\end{aligned} \tag{5}$$

It is believed that the solution to Eq. (2) has the following series form:

$$\Phi = \sum_{n=0}^{\infty}\Phi_n\,, \tag{6}$$

As a result of applying Eq. (6) and the inverse of ET to Eq. (5), we can now determine,

$$\sum_{n=0}^{\infty}\Phi = G - E^{-1}\{p^{\alpha}E[R[\Phi]+N[\Phi]]\}, \tag{7}$$

When, G an expression is made from a source word and the necessary ICs,

This method relies on how we choose the initial iteration Φ_0 that offers the exact solution in a constrained number of steps.

To discover the solution iteratively, apply the relations listed below.

$$\Phi_{n+1} = E^{-1}\{p^{\alpha}E[R[\Phi_n^*]+N[\Phi_n^*]]\}\,,$$

$$\Phi_0 = G, \tag{8}$$

From Eqs. (8) and (4), we can infer that:

$\Phi_0,\ \ \Phi_1,\ \ \Phi_2, \ldots,$

The solution can then be inferred from Eq. (6).

We demonstrate that FBPMs that are under the ICs, BCs, and ET may be resolved using the suggested strategy.

4. CONVERGENCE ANALYSIS CONSIDERED PROBLEM

This section examines the FBPM's convergence for the specified problem, as stated in Eq. (1). To do this, we apply the operator's Eq. (1) as:

$$T(\Phi) = D_t\Phi = (D_x^2+D_y^2)\,\Phi^2 + h\Phi^a - rh\Phi^{a+b},$$

Let $H\in L^2[T],\ \forall\Phi\in H,$ [25] where

$H\in L_{\Phi}^2[(m,n)\times[0,T]],$ such that,

$\Phi := [(m,n)\times[0,T]] \to R^3,$

with $m << 0$ and $B=[(m,n)\times[0,T]],$

where $\|\Phi\|_H^2 = \int_B \Phi^2\,dxdydt,$ then $E_t^{-1}\{E_t[\Phi(x,y,t)]\} < \infty$

We now assume the following in order to demonstrate T, to be semi-continuous [25]:

Assumption:

H_1 for $\sigma>0,$ exist a constant $\beta>0,$ and $\forall\Phi_1,\Phi_2\in H,$ with $k\|\Phi_1+\Phi_2\|\le\sigma,$

we obtain

$$\|T(\Phi_1)-T(\Phi_2)\| \le \beta\|\Phi_1-\Phi_2\|, \forall\Phi_1,\Phi_2\in H.$$

Theorem 2: (Convergence Condition)[26]. Without initial and BCs convergent to a specific solution, the problem under consideration is examined in Eq. (1).

Making use of the above Assumption for operator $T(\Phi)$, in Eq. (1), to obtain,

$$\begin{aligned}T(\Phi_1)-T(\Phi_2) &= (D_x^2+D_y^2)\,\Phi_1^2 + h\Phi_1^a - rh\Phi_1^{a+b}\\ &-\{(D_x^2+D_y^2)\,\Phi_2^2 + h\Phi_2^a - rh\Phi_2^{a+b}\} = D_x^2(\Phi_1^2-\Phi_2^2)\\ &+D_y^2(\Phi_1^2-\Phi_2^2) + h(\Phi_1^a-\Phi_2^a) - rh(\Phi_1^{a+b}-\Phi_2^{a+b}),\end{aligned}$$

By using the norm, we can get:

$$\begin{aligned}\|T(\Phi_1)-T(\Phi_2)\| &= \|D_x^2(\Phi_1-\Phi_2)(\Phi_1+\Phi_2)\|\\ &+\|D_y^2(\Phi_1-\Phi_2)(\Phi_1+\Phi_2)\|\\ &+h\|(\Phi_1^a-\Phi_2^a)\| - rh\|(\Phi_1^{a+b}-\Phi_2^{a+b})\|,\end{aligned}$$

By utilizing the conditions on the operators $D_x^2, D_y^2,$ in H, $\exists\ \varsigma_1,\varsigma_2>0,$ and if $a=b=1,$ we can define,

$$\begin{aligned}&D_x^2(\Phi_1-\Phi_2)(\Phi_1+\Phi_2) \le \varsigma_1\|\Phi_1-\Phi_2\|,\\ &D_y^2(\Phi_1-\Phi_2)(\Phi_1+\Phi_2) \le \varsigma_2\|\Phi_1-\Phi_2\|,\end{aligned}$$

Therefore,

$$\begin{aligned}&\|T(\Phi_1)-T(\Phi_2)\| \le \varsigma_1\|\Phi_1-\Phi_2\| + \varsigma_2\|\Phi_1-\Phi_2\|\\ &+h\|\Phi_1-\Phi_2\| - r\sigma\|\Phi_1-\Phi_2\| \ \Rightarrow\\ &\|T(\Phi_1)-T(\Phi_2)\| \le (\varsigma_1+\varsigma_2+h-r\sigma)\|\Phi_1-\Phi_2\|,\end{aligned}$$

Let,

$d = \varsigma_1+\varsigma_2+h-r\sigma > 0,$ then we can write,

$$\|T(\Phi_1)-T(\Phi_2)\| \le d\|\Phi_1-\Phi_2\|.$$

Thus, the assumption is met. As a result, the suggested approach converges.

5. ILLUSTRATIVE EXAMPLES

In this section, we will use three numerical examples to illustrate the efficiency and dependability of the method.

Example 1: Take a look at FBPM in one dimension,

$$D_t^{\alpha}\Phi(x,t) = \frac{\partial^2}{\partial x^2}(\Phi^2) + \Phi\left(1-\frac{4}{9}\Phi\right), 0<\alpha\le 1, \quad 0\le x\le d, \quad t\ge 0, \quad d,j\in\mathbb{R}, \tag{9}$$

With the IC and BCs,

$$\Phi(x,0) = e^{\frac{x}{3}}, \quad \Phi(0,t) = \in_{\alpha}(t^{\alpha}), \quad \Phi(d,t) = e^{\frac{d}{3}} \in_{\alpha}(t^{\alpha}), \tag{10}$$

By combining the IC, ET, and Eq. (9), the following result is produced:

$$\frac{1}{p^{\alpha}}E[\Phi(x,t)] - \Phi(x,0)p^{2-\alpha} - E[\Phi(x,t)] = E\left[\frac{\partial^2}{\partial x^2}(\Phi^2) - \frac{4}{9}\Phi^2\right],$$

$$\Rightarrow E[\Phi(x,t)] = \frac{p^2 e^{\frac{x}{3}}}{1-p^{\alpha}} + \frac{p^{\alpha}}{1-p^{\alpha}}E\left[\frac{\partial^2}{\partial x^2}(\Phi^2) - \frac{4}{9}\Phi^2\right],$$

Inverse ET suggests that:

$$\Phi((x,t)) = e^{\frac{x}{3}} \in_{\alpha}(t^{\alpha}) + E^{-1}\left\{\frac{p^{\alpha}}{1-p^{\alpha}}E\left[\frac{\partial^2}{\partial x^2}(\Phi^2) - \frac{4}{9}\Phi^2\right]\right\},$$

The following diagram illustrates the iteration formula using an initial approximation.

$$\Phi_{n+1}(x,t) = E^{-1}\left\{\frac{p^{\alpha}}{1-p^{\alpha}}E\left[\frac{\partial^2}{\partial x^2}(\Phi_n^*)^2 - \frac{4}{9}(\Phi_n^*)^2\right]\right\}, \tag{11}$$

with, $\Phi_0(x,t) = e^{\frac{x}{3}} \in_{\alpha}(t^{\alpha})$,

Utilize the BCs in Eq. (4) and $n=0$, to ascertain:

$$\Phi_0^*(x,t) = \Phi_0(x,t) + (d-x)(\Phi(0,t)-\Phi_0(0,t)) + x\big(\Phi(d,t)-\Phi_0(d,t)\big) = e^{\frac{x}{3}} \in_{\alpha}(t^{\alpha}),$$

Eq. (11), give:

$$\Phi_1(x,t) = E^{-1}\left\{\frac{p^{\alpha}}{1-p^{\alpha}}E\left[\frac{\partial^2}{\partial x^2}(\Phi_0^*)^2 - \frac{4}{9}(\Phi_0^*)^2\right]\right\} = 0,$$

Then,

$$\Phi_1 = 0, \quad \Phi_2 = 0, \quad \Phi_3 = 0, \ \dots,$$

Using Eq. (6), to find the solution of Eq. (9),

$$\Phi(x,t) = \sum_{n=0}^{\infty}\Phi_n(x,t) = e^{\frac{x}{3}} \in_{\alpha}(t^{\alpha}), \text{if } \alpha = 1, \text{then: } \Phi(x,t) = e^{\frac{x}{3}+t}.$$

Example 2: Consider the FBPM in two-dimensional,

$$D_t^{\alpha}\Phi = \frac{\partial^2}{\partial x^2}(\Phi^2) + \frac{\partial^2}{\partial y^2}(\Phi^2) + \Phi, \ 0<\alpha\le 1, 0\le x,y\le \pi, \ t\ge 0, \tag{12}$$

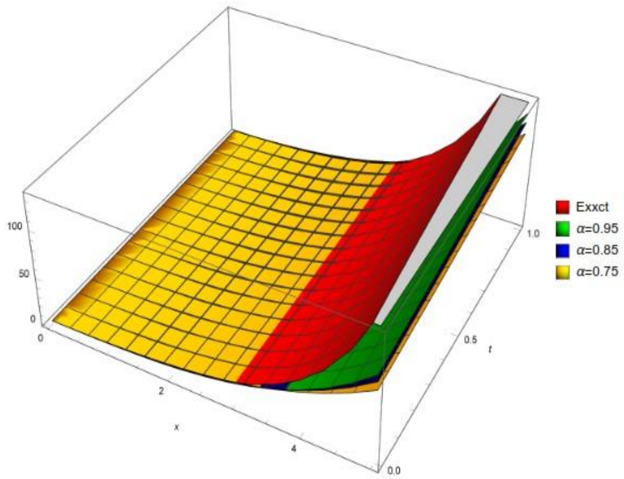
With the IC and BCs,

$$\Phi(x,y,0) = \sqrt{sinxsinhy}, \quad \Phi(0,y,t) = 0, \quad \Phi(\pi,y,t) = 0, \quad \Phi(x,0,t) = 0, \quad \Phi(x,\pi,t) = \sqrt{sinxsinh\pi}.\in_{\alpha}(t^{\alpha}), \tag{13}$$

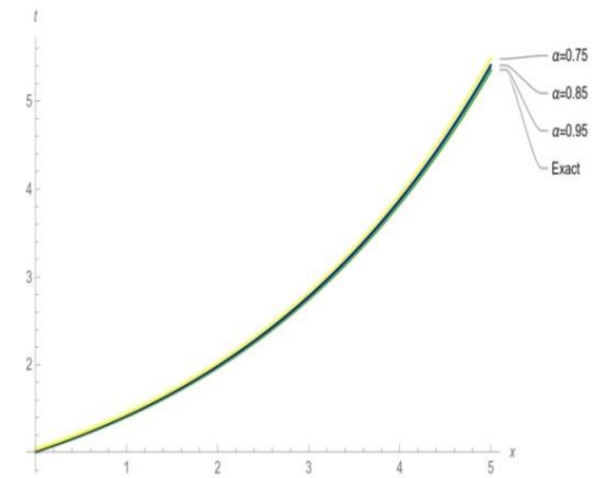
By combining the IC, ET, and Eq. (12), the following result is produced:

$$\frac{1}{p^{\alpha}}E[\Phi] - \Phi(x,y,0)p^{2-\alpha} - E[\Phi(x,y,t)] = E\left[\frac{\partial^2}{\partial x^2}(\Phi^2) + \frac{\partial^2}{\partial y^2}(\Phi^2)\right],$$

$$\Rightarrow E[\Phi] = \frac{p^2}{1-p^{\alpha}}\sqrt{\text{sinxsinhy}} + \frac{p^{\alpha}}{1-p^{\alpha}}E\left[\frac{\partial^2}{\partial A^2}(\Phi^2) + \frac{\partial^2}{\partial y^2}(\Phi^2)\right],$$



(1)



(2)

Fig. 1,2. 3D and 2D graph representations of the exact and approximate solutions for example 1, when $t = 0.01, \alpha = 0.95, 0.85, 0.75$

Tab. 1. The numirical outcome of example 1 is determinted by comparing the exact and approximate solutions for the two-terms approximation

	t	x	$\alpha = 0,75$	$\alpha = 0,85$	$\alpha = 0,95$	Exact
		0	1.03517	1.02136	1.01293	1.01005
		0.1	1.07026	1.05598	1.04727	1.04429
		0.2	1.10654	1.09177	1.08277	1.07968
		0.3	1.14404	1.12878	1.11947	1.11628
		0.4	1.18282	1.16704	1.15741	1.15411
$\Phi(x,t)$	0.01	0.5	1.22291	1.20659	1.19664	1.19323
		0.6	1.26436	1.24749	1.2372	1.23368
		0.7	1.30722	1.28978	1.27914	1.27549
		0.8	1.35153	1.33349	1.32249	1.31873
		0.9	1.39734	1.37869	1.36732	1.36343
		1	1.4447	1.42542	1.41366	1.40964

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Inverse ET suggests that:

$$\Phi = \in_\alpha (t^\alpha)\sqrt{sinxsinhy} + E^{-1}\left\{\frac{p^\alpha}{1-p^\alpha}E\left[\frac{\partial^2}{\partial x^2}(\Phi^2) + \frac{\partial^2}{\partial y^2}(\Phi^2)\right]\right\},$$

The following diagram illustrates the iteration formula using an initial approximation

$$\Phi_{n+1} = E^{-1}\left\{\frac{p^\alpha}{1-p^\alpha}E\left[\frac{\partial^2}{\partial x^2}(\Phi_n^*)^2 + \frac{\partial^2}{\partial y^2}(\Phi_n^*)^2\right]\right\}, \quad (14)$$

With, $\Phi_0 = \in_\alpha (t^\alpha)\sqrt{\text{sinxsinhy}}$, use the BCs in Eq. (4) and $n = 0$ to find:

$$\Phi_0^* = \Phi_0 + (\pi - x)(\Phi(0,y,t) - \Phi_0(0,y,t)) + x(\Phi(\pi,y,t) - \Phi_0(\pi,y,t)) + (\pi - y)(\Phi(x,0,t) - \Phi_0(x,0,t)) + y(\Phi(x,\pi,t) - \Phi_0(x,\pi,t)) = \in_\alpha (t^\alpha)\sqrt{\text{sinxsinhy}},$$

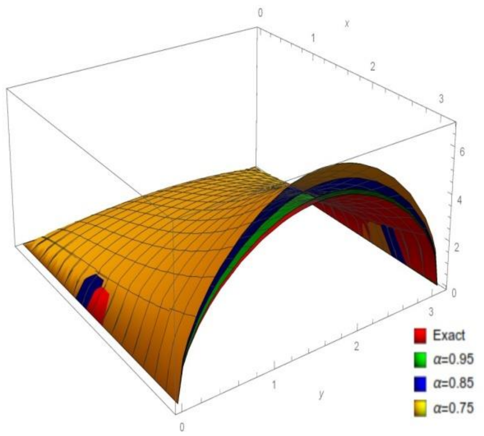
From Eq. (14), we get:

$$\Phi_1 = E^{-1}\left\{\frac{p^\alpha}{1-p^\alpha}E\left[\frac{\partial^2}{\partial x^2}(\Phi_0^*)^2 + \frac{\partial^2}{\partial y^2}(\Phi_0^*)^2\right]\right\} = 0,$$

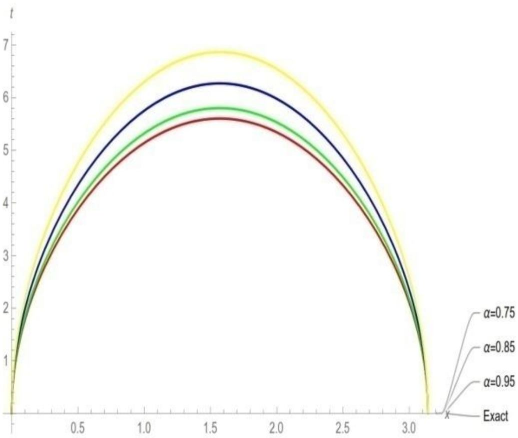
Then, $\Phi_1 = 0, \quad \Phi_2 = 0, \quad \Phi_3 = 0, \ \ldots ,$

The solution to Eq. (12) can then be found by applying Eq. (6), $\Phi = \sum_{n=0}^{\infty}\Phi_n = \in_\alpha (t^\alpha)\sqrt{\text{sinxsinhy}},$

when, $\alpha = 1$, then $\Phi = e^t\sqrt{\text{sinxsinhy}}$.



(3)



(4)

Fig. 3,4: 3D and 2D graph representations of the exact and approximate solutions for example 2, when $t = 0.5, \alpha = 0.95, 0.85, 0.75$

Tab. 2. The numerical outcames of example 2 is determinted by comparing the exact and approximate solutions for the two-terms approximation.

	t	y	x	α = 0,75	α = 0,85	α = 0,95	Exact
			0	0.	0.	0.	0.
			$\pi/6$	4.85575	4.43446	4.1024	3.96186
			$\pi/3$	6.39052	5.83608	5.39906	5.21411
$\Phi(y,t)$	0,5	π	$\pi/2$	6.86706	6.27127	5.80167	5.60292
			$2\pi/3$	6.39052	5.83608	5.39906	5.21411
			$5\pi/6$	4.85575	4.43446	4.1024	3.96186
			π	0.	0.	0.	0.

Example 3: Think of the FBPM in two dimensions,

$$D_t^\alpha\Phi(x,y,t) = \frac{\partial^2}{\partial x^2}(\Phi^2) + \frac{\partial^2}{\partial y^2}(\Phi^2) + k\Phi, \ 0 < \alpha \le 1, \ 0 \le x \le d, 0 \le y \le j, \ t \ge 0, \ k,d,j \in \mathbb{R}, \quad (15)$$

With the IC and BCs,

$$\Phi(x,y,0) = \sqrt{xy}, \Phi(0,y,t) = 0, \ \Phi(d,y,t) = \sqrt{dy}\in_\alpha ((kt)^\alpha), \ \Phi(x,0,t) = 0, \ \Phi(x,j,t) = \sqrt{jx}\in_\alpha ((kt)^\alpha), \quad (16)$$

The recurrence relationship will continue to exist in the same manner, using an initial approximation, as shown in the following:

$$\Phi_{n+1} = E^{-1}\left\{\frac{p^\alpha}{1-p^\alpha}E\left[\frac{\partial^2}{\partial x^2}(\Phi_n^*)^2 + \frac{\partial^2}{\partial y^2}(\Phi_n^*)^2\right]\right\}, \quad (17)$$

With, $\Phi_0 = \in_\alpha ((kt)^\alpha)\sqrt{xy}$, use the BCs in Eq. (4) and but, $n = 0$ to get:

$$\Phi_0^* = \Phi_0 + (d - x)(\Phi(0,y,t) - \Phi_0(0,y,t)) + x(\Phi(d,y,t) - \Phi_0(d,y,t)) + (j - y)(\Phi(x,0,t) - \Phi_0(x,0,t)) + y(\Phi(x,j,t) - \Phi_0(x,j,t)) = \in_\alpha ((kt)^\alpha)\sqrt{xy},$$

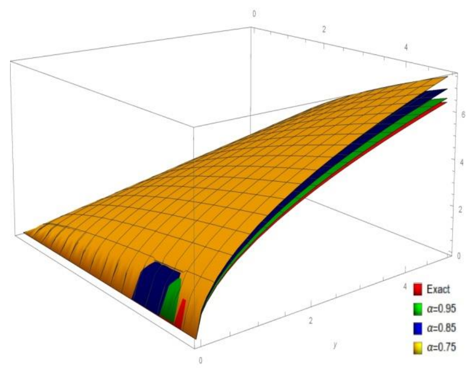
The following is taken from Eq. (17):

$$\Phi_1 = E^{-1}\left\{\frac{p^\alpha}{1-p^\alpha}E[0]\right\} = 0$$

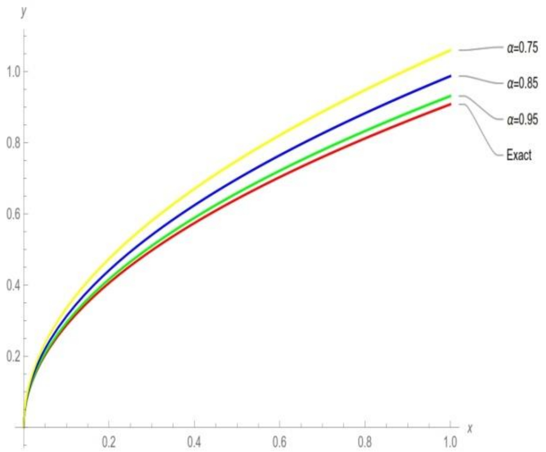
then, $\Phi_1 = 0, \quad \Phi_2 = 0, \quad \Phi_3 = 0, \ \ldots ,$

The solution to Eq. (15) can therefore be found using Eq. (6).

$\Phi = \sum_{n=0}^{\infty}\Phi_n = \in_\alpha \left((kt)^\alpha\right)\sqrt{xy}$, if, $\alpha = 1$, then: $\Phi = e^{kt}\sqrt{xy}$.



(5)



(6)

Fig. 5,6. 3D and 2D graph representations of the exact and approximate solutions for example 3, when $t = 0.5, \alpha = 0.95, 0.85, 0.75$.

Tab. 3. The numerical outcames of example 3 is determinted by comparing the exact and approximate solutions for the two-terms approximation.

	t	y	x	$\alpha = 0,75$	$\alpha = 0,85$	$\alpha = 0,95$	Exact
			0	1.03517	1.02136	1.01293	1.01005
			0.1	1.07026	1.05598	1.04727	1.04429
			0.2	1.10654	1.09177	1.08277	1.07968
			0.3	1.14404	1.12878	1.11947	1.11628
			0.4	1.18282	1.16704	1.15741	1.15411
$\Phi(y,t)$	0.5	0.5	0.5	1.22291	1.20659	1.19664	1.19323
			0.6	1.26436	1.24749	1.2372	1.23368
			0.7	1.30722	1.28978	1.27914	1.27549
			0.8	1.35153	1.33349	1.32249	1.31873
			0.9	1.39734	1.37869	1.36732	1.36343
			1	1.4447	1.42542	1.41366	1.40964

6. NUMERICAL RESULTS

This work resolves the FBPM with ICs and BCs using a novel analytical method based on ET. Results obtained using earlier methods are not equivalent to those obtained with current methods. Multiple parameter values are given in Eqs. (9), (12), and (15) to offer a variety of solutions. A variety of solutions can be produced by permitting the arbitrary parameters to have different values in the solutions. The responses gathered are categorized. Visual representations in 2D and 3D are also developed. The following information can be used to describe these plots: Figs. 1, 2, and 3 depict lone waves in different arrangements. Fig. 1 was produced for the values in Eq. (9), showing: at $\alpha = 0.95, 0.85, 0.75$, $t = 0.01$.This mixture falls within the periodic category. Fig. 2 was created using the data $\alpha = 0.95$, $0.85, 0.75$, $t = 0.5$, in Eq. (12). Fig. 3 was created using$\alpha = 0.95, 0.85, 0.75, t = 0.5$, in Eq. (15).

The exact solution, which was determined by comparing the values of the exact and approximate solutions of FBPM discovered in this issue for various values of the variables,$0 < x \leq 1, t = 0.01$, is provided in Tab.1. Tab.2 contains the approximate solutions, which were ascertained by comparing the values of the exact solutions of FBPM for various values of the variables, $0 < x \leq \pi, y = \pi, t = 0.5. 0 < x \leq 1, y = 0.5, t = 0.5$. We were able to determine the exact solution, which is presented in Tab. 3. The suggested approaches were found to be both workable and efficient. Using the Wolfram Mathematica program, the simulations were run and the outcomes were examined.

7. CONCLUSION

In this paper, analytical evaluations of non-linear fractional biological population models with ICs and BCs are carried out. The Riemann–Liouville FI operator generates partially specified fractional derivatives. For the speedy and effective resolution of numerous difficulties, a novel approach based on ET is proposed. Three examples are offered to show the value of the suggested approach. The solutions might be handled in a very simple way. The outstanding ability of the approach to solve non-linear FBPM utilising ICs and BCs allows it to be modified to address a variety of boundary value problems. In addition, 2D and 3D graphs were employed to demonstrate how the suggested approach contributed to the outcomes.

Figures and tables demonstrate that the shapes of the solutions discovered using the suggested method is similar to those of the precise solution when the same parameters were chosen. The proposed method can also be extended to solve additional FPDEs that arise in applied research, according to the ease with which it can be put into practice. In the future, we propose adapting the considered novel approach with the ET scheme for the analysis of nonlinear partial differential equations and certain advanced integral-related problems in fluid dynamics and elasticity and investigating the proposed method's stability and error analysis in forthcoming articles. Finally, we affirm that, subject to ICs and BCs, the proposed technique is valid and applicable to all non-linear FPDEs.

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