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Real solutions of polynomial equations*

by

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Abstract: In this paper the problem of existence of solution of polynomial equations over the field of real numbers is considered. In particular, the explicit necessary and sufficient conditions are established for the equation $A(z)X + B(z)Y = C(z)$ in polynomial matrices to have a solution for X and Y over the field of real numbers, with X being non-singular, for every polynomial matrix $C(z)$ from a given class.

Keywords: polynomial equations, polynomial matrices, real solutions

1. Introduction

Let R be the field of real numbers. Also, let R$[z]$ be the ring of polynomials with coefficients in R. Let $A(z)$ and $B(z)$ be fixed matrices over R$[z]$ with dimensions mxk and m x p, respectively. Further, let δ_ι for $i = 1, 2, ..., m$, be the row degrees of $[A(z), B(z)]$. Then, let L be the class of all matrices $C(z)$ over R$[z]$ with dimensions mxq and row degrees less than or equal to δ_ι for $i = 1, 2, ..., m$.

In this paper we study the following problem: do there exist matrices X and Y over R of appropriate dimensions, with X being non-singular, such that

$$A(z)X + B(z)Y = C(z) \tag{1}$$

for every polynomial matrix $C(z)$ from a given class L?

If so, give the necessary and sufficient conditions for the equation (1) to have a solution for X and Y over R with X being non-singular, for every polynomial matrix $C(z)$ from a given class L, and a procedure for the computation of the solution.

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Sufficient conditions, under which equation (1) will have a solution for X and Y over R with X being non-singular, for every polynomial matrix $C(z)$ from a given class L, have been established by Kucera and Zagalak (1992).

In this paper, explicit necessary and sufficient conditions are established for the equation (1) to have a solution for X and Y over R with X being non-singular, for every polynomial matrix $C(z)$ of dimension m x q, from a given class L and a procedure is established for the computation of the solution. The results of this paper may be useful in the study of some problems in the area of linear implicit systems, see Kucera, and Zagalak (1992), Korotka, Zagalak and Loiseau (2012), and Korotka, Zagalak, Loiseau and Kucera (2012).

2. Basic concepts and preliminary results

This section contains lemmas, which are needed to prove the main result of this paper and some basic notions from linear algebra and algebra of polynomial matrices that are used throughout the paper.

Let D and E be matrices over R of dimensions m x n, respectively. Let also rank$[D] = r$. Then, there exist non-singular matrices P and M over R of dimensions m x m and n x n, respectively, such that

$$D = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} M. \tag{2}$$

Let

$$P^{-1}E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{3}$$

Consider the linear equation

$$DX = E. \tag{4}$$

By using (2) and (3), and some algebraic manipulations, equation (4) can be rewritten as follows:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} MX \quad = \quad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{5}$$

It follows directly from (5) that the columns of the matrices D and E span the same linear space over R, or, equivalently, there exists a non-singular matrix$X$ over R of appropriate dimensions, which satisfies (4) and (5), if and only if the following conditions hold:

$$\text{rank } [E_1] = r \text{ and } E_2 = 0. \tag{6}$$

Let Z be a non-singular matrix of appropriate dimensions such that

$$E_1 = [I_r, 0]Z. \tag{7}$$

If conditions (6) are satisfied, then the non-singular matrix X over R of appropriate dimensions, satisfying (4) and (5), is given by

$$X = M^{-1} \begin{bmatrix} I_r & 0 \\ K & N \end{bmatrix} Z, \tag{8}$$

where K, N are arbitrary matrices over R of appropriate dimensions, with matrix N being non-singular.

Let $D(z)$ and $E(z)$ be polynomial matrices of dimensions m x n, respectively, with elements in R$[z]$. Let μ_ι for $i = 1, 2, ..., m$ be the row degrees of $D(z)$. Also, let the row degrees of the matrix $E\,(z)$ be less than or equal to μ_ι for $i = 1, 2, ..., m$. Consider the linear equation

$$D\,(z)\,X = E(z). \tag{9}$$

Let $d_i(z)$ and $e_i(z)$ for $i = 1, 2, ..., m$ be the rows of the polynomial matrices $D(z)$ and $E(z)$, respectively, with elements in R$[z]$. Write the row vectors $d_i(z)$ and $e_i(z)$ for $i = 1, 2, ..., m$ as follows

$$d_i(z) = \Sigma_{j=0}^{\mu_\iota} d_{ij} z^j \text{ and } \; e_i(z) = \Sigma_{j=0}^{\mu_\iota} e_{ij} z^j, \tag{10}$$

where d_{ij} and e_{ij} are row vectors over R of appropriate dimensions. Substituting (10) into (9) and using some algebraic manipulations we obtain the following system of linear equations:

$$DX = E. \tag{11}$$

Since equations (9) and (11) are equivalent, it follows directly from the above that the columns of the matrices $D(z)$ and $E(z)$ span the same linear space over R, or, equivalently, there exists a non-singular matrix X over R of appropriate dimensions that satisfies (9) and (11), if and only if conditions (6) hold. It is now, therefore, absolutely clear, when the columns of two polynomial matrices with elements in R[z] span the same linear space over R.

Taking into consideration the above, it is natural to pose the following question. When the equation (9) has a solution for every polynomial matrix $E(z)$ with elements in R$[z]$ and row degrees less than or equal to μ_ι for $i = 1, 2, ..., m$? The equations (9) and (11) are equivalent. This implies that equation (9) has a solution for every polynomial matrix $E(z)$ with elements in R$[z]$, if and only if the equation (11) has a solution for every matrix E over R or, equivalently, if and only if the matrix D over R has full row rank. It is well known from algebra of polynomial matrices that the columns of polynomial matrix $E(z)$ with elements in R$[z]$ span the linear space over R of all polynomial column vectors with row degrees less than or equal to μ_ι for $i = 1, 2, ..., m$, or, equivalently, the equation (9) has a solution for every polynomial matrix $E(z)$ with elements in R[z] and row degrees less than or equal to μ_ι for $i = 1, 2, ..., m$, if and only if the matrix D over R has full row rank.

Lemma 1 *Let A(z), B(z) and C(z) be fixed matrices with elements in R[z] of dimensions m x k , m x p and m x q, respectively. Then, equation (1) has a solution for X and Y over R if and only if the following condition holds:*

1. *The linear space over R spanned by the columns of C(z) is a subspace of the linear space over R spanned by the columns of [A(z), B(z)].*

Proof. Equation (1) can be rewritten as follows, see Kucera and Zagalak (1992):

$$[A(z),\ B(z)]\begin{bmatrix} X \\ Y \end{bmatrix} = C(z).$$

Since, by assumption, X and Y are matrices over R, the condition 1 of the Lemma follows directly from the above relationship and the proof is complete. ■

Lemma 2 *Let $k = q$, also let A(z), B(z) and C(z) be fixed matrices with elements in R[z] of dimensions $m\,x\,k$, $m\,x\,p$, and $m\,x\,q$, respectively. Then, equation (1) has a solution for X and Y over R with X being non-singular if and only if the following condition holds:*

1. *The columns of the matrices [A(z), B(z)] and [C(z), B(z)] span the same linear space over R.*

Proof. Let equation (1) have a solution for X and Y over R with X being non-singular, see Kucera and Zagalak (1992). Then

$$[A(z),\ B(z)]\begin{bmatrix} X & 0 \\ Y & I \end{bmatrix} = [C(z),\ B(z)].$$

Since, by assumption, matrix X is non-singular, the transformation matrix in the above equation is also non-singular and therefore the columns of the matrix $[A(z),\ B(z)]$ span the same linear space over R as those of $[C(z),\ B(z)]$. This is condition 1 of Lemma 2. To prove sufficiency, let T and Q be non-singular matrices over R of appropriate dimensions, such that

$$A(z)\,T = [A_1(z), A_2(z), 0],\ C(z)\,Q = [C_1(z), C_2(z), 0]$$

where the columns of the matrices $[A_1(z), A_2(z)]$ and $[C_1(z), C_2(z)]$ are linearly independent over R and where the columns of $A_1(z)$ and $C_1(z)$ do not belong to the linear column space over R of $B(z)$, while those of $A_2(z)$ and $C_2(z)$ do. Since, by assumption, the columns of the matrices $[A(z), B(z)]$ and $[C(z), B(z)]$ span the same linear space over R , then so do the columns of matrices $A_1(z)$ and $C_1(z)$. Hence, there exists a non-singular matrix X_1 over R of appropriate dimensions such that

$$A_1(z)X_1 = C_1(z)$$

and matrices X_2 and Y_2 over R, such that

$$B(z)\,X_2 = [C_2(z), 0], \quad B(z)\,Y_2 = [A_2(z), 0].$$

Thus, we have that

$$[A_1(z),\ (A_2(z),\ 0),\ B(z)] \begin{bmatrix} D & 0 \\ Y & I \end{bmatrix} = \quad [C_1(z),\ (C_2(z),\ 0), B(z)]$$

where

$$D = \text{diag}[X_1, I],\ Y = [0,\ (X_2 - Y_2)].$$

From the above it follows that the pair of matrices over R

$$X = \ T\ diag[X_1, I\,]Q^{-1}\,Y = [0,\ (X_2 - Y_2)]\,Q^{-1}$$

solves (1) and the matrix X is non-singular. This completes the proof. ∎

LEMMA 3 *Let δ_ι for $i =$ 1,2, ...,m be non-negative integers. Also let L be the class of all matrices C(z) over R[z] with dimensions m x q and row degrees less than or equal to δ_ι for i=1,2, ..., m. Let V be the linear space over R spanned by the columns of C(z). Then*

1. *The linear space V consists of all polynomial vectors v(z) with dimension $m \times 1$ and row degrees less than or equal to δ_ι for i=1,2, ..., m.*

PROOF. Let $c_i(z)$ for $i =$*1, 2, ..., q* be the columns of the matrix $C(z)$ from a class L. Let $v(z)$ be an element of V. Since by, assumption, V is the linear space over R spanned by the columns of $C(z)$, we have that

$$v(z) = \Sigma_{i=1}^{q} \lambda_\iota c_i(z)$$

where λ_ι for $i = 1, 2, \cdots, q$ are arbitrary real numbers. Since, by assumption, $C(z)$ is a matrix from a class L, the statement 1 of Lemma 3 follows directly from the above relationship and the proof is complete. ∎

LEMMA 4 *Let δ_ι for $i =$ 1,2, ...,m be non-negative integers. Also let L be the class of all matrices C(z) over R[z] with dimensions $m \times q$ and row degrees less than or equal to δ_ι for i=1,2, ..., m. Further, let B(z) be a fixed polynomial matrix over R[z]. Let also V be the linear space over R of all polynomial vectors v(z) with dimension $m \times 1$ and row degrees less than or equal to δ_ι for i=1,2, ..., m. Then, the columns of the matrix [C(z, B(z)] span V for every matrix C(z) from a class L if and only if the following condition holds:*

1. *The columns of the matrix B(z) span V.*

PROOF. Since, by assumption, L is the class of all matrices $C(z)$ over R[z] with dimensions m x q and row degrees less than or equal to δ_ι for $i =$*1,2, ..., m*, we assume, without any loss of generality, that

$$C\,(z) = K,$$

where K is an nonzero matrix over R. Then, the columns of the matrix $[K, B(z)]$ span V if and only if the columns of the matrices $B(z)$ span V and hence the proof is complete. ∎

Lemma 5 *The equation (1) has a solution for X and Y over R for every polynomial matrix C(z) from a class L, only if the following condition holds:*

1. *The columns of the matrix [A(z), B(z)] span V.*

Proof. Let equation (1) have a solution for X and Y over R for every polynomial matrix $C(z)$ from a given class L. Then, from Lemma 1, we have that the linear space over R, spanned by the columns of $C(z)$, is a subspace of the linear space over R spanned by the columns of $[A(z), B(z)]$. Since, by Lemma 3, the columns of $C(z)$ span V, we conclude that the columns of $[A(z), B(z)]$ span also V and thus the proof is complete. ■

3. Main results

The theorem that follows is the main result of this paper and gives the necessary and sufficient conditions for the equation (1) to have a solution for X and Y over R with X being non-singular, for every polynomial matrix $C(z)$ from a given class L.

Theorem 1 *Let $k = q$. Then, equation (1) has a solution for X and Y over R with X being non-singular for every polynomial matrix C(z) from a class L if and only if the following condition holds:*

1. *The columns of the matrix B(z) span V.*

Proof. Let equation (1) have a solution for X and Y over R with X being non-singular for every polynomial matrix $C(z)$ from a given class L. Since, by assumption, the class L consists of all matrices $C(z)$ over R$[z]$ with dimensions $m \times q$ and row degrees less than or equal to δ_ι for $i = 1, 2, \ldots, m$, we have from Lemma 5 that the columns of the matrix $[A(z), B(z)]$ span V. Equation (1) can be rewritten as follows

$$[A(z), B(z)] \begin{bmatrix} X & 0 \\ Y & I \end{bmatrix} = [C(z), B(z)].$$

Since X is non-singular, the transformation matrix in the above relationship is also non-singular and therefore the columns of the matrix $[A(z), B(z)]$ span the same linear space over R as the columns of $[C(z), B(z)]$. Since the columns of the matrix $[A(z), B(z)]$ span V, the columns of $[C(z), B(z)]$ span also V. From Lemma 4 we have that the columns of $[C(z), B(z)]$ span V for every polynomial matrix $C(z)$ over $R[z]$ from a given class L if and only if the columns of $B(z)$ span V. This is condition 1 of the Theorem.

Let condition 1 of the Theorem hold. Since the columns of the matrix $B(z)$ span V, the columns of the matrices $[A(z), B(z)]$ and $[C(z), B(z)]$ span also V and by Lemma 2, equation (1) has a solution for X and Y over R with X being non-singular for every polynomial matrix $C(z)$ from a given class L. This completes the proof. ■

REMARK 1 *In this paper an explicit necessary and sufficient condition has been established for the equation (1) to have a solution for X and Y over R with X being non-singular, for every polynomial matrix $C(z)$ from a given class L and an algorithm is given for the computation of the solution. This clearly demonstrates the originality of the contribution of Theorem 1, provided in this paper, with respect to existing results, see Kucera and Zagalak (1992), where restrictive sufficient conditions have been established for the solution of the same problem.*

4. Computational algorithm

Given: $A\,(z)\,,\ B\,(z)$ and δ_ι for $i = 1, 2, \ldots, m$.

Find: X and Y over R with X being non-singular, for every polynomial matrix $C(z)$ from a given class L.

Step 1: Check condition (1) of Theorem 1. If this condition is satisfied go to Step 2. If condition of Theorem 1 is not satisfied go to, Step 3.

Step 2: Let L be an arbitrary non-singular real matrix of size $q \times q$. Set $X = L$. Since δ_ι for $i = 1, 2, \ldots, m$, are the row degrees of $[A(z), B(z)]$, we conclude that the row degrees of the matrix $A\,(z)$ are less than or equal to δ_ι for $i = 1, 2, \ldots, m$. Since the matrix L is non-singular, the row degrees of the matrix $A\,(z)\,L$ are less than or equal to δ_ι for $i = 1, 2, \ldots., m$, and therefore the row degrees of the matrix $[C(z) - A\,(z)\,L]$ are also less than or equal to δ_ι for $i = 1, 2, \ldots., m$. Hence, condition (1) of Theorem 1 guarantees that the following linear equation

$$B\,(z)\,Y = C\,(z) - \ A\,(z)\,L$$

has a solution for Y over R for every non-singular matrix L over R and for every polynomial matrix $C(z)$ from a given class L. Solve the above equation and find Y.

Step 3: Our problem has no solution.

5. A numerical example

Consider equation (1) with matrices $A(z), B(z)$ and $C(z)$ given by

$$A(z) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

$$B(z) = \begin{bmatrix} z & 1 & 0 & 0 \\ 0 & 0 & \mathrm{z} & 1 \end{bmatrix}$$

$$C(z) = \begin{bmatrix} z & 1 \\ 2 & z \end{bmatrix}.$$

The row degrees of the polynomial matrix $[A(z), B(z)]$ are $\delta_\iota = 1$ for $i = 1, 2$.

Our aim is to find the solution to equation (1) for X and Y over R with X being non-singular. We shall follow the Steps of the computational algorithm given in the preceding section. Let $b_1(z)$ and $b_2(z)$ be the rows of the polynomial matrix $B(z)$. In order to execute Step 1, write

$$b_1(z) = [1\,0\,0\,0]z + [0\,1\,0\,0], b_2(z) = [0\,0\,1\,0]z + [0\,0\,0\,1].$$

Of the coefficients of the matrix polynomials $b_1(z)$ and $b_2(z)$, we form the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the above matrix is non-singular, we conclude that the columns of the matrix $B(z)$ span V (see paragraph after equation (11) in Section 2). Hence, by Theorem 1, equation (1) has a solution for X and Y over R with X being non-singular.

To carry out Step 2 set

$$X = L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

form the matrix

$$C(z) - A(z)X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

and solve the equation

$$B(z)Y = C(z) - A(z)L$$

or, equivalently, the equation

$$\begin{bmatrix} z & 1 & 0 & 0 \\ 0 & 0 & z & 1 \end{bmatrix} Y = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

The solution for Y of this equation is given by

$$Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

6. Conclusions

In this paper, an explicit necessary and sufficient condition has been established for the equation (1) to have a solution, for X and Y over the field of real numbers, with X being non-singular for every polynomial matrix $C(z)$ from a given class. Furthermore, a procedure is given for the computation of the solution. We believe that the results of this paper may be useful in the study of some problems in the area of linear implicit systems.

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