Null controllability from the exterior of a one-dimensional nonlocal heat equation[*][†]

by

Mahamadi Warma[1] and Sebastián Zamorano[2]

[1]George Mason University, Department of Mathematical Sciences,
Fairfax, VA 22030 (USA)
mwarma@gmu.edu
mjwarma@gmail.com
[2]Universidad de Santiago de Chile,
Departamento de Matemática y Ciencia de la Computación,
Facultad de Ciencia, Casilla 307-Correo 2, Santiago, Chile
sebastian.zamorano@usach.cl

Abstract: We consider the null controllability problem from the exterior for the one dimensional heat equation on the interval $(-1, 1)$, associated with the fractional Laplace operator $(-\partial_x^2)^s$, where $0 < s < 1$. We show that there is a control function, which is localized in a nonempty open set $\mathcal{O} \subset (\mathbb{R} \setminus (-1, 1))$, that is, at the exterior of the interval $(-1, 1)$, such that the system is null controllable at any time $T > 0$ if and only if $\frac{1}{2} < s < 1$.

Keywords: fractional heat equation, exterior control, null controllability, controllability to the trajectories

1. Introduction and main results

In the present paper we consider a nonlocal version of the boundary controllability problem for the heat equation in the one dimensional case. The standard problem of finding a boundary control for the heat equation is a well-known topic and has been studied by several authors. We refer, for example, to the pioneering works of MacCamy, Mize and Seidman (1968, 1969), and the books of Zuazua (2006) and Lions (1988), and the references therein, for a complete analysis and review on this topic.

We shall now describe our problem and state the main result. We consider the fractional heat equation in the interval $(-1, 1)$. That is,

$$\begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1, 1) \times (0, T), \\ u = g\chi_{\mathcal{O} \times (0,T)} & \text{in } (\mathbb{R} \setminus (-1, 1)) \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } (-1, 1). \end{cases} \tag{1.1}$$

In (1.1), $u = u(x, t)$ is the state to be controlled, $T > 0$ and $0 < s < 1$ are real numbers, $g = g(x, t)$ is the exterior control function, which is localized in a nonempty open subset $\mathcal{O}$ of $(\mathbb{R} \setminus (-1, 1))$, and $(-\partial_x^2)^s$ denotes the fractional Laplace operator, given formally for a smooth function u by the following singular integral:

$$(-\partial_x^2)^s u(x) := C_s \, \mathrm{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \, dy, \quad x \in \mathbb{R}.$$

We refer to Section 2 for the precise definition. We would like to mention here that it has been shown in Warma (2019) that a boundary control (the case where the control g is localized in a nonempty subset of the boundary) does not make sense for the fractional Laplace operator. By Warma (2019), for the fractional Laplace operator, the classical boundary control problem must be replaced by an exterior control problem. That is, the control function must be localized outside the open set $(-1, 1)$ as it is formulated in (1.1).

We shall show that for every $u_0 \in L^2(-1, 1)$ and $g \in L^2((0, T); H^s(\mathbb{R} \setminus (-1, 1)))$, the system (1.1) has a weak solution $u \in L^2((0, 1); L^2(-1, 1))$ (see Section 3). In that case the set of reachable states is given by

$$\mathcal{R}(u_0, T) = \left\{ u(\cdot, T) : \ g \in L^2((0, T); H^s(\mathbb{R} \setminus (-1, 1))) \right\}.$$

We say that the system (1.1) is null controllable at time $T > 0$, if $0 \in \mathcal{R}(u_0, T)$. The system is said to be exactly controllable at $T > 0$, if $\mathcal{R}(u_0, T) = L^2(-1, 1)$. We say that the system (1.1) is controllable to the trajectories at $T > 0$, if for any trajectory $\tilde{u}$, a solution of (1.1) with initial datum $\tilde{u}_0 \in L^2(-1, 1)$ and without control ($g \equiv 0$), and for every initial datum $u_0 \in L^2(-1, 1)$, there exists a control function $g \in L^2((-1, 1); H^s(\mathbb{R} \setminus (-1, 1)))$ such that the associated weak solution u of (1.1) satisfies

$$u(\cdot, T) = \tilde{u}(\cdot, T), \quad \text{a.e. in } (-1, 1).$$

The system is said to be approximately controllable at time $T > 0$, if $\mathcal{R}(u_0, T)$ is dense in $L^2(-1, 1)$. We refer to Section 2 for the definition of the function spaces involved.

We would like to mention that like in the classical local case ($s = 1$), discussed in Zuazua (2006, Chapter 2), we have the following situation for the

nonlocal case. We observe that solutions of (1.1) are of class C^∞ far from $(\mathbb{R} \setminus (-1,1))$ at time $t = T$. This shows that the elements of $\mathcal{R}(u_0, T)$ are C^∞ functions in $(-1,1)$. Thus, the exact controllability may not hold. For this reason, we shall study the null controllability of the system. However, since the system (1.1) is linear, the null controllability is equivalent to the controllability to trajectories.

The following theorem is the main result of the paper:

THEOREM 1 *Let $0 < s < 1$ and let $\mathcal{O} \subset (\mathbb{R} \setminus (-1,1))$ be an arbitrary nonempty open set. Then the following assertions hold.*
  1. If $\frac{1}{2} < s < 1$, then the system (1.1) is null controllable at any time $T > 0$.
  2. If $0 < s \le \frac{1}{2}$, then the system (1.1) is not null controllable at time $T > 0$.

We would like to mention that in the proof of Theorem 1, we shall heavily exploit the fact that the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ of the realization of $(-\partial_x^2)^s$ in $L^2(-1,1)$ with the zero Dirichlet exterior condition (see Section 2) satisfy the following asymptotics (see, e.g., Kwasnicki, 2012):

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-2s)\pi}{8} \right)^{2s} + O\left(\frac{1}{n} \right) \quad \text{as } n \to \infty. \tag{1.2}$$

Recall that by Theorem 1(2.), the system (1) is not null controllable at time $T > 0$, if $0 < s \le \frac{1}{2}$. It has been recently shown in Warma (2019) that the system is indeed approximately controllable at any time $T > 0$. The result, obtained in Warma (2019) is more general since it includes the N-dimensional case and the fractional diffusion equation, that is, the case where $\partial_t u$ is replaced by the Caputo time fractional derivative $\mathbb{D}_t^\alpha u$ of order $0 < \alpha \le 1$. Of course, the case of $\alpha = 1$ includes (1.1).

The null controllability from the interior (that is, the case where the control function is localized in a nonempty subset ω of $(-1,1)$) of the one-dimensional fractional heat equation has been recently investigated in Biccari and Hernández-Santamaria (2019), where the authors have shown that the system is null controllable at any time $T > 0$ if and only if $\frac{1}{2} < s < 1$. The interior null controllability of the Schrödinger and wave equations has been studied in Biccari (2018). The approximate controllability from the exterior of the super-diffusive system, that is, the case where u_{tt} is replaced by the Caputo time fractional derivative $\mathbb{D}_t^\alpha u$ of order $1 < \alpha < 2$, has been very recently considered in Louis-Rose and Warma (2020). The case of the (possible) strong damping fractional wave equation has been investigated in Warma and Zamorano (2020).

The fractional heat equation (1.1), defined over the entire real line, arises from a probabilistic process in which a particle makes long jumps random walks with a small probability, see, for instance, Bucur and Valdinoci (2016)

or Valdinoci (2009). Besides, this type of process occurs in real life phenomena quite often, see for example the biological observations in Viswanathan et al. (1996) and the study, related to marine predators in Humphries et al. (2010).

Regarding the exterior control problem, in many real life applications control is placed outside (disjoint from) the observation domain Ω, where the PDE is satisfied. Some examples of control problems where this situation may arise are (but not limited to): acoustic testing, when the loudspeakers are placed far from the aerospace structures (Larkin and Whalen, 1999); magnetotellurics (MT), which is a technique to infer earth's subsurface electrical conductivity from surface measurements (Unsworth, 2005; Weiss, Waanders and Antil, 2019); magnetic drug targeting (MDT), where drugs with ferromagnetic particles in suspension are injected into the body and the external magnetic field is then used to steer the drug to relevant areas, for example, solid tumors (Antil, Nochetto and Venegas, 2018a,b; Lübbe et al., 1996); and electroencephalography (EEG), which is used to record electrical activities in brain (Niedermayer and da Silva, 2005; Williams, Karacan and Hursch, 1974), in case one accounts for the neurons disjoint from the brain, one will obtain an external control problem. Besides, we should mention that some preliminary results concerning numerical analysis have been obtained in the recent work of Antil, Khatri and Warma (2019).

The study of fractional order operators and nonlocal PDEs is nowadays a topic of interest for the mathematicians as well as natural and engineering scientist communities, due to the numerous applications that nonlocal PDEs provide. A motivation for this growing interest stems from the large number of possible applications in the modeling of several complex phenomena, for which a local approach turns out to be inappropriate or limited. Indeed, there is an ample spectrum of situations, in which a nonlocal equation gives a significantly better description than a local PDE of the problem one wants to analyze. Among others, we mention applications in turbulence, anomalous transport and diffusion, elasticity, image processing, porous media flow, wave propagation in heterogeneous high contrast media (see, e.g., Antil and Bartels, 2017; Bogdan, Burdzy and Chen, 2003; Valdinoci, 2009; Servadel and Valdinoci, 2014 and the references therein). Also, it is well known that the fractional Laplace operator is the generator of the so-called s-stable Lévy process, and it is often used in stochastic models with applications, for instance, in mathematical finance (see, e.g., Antil, Khatri and Warma, 2019; Du et al., 2013). One of the main differences between nonlocal models and classical PDEs is that the fulfillment of a nonlocal equation at a point involves the values of the function far away from that point. We refer to Bogdan, Burdzy and Chen (2003), Caffarelli, Roquejoffre and Sire (2010) and Caffarelli and Silvestre (2007) and the references contained there for more applications and information on this topic.

The rest of the paper is structured as follows. In Section 2 we introduce the

function spaces needed to study our problem and we give some intermediate known results that are needed in the proof of our main results. In Section 3 we show the well-posedness of the system (1.1) and its associated dual system and we give an explicit representation of solutions in terms of series for both problems. Finally, in Section 4 we give the proof of our main results.

2. Preliminary results

In this section we provide the notation used and recall some known results as they are needed in the proof of our main results. We start with fractional order Sobolev spaces.

For $0 < s < 1$ and $\Omega \subset \mathbb{R}$ being an arbitrary open set, we let

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \, dxdy < \infty \right\},$$

and we endow it with the norm defined by

$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega |u(x)|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

We set

$$\widetilde{H}^s_0(\Omega) := \left\{ u \in H^s(\mathbb{R}) : \ u = 0 \ \text{ a.e. in } \ \mathbb{R} \setminus \Omega \right\}.$$

We shall denote by $\widetilde{H}^{-s}(\Omega)$ the dual space of $\widetilde{H}^s_0(\Omega)$, that is, $\widetilde{H}^{-s}(\Omega) = (\widetilde{H}^s_0(\Omega))^\star$.

For more information on fractional order Sobolev spaces, we refer to Di Nezza, Palatucci and Valdinoci (2012), Warma (2015) and the references provided there.

Next, we give a rigorous definition of the fractional Laplace operator. To do this, we need the following function space:

$$\mathcal{L}^1_s(\mathbb{R}) := \left\{ u : \mathbb{R} \to \mathbb{R} \ \text{ measurable and } \ \int_\mathbb{R} \frac{|u(x)|}{(1 + |x|)^{1+2s}} \, dx < \infty \right\}.$$

For $u \in \mathcal{L}^1_s(\mathbb{R})$ and $\varepsilon > 0$ we set

$$(-\partial_x^2)^s_\varepsilon u(x) := C_s \int_{\{y \in \mathbb{R} : \, |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \, dy, \ \ x \in \mathbb{R},$$

where C_s is a normalization constant, given by

$$C_s := \frac{s 2^{2s} \Gamma\left(\frac{2s+1}{2}\right)}{\pi^{\frac{1}{2}} \Gamma(1 - s)}. \tag{2.1}$$

The *fractional Laplacian* $(-\partial_x^2)^s$ is defined for $u \in \mathcal{L}_s^1(\mathbb{R})$ by the following singular integral:

$$(-\partial_x^2)^s u(x) := C_s \,\text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \, dy = \lim_{\varepsilon \downarrow 0} (-\partial_x^2)_\varepsilon^s u(x), \quad x \in \mathbb{R}, \quad (2.2)$$

provided that the limit exists. We notice that if $u \in \mathcal{L}_s^1(\mathbb{R})$, then $v := (-\partial_x^2)_\varepsilon^s u$ exists for every $\varepsilon > 0$, v being also continuous at the continuity points of u. For more details on the fractional Laplace operator we refer to Caffarelli and Silvestre (2007), Di Nezza, Palatucci and Valdinoci (2012), Gal and Warma (2017), and Warma (2015) and the references contained there.

Next, we consider the realization of $(-\partial_x^2)^s$ in $L^2(-1, 1)$ with the zero Dirichlet exterior condition $u = 0$ in $\mathbb{R} \setminus (-1, 1)$. More precisely, we consider the closed and bilinear form $\mathcal{F} : \widetilde{H}_0^s(-1, 1) \times \widetilde{H}_0^s(-1, 1) \to \mathbb{R}$, given by

$$\mathcal{F}(u, v) := \frac{C_s}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} \, dxdy, \quad u, v \in \widetilde{H}_0^s(-1, 1).$$

Let $(-\partial_x^2)_D^s$ be the selfadjoint operator on $L^2(-1, 1)$ associated with $\mathcal{F}$ in the sense that

$$\begin{cases} D((-\partial_x^2)_D^s) = \\ \left\{ u \in \widetilde{H}_0^s(-1, 1), \, \exists \, f \in L^2(-1, 1), \, \mathcal{F}(u, v) = (f, v)_{L^2(-1,1)} \, \forall \, v \in \widetilde{H}_0^s(-1, 1) \right\}, \\ (-\partial_x^2)_D^s u = f. \end{cases}$$

It is easy to see that

$$D((-\partial_x^2)_D^s) = \left\{ u \in \widetilde{H}_0^s(-1, 1) : \, (-\partial_x^2)^s u \in L^2(-1, 1) \right\},$$
$$(-\partial_x^2)_D^s u = ((-\partial_x^2)^s u)|_{(-1,1)}. \tag{2.3}$$

Then, $(-\partial_x^2)_D^s$ is the realization of $(-\partial_x^2)^s$ in $L^2(-1, 1)$ with the condition $u = 0$ in $\mathbb{R} \setminus (-1, 1)$. It is well known (see, e.g., Claus and Warma, 2020) that the operator $-(-\partial_x^2)_D^s$ generates a strongly continuous submarkovian semigroup $(e^{-t(-\partial_x^2)_D^s})_{t \geq 0}$ on $L^2(-1, 1)$. It has been shown in Servadel and Valdinoci (2014) that $(-\partial_x^2)_D^s$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ satisfying $\lim_{n \to \infty} \lambda_n = \infty$. In addition, if $\frac{1}{2} \leq s < 1$, then the eigenvalues are of finite multiplicity. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be the orthonormal basis of eigenfunctions, associated with the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$. Then $\varphi_n \in D((-\partial_x^2)_D^s)$ for every $n \in \mathbb{N}$, $\{\varphi_n\}_{n \in \mathbb{N}}$ is total in $L^2(-1, 1)$ and satisfies

$$\begin{cases} (-\partial_x^2)^s \varphi_n = \lambda_n \varphi_n & \text{in } (-1, 1), \\ \varphi_n = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases} \tag{2.4}$$

Next, for $u \in H^s(\mathbb{R})$ we introduce the *nonlocal normal derivative* $\mathcal{N}_s$ given by

$$\mathcal{N}_s u(x) := C_s \int_{-1}^{1} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \, dy, \quad x \in \mathbb{R} \setminus \overline{(-1,1)}, \tag{2.5}$$

where C_s is the constant given in (2.1). We notice that since equality is to be understood a.e., we have that (2.5) is the same as for a.e. $x \in \mathbb{R} \setminus (-1,1)$. By Ghosh, Salo and Uhlmann (2018, Lemma 3.2), for every $u \in H^s(\mathbb{R})$, we have that $\mathcal{N}_s u \in H^s_{\mathrm{loc}}(\mathbb{R} \setminus (-1,1))$. We would also like to mention that the operator $\mathcal{N}_s$ has been called "interaction operator" in Antil, Khatri and Warma (2019) and Du et al. (2013). Several properties of $\mathcal{N}_s$ have been studied in Claus and Warma (2020) and Dipierro, Ros-Oton and Valdinoci (2012).

The following unique continuation property, which shall play an important role in the proof of Theorem 1 has been recently obtained in Warma (2019, Theorem 16).

LEMMA 1 *Let $\lambda > 0$ be a real number and $\mathcal{O} \subset (\mathbb{R} \setminus (-1,1))$ an arbitrary nonempty open set. If $\varphi \in D((-\partial_x^2)_D^s)$ satisfies*

$$(-\partial_x^2)_D^s \varphi = \lambda \varphi \ \ in \ \ (-1,1) \ \ and \ \ \mathcal{N}_s \varphi = 0 \ \ in \ \ \mathcal{O},$$

then $\varphi = 0$ in $\mathbb{R}$.

For more details on the Dirichlet problem associated with the fractional Laplace operator we refer the interested reader to Biccari, Warma and Zuazua (2017), Grubb (2015), Ros-Oton and Serra (2014a,b) and Warma (2019), and the references provided there.

We conclude this section with the following integration by parts formula:

LEMMA 2 *Let $u \in \widetilde{H}_0^s(-1,1)$ be such that $(-\partial_x^2)^s u \in L^2(-1,1)$ and $\mathcal{N}_s u \in L^2(\mathbb{R} \setminus (-1,1))$. Then, for every $v \in H^s(\mathbb{R})$, the following identity*

$$\frac{C_s}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} dx dy =$$
$$\int_{-1}^{1} v(x)(-\partial_x^2)^s u(x) \, dx + \int_{\mathbb{R} \setminus (-1,1)} v(x) \mathcal{N}_s u(x) \, dx, \tag{2.6}$$

holds.

We refer to Dipierro, Ros-Oton and Valdinoci (2012, Lemma 3.3) (see also Warma, 2019, Proposition 3.7) for the proof and more details.

3.   Well-posedness of the parabolic problem

This section is devoted to the well-posedness and the explicit representation in terms of series for solutions to the system (1.1) and its associated dual system.

Throughout the remainder of the article, $\{\varphi_n\}_{n\in\mathbb{N}}$ denotes the orthonormal basis of eigenfunctions of the operator $(-\partial_x^2)_D^s$ associated with the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, and $(e^{-t(-\partial_x^2)_D^s})_{t\geq 0}$ denotes the strongly continuous semigroup on $L^2(-1,1)$, generated by the operator $-(-\partial_x^2)_D^s$.

Furthermore, for a given measurable set $E \subseteq \mathbb{R}^N$ $(N \geq 1)$, we shall denote by $(\cdot,\cdot)_{L^2(E)}$ the scalar product in $L^2(E)$ and by $\mathcal{D}(E)$ we mean the space of all continuously infinitely differentiable functions with compact support in E. For given $u \in L^2(-1,1)$ and $n \in \mathbb{N}$, we shall let $u_n := (u,\varphi_n)_{L^2(-1,1)}$. Finally, given a Banach space $\mathbb{X}$ and its dual $\mathbb{X}^\star$, we shall denote by $\langle\cdot,\cdot\rangle_{\mathbb{X}^\star,\mathbb{X}}$ (simply $\langle\cdot,\cdot\rangle$ if there is no confusion) they duality pairing.

3.1.   Representation of solutions to the system (1.1)

Let $T > 0$ be a fixed real number, $u_0 \in L^2(-1,1)$, $g \in L^2((0,T);H^s(\mathbb{R}\backslash(-1,1)))$, and consider the following two systems:

$$\begin{cases} \partial_t v + (-\partial_x^2)^s v = 0 & \text{in } (-1,1)\times(0,T), \\ v = 0 & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T), \\ v(\cdot,0) = u_0 & \text{in } (-1,1), \end{cases} \qquad (3.1)$$

and

$$\begin{cases} \partial_t w + (-\partial_x^2)^s w = 0 & \text{in } (-1,1)\times(0,T), \\ w = g & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T), \\ w(\cdot,0) = 0 & \text{in } (-1,1). \end{cases} \qquad (3.2)$$

Notice that the system (3.1) can be rewritten as the following Cauchy problem:

$$\begin{cases} \partial_t v + (-\partial_x^2)_D^s v = 0 & \text{in } (-1,1)\times(0,T), \\ v(\cdot,0) = u_0 & \text{in } (-1,1). \end{cases}$$

Hence, using semigroup theory and the spectral theorem for selfadjoint operators, one arrives at the following result.

PROPOSITION 1 *For every $u_0 \in L^2(-1,1)$, there is a unique function*

$$v \in C([0,T];L^2(-1,1)) \cap L^2((0,T);\widetilde{H}_0^s(-1,1) \cap H^1((0,T);\widetilde{H}^{-s}(-1,1))$$

satisfying (3.1) *and it is given for a.e. $x \in (-1,1)$ and every $t \in [0,T]$ by*

$$v(x,t) = e^{-t(\partial_x^2)_D^s} u_0(x) = \sum_{n=1}^{\infty} u_{0,n} e^{-\lambda_n t}\varphi_n(x). \qquad (3.3)$$

Next, we consider the system (3.2).

DEFINITION 1 *Let* $g \in L^2((0,T); H^s(\mathbb{R} \setminus (-1,1)))$. *By a weak solution of* (3.2), *we mean a function* $w \in L^2((0,T); H^s(\mathbb{R}))$ *such that* $w = g$ *a.e. in* $(\mathbb{R} \setminus (-1,1)) \times (0,T)$ *and the identity*

$$\int_0^T \langle -\partial_t \phi + (-\partial_x^2)^s \phi, w \rangle \, dt = \int_{-1}^1 w(x,T)\phi(x,T) \, dx + \int_0^T \int_{\mathbb{R} \setminus (-1,1)} g \mathcal{N}_s \phi \, dx dt$$

holds, for every function $\phi \in C([0,T]; L^2(-1,1)) \cap L^2((0,T); \widetilde{H}_0^s(-1,1)) \cap H^1((0,T); \widetilde{H}^{-s}(-1,1))$ *with* $\mathcal{N}_s \phi \in L^2((0,T); L^2(\mathbb{R} \setminus (-1,1)))$.

We have the following existence result, whose proof is inspired by the local case contained in the monograph of Lasiecka and Triggiani (2000, pp. 180-185).

PROPOSITION 2 *For every* $g \in L^2((0,T); H^s(\mathbb{R} \setminus (-1,1)))$, *the system* (3.2) *has a weak solution* $w \in L^2((0,T); H^s(\mathbb{R}))$, *given by*

$$w(x,t) = \sum_{n=1}^\infty \left(\int_0^t (g(\cdot,\tau), \mathcal{N}_s \varphi_n)_{L^2(\mathbb{R} \setminus (-1,1))} e^{-\lambda_n(t-\tau)} d\tau \right) \varphi_n(x). \qquad (3.4)$$

PROOF. Recall that the operator,

$$(-\partial_x^2)_D^s : D((-\partial_x^2)_D^s) \to L^2(-1,1), \quad u \mapsto (-\partial_x^2)_D^s u := (-\partial_x^2)^s u \text{ in } (-1,1),$$

defined in (2.3), is a self-adjoint operator on $L^2(-1,1)$. We denote by $\left(D((-\partial_x^2)_D^s) \right)^\star$ the dual space of $D((-\partial_x^2)_D^s)$ with respect to the pivot space $L^2(-1,1)$, so that $D((-\partial_x^2)_D^s) \hookrightarrow L^2(-1,1) \hookrightarrow \left(D((-\partial_x^2)_D^s) \right)^\star$.

Let $\mathbb{D}$ be the nonlocal Dirichlet map given by

$$\mathbb{D}g = u \iff (-\partial_x^2)^s u = 0 \text{ in } (-1,1) \text{ and } u = g \text{ in } \mathbb{R} \setminus (-1,1). \qquad (3.5)$$

It is well known (see, e.g., Antil, Khatri and Warma, 2019, or Warma, 2019) that for every $g \in H^s(\mathbb{R} \setminus (-1,1))$, there is a unique function $u \in H^s(\mathbb{R})$ satisfying (3.5).

Next, let the operator $\mathbb{B}$ be given by

$$\mathbb{B} : H^s(\mathbb{R} \setminus (-1,1)) \to \left(D((-\partial_x^2)_D^s) \right)^\star, \ g \mapsto \mathbb{B}g := -(-\partial_x^2)_D^s \mathbb{D}g. \qquad (3.6)$$

Firstly, we claim that for every $u \in D((-\partial_x^2)_D^s)$ and $g \in H^s(\mathbb{R} \setminus (-1,1))$ we have

$$\int_{-1}^1 u \mathbb{B}g \, dx = \int_{\mathbb{R} \setminus (-1,1)} g \mathcal{N}_s u \, dx. \qquad (3.7)$$

Indeed, let $u \in D((-\partial_x^2)_D^s)$ and $g \in H^s(\mathbb{R} \setminus (-1,1))$. By applying the integration by parts formula (2.6) and using (3.5)-(3.6), we get that

$$\int_{-1}^1 u \mathbb{B} g \, dx = - \int_{-1}^1 \mathbb{D}g(-\partial_x^2)^s u =$$

$$- \int_{-1}^1 u(-\partial_x^2)^s \mathbb{D}g \, dx + \int_{\mathbb{R} \setminus (-1,1)} \mathbb{D}g \mathcal{N}_s u \, dx - \int_{\mathbb{R} \setminus (-1,1)} u \mathcal{N}_s(\mathbb{D}g) \, dx$$

$$= \int_{\mathbb{R} \setminus (-1,1)} \mathbb{D}g \mathcal{N}_s u \, dx = \int_{\mathbb{R} \setminus (-1,1)} g \mathcal{N}_s u \, dx, \tag{3.8}$$

where we have also used the facts that $u = 0$ in $\mathbb{R} \setminus (-1,1)$ (since $u \in D((\partial_x^2)_D^s) \subset \widetilde{H}_0^s(-1,1)$) and $(\mathbb{D}g)|_{\mathbb{R} \setminus (-1,1)} = g$ by (3.5). We have thus shown the claim (3.7).

Secondly, with the above setting, proceeding as in the local case (see Lasiecka and Triggiani, 2000, pp. 180-185, and the references therein), using semigroup theory, (3.1) and the spectral theorem for selfadjoint operators, we can deduce that for every function $g \in L^2((0,T); H^s(\mathbb{R} \setminus (-1,1)))$ there exists a function $w \in L^2((0,T); H^s(\mathbb{R}))$, which is a weak solution of (3.2) and is given by

$$w(x,t) = \int_0^t e^{-(t-\tau)(-\partial_x^2)_D^s}(\mathbb{B}g)(x,\tau) \, d\tau$$

$$= \sum_{n=1}^\infty \left(\int_0^t ((\mathbb{B}g)(\cdot,\tau), \varphi_n)_{L^2(-1,1)} e^{-\lambda_n(t-\tau)} d\tau \right) \varphi_n(x)$$

$$= \sum_{n=1}^\infty \left(\int_0^t (g(\cdot,\tau), \mathcal{N}_s \varphi_n)_{L^2(\mathbb{R} \setminus (-1,1))} e^{-\lambda_n(t-\tau)} d\tau \right) \varphi_n(x).$$

We have thus shown (3.4) and the proof is finished. ∎

We have the following result on existence and explicit representation in terms of series of solutions to (1.1).

THEOREM 2 *Let $T > 0$. Then for every $u_0 \in L^2(-1,1)$ and $g \in L^2((0,T); H^s(\mathbb{R} \setminus (-1,1)))$, the system (1.1) has a weak solution $u \in L^2((0,T); L^2(-1,1))$, given by*

$$u(x,t) = \sum_{n=1}^\infty u_{0,n} e^{-\lambda_n t} \varphi_n(x) + \sum_{n=1}^\infty \left(\int_0^t (g(\cdot,\tau), \mathcal{N}_s \varphi_n)_{L^2(\mathcal{O})} e^{-\lambda_n(t-\tau)} d\tau \right) \varphi_n(x). \tag{3.9}$$

PROOF. Let $u_0 \in L^2(-1,1)$ and $g \in L^2((0,T); H^s(\mathbb{R} \setminus (-1,1)))$. Let $v \in C([0,T]; L^2(-1,1))$ be the weak solution of (3.1) and $w \in L^2((0,T); H^s(\mathbb{R}))$ the weak solution of (3.2) with g replaced by $g\chi_{\mathcal{O} \times (0,T)}$. Set $u := u + v$. Then, it is clear that $u \in L^2((0,T); L^2(-1,1))$ and is a weak solution of (1.1). The representation (3.9) follows directly from (3.3) and (3.4). The proof is finished. ∎

We conclude this section with the following remark.

REMARK 1 *We make the following observations.*
   1. *Theorem 2 is the fractional version of the classical local heat equation with inhomogeneous boundary data, and it is the so-called* boundary control semigroup *formula. We refer, for instance, to the book of Lasiecka and Triggiani (2000) and the paper of Fattorini (1975) for more details on the local case.*
   2. *The representation* (3.9) *of solutions to the system* (1.1) *is contained in Warma (2019) for the case where ∂_t is replaced with the Caputo time-fractional derivative $\mathbb{D}_t^\alpha$ ($0 < \alpha \leq 1$), and for a smooth function $g \in \mathcal{D}((0,T) \times \mathbb{R} \setminus (-1,1))$. In that case, since the function g is smooth, one has that the solution $u \in C([0,T]; L^2(-1,1))$. This is not the case here, since $g \in L^2((0,T); H^s(\mathbb{R} \setminus (-1,1)))$ and it is not smooth enough. However, proceeding as in Lasiecka and Triggiani (2000, pp. 180-185) and the references therein, the time regularity of the solution u can be improved. Since this is not the goal of the present paper, and since such a result and the representation* (3.9) *are not needed in the proof of our main results, we will not go into respective details.*

3.2. Representation of solutions to the associated dual system

Using the classical integration by parts formula, we obtain that the following backward system,

$$
\begin{cases}
-\partial_t \psi + (-\partial_x^2)^s \psi = 0 & \text{in } (-1,1) \times (0,T), \\
\psi = 0 & \text{in } (\mathbb{R} \setminus (-1,1)) \times (0,T), \\
\psi(\cdot, T) = \psi_0 & \text{in } (-1,1),
\end{cases}
\tag{3.10}
$$

can be viewed as the dual system associated with (1.1).

We have the following existence result:

THEOREM 3 *Let $T > 0$ be a real number and $\psi_0 \in L^2(-1,1)$. Then the system* (3.10) *has a unique weak solution $\psi \in C([0,T]; L^2(-1,1))$ given for a.e. $x \in (-1,1)$ and every $t \in [0,T]$ by*

$$
\psi(x,t) = \sum_{n=1}^\infty \psi_{0,n} e^{-\lambda_n(T-t)} \varphi_n(x).
\tag{3.11}
$$

In addition, the following assertions hold:
   1. *There is a constant $C > 0$ such that for all $t \in [0,T]$,*

$$
\|\psi(\cdot, t)\|_{L^2(-1,1)} \leq C \|\psi_0\|_{L^2(-1,1)}.
\tag{3.12}
$$

   2. *For every $t \in [0,T)$ fixed, $\mathcal{N}_s \psi(\cdot, t)$ exists, belongs to $L^2(\mathbb{R} \setminus (-1,1))$ and is given by*

$$
\mathcal{N}_s \psi(x,t) = \sum_{n=1}^\infty \psi_{0,n} e^{-\lambda_n(T-t)} \mathcal{N}_s \varphi_n(x),
\tag{3.13}
$$

where we recall that $\psi_{0,n} := (\psi_0, \varphi_n)_{L^2(-1,1)}$.

Proof. Using the spectral theorem for selfadjoint operators with compact resolvent, we reduce the problem to the one of looking for a solution ψ of the form

$$\psi(x,t) = \sum_{n=1}^{\infty} (\psi(\cdot,t), \varphi_n)_{L^2(-1,1)} \varphi_n(x).$$

Replacing this expression in (3.10) and letting $\psi_n(t) := (\psi(\cdot,t), \varphi_n)_{L^2(-1,1)}$, we get that $\psi_n(t)$ solves the following ordinary differential equation:

$$-\psi_n'(t) + \lambda_n \psi_n(t) = 0, \ t \in (0,T); \quad \text{and} \ \ \psi_n(T) = \psi_{0,n}.$$

It is straightforward to show that ψ is given by (3.11). Noticing that $\psi(x,t) = e^{-(T-t)(-\partial_x^2)_D^s} \psi_0(x)$ (where we recall that $(e^{-t(-\partial_x^2)_D^s})_{t \geq 0}$ is a strongly continuous semigroup on $L^2(-1,1)$, generated by the operator $-(\partial_x^2)_D^s$), and using semigroup theory, leads us to the well known statement that $\psi \in C([0,T]; L^2(-1,1))$. The estimate (3.12) and the identity (3.13) can also be easily justified. The proof is finished. ∎

We conclude this section with the following remark:

Remark 2 *From the semigroup theory, it is well known that the solution $\psi \in C([0,T]; L^2(-1,1))$ of the backward system (3.10) enjoys the following regularity:*

$$\psi \in C([0,T]; L^2(-1,1)) \cap L^2((0,T); \widetilde{H}_0^s(-1,1)) \cap H^1((0,T); \widetilde{H}^{-s}(-1,1)).$$

4. Proof of the main result

In this section we give the proof of the main result of this work, namely Theorem 1. In order to do this, we need first to establish some auxiliary results that will be used in the proof.

Lemma 3 *The system (1.1) is null controllable at time $T > 0$ if and only if for every initial datum $u_0 \in L^2(-1,1)$ there exists a control function $g \in L^2((0,T); \widetilde{H}_0^s(\mathcal{O}))$ such that the solution ψ of the dual system (3.10) satisfies*

$$\int_{-1}^{1} u_0(x)\psi(x,0) \ dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t) \ dxdt, \tag{4.1}$$

for every $\psi_0 \in L^2(-1,1)$.

Proof. Let $u_0 \in L^2(-1,1)$ and $g \in L^2((0,T); \widetilde{H}_0^s(\mathcal{O}))$. We write the solution u of (1.1) as $u := v + w$ where v and w are the solutions of (3.1) and (3.2), respectively. Let ψ be the solution of the dual problem (3.10). Taking ψ as a

test function in Definition 1 of a weak solution to the system (3.2), using the integration by parts formula (2.6), noticing that

$$-\psi_t + (-\partial_x^2)^s\psi = 0 \text{ in } (-1,1) \times (0,T),$$

and that $\psi = 0$ in $(\mathbb{R} \setminus (-1,1)) \times (0,T)$, we obtain that

$$
\begin{aligned}
0 &= \int_0^T \langle v_t(\cdot,t) + (-\partial_x^2)^s v(\cdot,t), \psi(\cdot,t) \rangle \, dt \\
&\quad + \int_0^T \langle -\psi_t(\cdot,t) + (-\partial_x^2)^s \psi(\cdot,t), w(\cdot,t) \rangle \, dt \\
&= \int_{-1}^1 \Big(v(x,T)\psi(x,T) - v(x,0)\psi(x,0) \Big) \, dx \\
&\quad + \int_0^T \int_{\mathbb{R}\setminus(-1,1)} \Big(v(x,t)\mathcal{N}_s\psi(x,t) - \psi(x,t)\mathcal{N}_s v(x,t) \Big) \, dxdt \\
&\quad + \int_{-1}^1 w(x,T)\psi(x,T) \, dx + \int_0^T \int_{\mathbb{R}\setminus(-1,1)} w(x,t)\mathcal{N}_s\psi(x,t) \, dxdt \\
&= -\int_{-1}^1 v(x,0)\psi(x,0) \, dx + \int_{-1}^1 \Big(v(x,T) + w(x,T) \Big)\psi(x,T) \, dx \\
&\quad + \int_0^T \int_{\mathbb{R}\setminus(-1,1)} \Big(v(x,t) + w(x,t) \Big)\mathcal{N}_s\psi(x,t) \, dxdt. \tag{4.2}
\end{aligned}
$$

Since $v(x,0) = u(x,0) = u_0(x)$ for a.e. $x \in (-1,1)$, and $u(x,t) = v(x,t) + w(x,t)$ for a.e. $(x,t) \in (-1,1) \times (0,T]$, and $u = g\chi_{\mathcal{O}\times(0,T)}$ in $(\mathbb{R} \setminus (-1,1)) \times (0,T)$, it follows from (4.2) that

$$\int_{-1}^1 u(x,0)\psi(x,0) \, dx - \int_{-1}^1 u(x,T)\psi(x,T) \, dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t) \, dxdt. \tag{4.3}$$

Now, if (4.1) holds, then it follows from (4.3) that

$$\int_{-1}^1 u(x,T)\psi(x,T) \, dx = 0.$$

Thus, we can deduce that $u(\cdot,T) = 0$ in $(-1,1)$ and the system (1.1) is null controllable.

Conversely, if the system (1.1) is null controllable, that is, if $u(\cdot,T) = 0$ in $(-1,1)$, then (4.1) follows from (4.3) and the proof is finished. ∎

Next, using classical duality arguments, we can establish the following criterion for null controllability:

Lemma 4 *Let $\mathcal{O} \subset (\mathbb{R} \setminus (-1,1))$ be an arbitrary nonempty open set. Then the following assertions are equivalent:*

1. *The system* (1.1) *is null controllable at time $T > 0$ and there is a constant $C > 0$ such that*

$$\|g\|_{L^2((0,T);\widetilde{H}_0^s(\mathcal{O}))} \leq C\|u_0\|_{L^2(-1,1)}. \tag{4.4}$$

2. *For every $T > 0$ and $\psi_0 \in L^2(-1,1)$, let ψ be the unique weak solution of the dual system* (3.10) *with final datum ψ_0. Then, there is a constant $C = C(T) > 0$ such that the following observability inequality holds:*

$$\|\psi(\cdot,0)\|_{L^2(-1,1)}^2 \leq C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|^2 dx dt. \tag{4.5}$$

Proof. $(2.) \Rightarrow (1.)$: We start by proving that the observability inequality (4.5) implies the null controllability. Indeed, consider the linear subspace $\mathbb{H}$ of $L^2((0,T);\widetilde{H}^{-s}(\mathcal{O}))$, given by

$$\mathbb{H} := \left\{ \mathcal{N}_s\psi\Big|_{\mathcal{O}\times(0,T)} : \psi \text{ solves the system (3.10) with } \psi_0 \in L^2(-1,1) \right\}.$$

Next, we consider the linear functional $F : \mathbb{H} \to \mathbb{R}$ defined by

$$F(\mathcal{N}_s\psi) := \int_{-1}^1 u_0(x)\psi(x,0)dx,$$

where $u_0 \in L^2(-1,1)$. It follows from the observability inequality (4.5) that F is well defined and bounded on $\mathbb{H}$. More precisely, there is a constant $C > 0$ such that

$$|F(\mathcal{N}_s\psi)| \leq C\|u_0\|_{L^2(-1,1)}\|\mathcal{N}_s\psi\|_{L^2((0,T);\widetilde{H}^{-s}(\mathcal{O}))}.$$

It follows from the Hahn–Banach theorem that F can be extended to a bounded linear functional $\widetilde{F} : L^2((0,T);\widetilde{H}^{-s}(\mathcal{O})) \to \mathbb{R}$, such that

$$|\widetilde{F}v| \leq C\|u_0\|_{L^2(-1,1)}\|v\|_{L^2((0,T);\widetilde{H}^{-s}(\mathcal{O}))}, \quad \forall v \in L^2((0,T);\widetilde{H}^{-s}(\mathcal{O})). \tag{4.6}$$

By the Riesz representation theorem, there is a $g \in (L^2((0,T);\widetilde{H}^{-s}(\mathcal{O})))^* = L^2((0,T);\widetilde{H}_0^s(\mathcal{O}))$ such that

$$\widetilde{F}(\eta) = \int_0^T \langle \eta(\cdot,t), g(\cdot,t) \rangle \, dt, \quad \forall\, \eta \in L^2((0,T);\widetilde{H}^{-s}(\mathcal{O})).$$

Moreover, we have that

$$\|\widetilde{F}\| = \|g\|_{L^2((0,T);\widetilde{H}_0^s(\mathcal{O}))}.$$

Thus, we can deduce from (4.6) that

$$\|g\|_{L^2((0,T);\widetilde{H}_0^s(\mathcal{O}))} \leq C\|u_0\|_{L^2(-1,1)}.$$

Notice that $\mathcal{N}_s\psi \in L^2((0,T);L^2(\mathcal{O})) \subset L^2((0,T);\widetilde{H}^{-s}(\mathcal{O}))$. Therefore, using the definition of F we can deduce that

$$F(\mathcal{N}_s\psi) = \int_{-1}^{1} u_0(x)\psi(x,0)dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)dxdt,$$

for every $\psi_0 \in L^2(-1,1)$. We have shown that there exists a control function $g \in L^2((0,T);\widetilde{H}_0^s(\mathcal{O}))$ satisfying (4.4) and

$$\int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)dxdt = \int_{-1}^{1} u_0(x)\psi(x,0)dx,$$

for every $\psi_0 \in L^2(-1,1)$. Thus, it follows from Lemma 3 that the system (1.1) is null controllable.

(1.) $\Rightarrow$ (2.): Now, we show that the null controllability implies the observability inequality (4.5). Recall that from Lemma 3 we have that for every $u_0 \in L^2(-1,1)$ there exists a control $g \in L^2((0,T);\widetilde{H}_0^s(\mathcal{O}))$ such that the unique solution ψ of the dual system (3.10) satisfies

$$\int_{-1}^{1} u_0(x)\psi(x,0)\ dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)\ dxdt,$$

for every $\psi_0 \in L^2(-1,1)$. Taking $u_0(x) = \psi(x,0)$ in the preceding identity, using (4.4) and Young's inequality, we get that

$$\int_{-1}^{1} |\psi(x,0)|^2 dx \leq \frac{C}{2\varepsilon}\|u_0\|_{L^2(-1,1)}^2 + \frac{\varepsilon}{2}\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|^2 dxdt,$$

for every $\varepsilon > 0$. Taking $\varepsilon = C$ and since $u_0(x) = \psi(x,0)$, we obtain (4.5). The proof is finished. ∎

Finally, for the proof of Theorem 1 we also need the following technical result:

LEMMA 5 *Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the orthogonal basis of normalized eigenfunctions of the operator $(-\partial_x^2)_D^s$, associated with the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Then, for every nonempty open set $\mathcal{O} \subset (\mathbb{R}\setminus(-1,1))$, there is a scalar $\eta > 0$ (independent of k) such that for every $k \in \mathbb{N}$, the function $\mathcal{N}_s\varphi_k$ is uniformly bounded from below by η in $L^2(\mathcal{O})$. Namely,*

$$\exists\, \eta > 0 \ \ such\ that\ \ \|\mathcal{N}_s\varphi_k\|_{L^2(\mathcal{O})} \geq \eta, \quad \forall\, k \in \mathbb{N}. \tag{4.7}$$

PROOF. We prove the result in several steps.

Step 1: Firstly, since $\varphi_k = 0$ in $\mathbb{R} \setminus (-1, 1)$ for every $k \in \mathbb{N}$, it follows from the definition of the operators $(-\partial_x^2)^s$ and $\mathcal{N}_s$ that for almost every $x \in \mathcal{O} \subseteq (\mathbb{R} \setminus (-1, 1))$, we have

$$(-\partial_x^2)^s \varphi_k(x) = C_s \text{P.V.} \int_{\mathbb{R}} \frac{\varphi_k(x) - \varphi_k(y)}{|x - y|^{1+2s}} \, dy =$$

$$C_s \int_{-1}^{1} \frac{\varphi_k(x) - \varphi_k(y)}{|x - y|^{1+2s}} \, dy = \mathcal{N}_s \varphi_k(x). \tag{4.8}$$

We have shown that $(\mathcal{N}_s \varphi_k)|_{\mathcal{O}} = ((-\partial_x^2)^s \varphi_k)|_{\mathcal{O}}$ for every $k \in \mathbb{N}$. We notice that (4.8) holds not only for φ_k, but for all functions in $\widetilde{H}_0^s(-1, 1)$.

Secondly, let us introduce the auxiliary function $q : \mathbb{R} \to [0, \infty)$, defined by:

$$q(x) := \begin{cases} 0 & x \in \left(-\infty, -\frac{1}{3}\right), \\ \frac{9}{2}\left(x + \frac{1}{3}\right)^2 & x \in \left(-\frac{1}{3}, 0\right), \\ 1 - \frac{9}{2}\left(x - \frac{1}{3}\right)^2 & x \in \left(0, \frac{1}{3}\right), \\ 1 & x \in \left(\frac{1}{3}, +\infty\right). \end{cases} \tag{4.9}$$

For any $\alpha > 0$, we define the function $F_\alpha : \mathbb{R} \to \mathbb{R}$ as follows:

$$F_\alpha(x) = F(\alpha x) := \sin\left(\alpha x + \frac{(1-s)\pi}{4}\right) - G(\alpha x),$$

where G is the Laplace transform of the function

$$\gamma(y) :=$$
$$\frac{\sqrt{4s}\sin(s\pi)}{2\pi} \frac{y^{2s}}{1 + y^{4s} - 2y^{2s}\cos(s\pi)} \exp\left(\frac{1}{\pi} \int_0^{+\infty} \frac{1}{1 + r^2} \log\left(\frac{1 - r^{2s}y^{2s}}{1 - r^2 y^2}\right) dr\right).$$

Next, we define the sequence of real numbers

$$\mu_k := \frac{k\pi}{2} - \frac{(1-s)\pi}{4}, \quad k \geq 1.$$

It has been shown in Kwasnicki (2011, Example 6.1) that F_{μ_k} is the solution of the system

$$\begin{cases} (-\partial_x^2)^s F_{\mu_k}(x) = \mu_k F_{\mu_k}(x) & x > 0, \\ F_{\mu_k}(x) = 0 & x \leq 0. \end{cases}$$

In other words, $\{F_{\mu_k}\}_{k \geq 1}$ are the eigenfunctions of $(-\partial_x^2)^s$ on the interval $(0, \infty)$ with the zero Dirichlet exterior condition, and $\{\mu_k\}_{k \geq 1}$ are the corresponding eigenvalues. Let us now define

$$\varrho_k(x) := q(-x)F_{\mu_k}(1+x) + (-1)^k F_{\mu_k}(1-x), \quad x \in \mathbb{R}, \quad k \geq 1.$$

Notice that $F_{\mu_k}(1+x) = 0$ for $x \leq -1$ and $F_{\mu_k}(1-x) = 0$ for $x \geq 1$. This fact, together with the definition (4.9) of the function q imply that, for all $k \geq 1$, $\varrho_k(x) = 0$ for $x \in \mathbb{R} \setminus (-1, 1)$. In addition, it follows from Kwasnicki (2012, Lemma 1) that $\{\varrho_k\}_{k \in \mathbb{N}} \subset D((-\partial_x^2)_D^s)$ and there is a constant $C > 0$ such that

$$\left| (-\partial_x^2)^s \varrho_k(x) - \mu_k^{2s} \varrho_k(x) \right| \leq \frac{C(1-s)}{\sqrt{2s}} \mu_k^{-1}, \quad \text{for all} \quad x \in (-1, 1), \ k \geq 1.$$

Furthermore, by Kwasnicki (2012, Proposition 1), there is a constant $C > 0$ such that for every $k \geq 1$, we have

$$\|\varrho_k - \varphi_k\|_{L^2(-1,1)} \leq \begin{cases} \frac{C(1-s)}{k} & \text{when } \frac{1}{2} \leq s < 1, \\ \frac{C(1-s)}{k^{2s}} & \text{when } 0 < s < \frac{1}{2}. \end{cases} \tag{4.10}$$

Step 2: Now, let $\mathcal{O} \subset \mathbb{R} \setminus (-1, 1)$ be an arbitrary nonempty open set and assume that for every $\eta > 0$ there exists $k \in \mathbb{N}$ such that

$$\|\mathcal{N}_s \varphi_k\|_{L^2(\mathcal{O})} < \eta. \tag{4.11}$$

It follows from (4.11) that there is a subsequence $\{\varphi_{k_n}\}_{n \in \mathbb{N}}$ such that

$$\|\mathcal{N}_s \varphi_{k_n}\|_{L^2(\mathcal{O})} < \frac{1}{n}, \tag{4.12}$$

for n large enough. Since $L^2(\mathcal{O}) \hookrightarrow \widetilde{H}^{-s}(\mathcal{O})$, it follows from (4.12) that there is a constant $C > 0$ (independent of n) such that for n large enough, we have

$$\|\mathcal{N}_s \varphi_{k_n}\|_{\widetilde{H}^{-s}(\mathcal{O})} \leq \frac{C}{n}. \tag{4.13}$$

Step 3: Using the triangle inequality, we get that there is a constant $C > 0$ such that

$$\|\varrho_{k_n} - \varphi_{k_n}\|_{\widetilde{H}_0^s(-1,1)}^2 \leq C \|(-\partial_x^2)^s \varrho_{k_n} - (-\partial_x^2)^s \varphi_{k_n}\|_{L^2(-1,1)}^2$$

$$\leq C \Big(\|(-\partial_x^2)^s \varrho_{k_n} - \mu_{k_n}^{2s} \varrho_{k_n}\|_{L^2(-1,1)}^2 + \|\varrho_{k_n}(\mu_{k_n}^{2s} - \lambda_{k_n})\|_{L^2(-1,1)}^2$$

$$+ \|\lambda_{k_n} \varrho_{k_n} - (-\partial_x^2)^s \varphi_{k_n}\|_{L^2(-1,1)}^2 \Big). \tag{4.14}$$

It follows from (4.14) and Step 1 that there is a constant $C_{k_n}(s) > 0$, which converges to zero as $n \to \infty$, such that

$$\|\varrho_{k_n} - \varphi_{k_n}\|^2_{\widetilde{H}^s_0(-1,1)} \le C_{k_n}(s).$$

Let the operator L be defined by

$$L : \widetilde{H}^s_0(-1,1) \to \widetilde{H}^{-s}(\mathcal{O}), \ v \mapsto Lv := ((-\partial_x^2)^s v)|_\mathcal{O} = (\mathcal{N}_s v)|_\mathcal{O},$$

where we recall that $\widetilde{H}^{-s}(\mathcal{O}) = (\widetilde{H}^s_0(\mathcal{O}))^\star$. It has been shown in Ghosh et al. (2018, Lemma 2.2) that the operator L is compact, injective with dense range. Let $B_1 := \overline{B}(\varrho_{k_n}, C_{k_n}(s))$ be the closed ball in $\widetilde{H}^s_0(-1,1)$ with center in ϱ_{k_n} and radius $C_{k_n}(s)$. Since L is a compact operator, we have that the image of B_1, namely $L(B_1)$, is totally bounded in $\widetilde{H}^{-s}(\mathcal{O})$. Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and $\{\psi_1, \ldots, \psi_N\} \subseteq B_1$ such that

$$L(B_1) \subseteq \bigcup_{j=1}^N \overline{B}_{\widetilde{H}^{-s}(\mathcal{O})}(L(\psi_j), \varepsilon).$$

We observe that the eigenfunction φ_{k_n} belongs to B_1. Thus, there exists $j \in \{1, \ldots, N\}$ such that

$$L(\varphi_{k_n}) \in \overline{B}_{\widetilde{H}^{-s}(\mathcal{O})}(L(\psi_j), \varepsilon).$$

We have shown that for n large enough,

$$\|L(\varphi_{k_n}) - L(\psi_j)\|_{\widetilde{H}^{-s}(\mathcal{O})} \le \varepsilon.$$

Since $\psi_j \in B_1$, firstly we obtain that $\varphi_{k_n} \to \psi_j$, as $n \to \infty$ in $\widetilde{H}^s_0(-1,1)$ and, secondly we have that ψ_j is an element of the spectrum $\{(\varphi_k, \lambda_k)\}_{k \ge 1}$. That is, ψ_j is a solution of (2.4). Finally, as $L(\varphi_{k_n})$ converges to zero in $\widetilde{H}^{-s}(\mathcal{O})$ (by (4.13)), we can deduce that $L(\psi_j) = \mathcal{N}_s \psi_j = (-\partial_x^2)^s \psi_j = 0$ a.e. in $\mathcal{O}$. It follows from the unique continuation property (Lemma 1) that $\psi_j = 0$ a.e. in $\mathbb{R}$, which is a contradiction. The proof of the lemma is finished. ∎

Now we are ready to give the proof of our main result.

Proof of Theorem 1. Let u be the weak solution of (1.1) and ψ the weak solution of the dual problem (3.10). Recall that by Lemma 3, the system (1.1) is null controllable if and only if the identity (4.1) holds. Moreover, from Lemma 4, (4.1) is equivalent to the observability inequality (4.5) for the dual system, that is, there exists a constant $C > 0$ such that

$$\|\psi(\cdot, 0)\|^2_{L^2(-1,1)} \le C \int_0^T \int_\mathcal{O} |\mathcal{N}_s \psi(x,t)|^2 \ dxdt. \tag{4.15}$$

From Section 3.2, the solution ψ of (3.10) is given by

$$\psi(\cdot, t) = \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n(T-t)} \varphi_n(x),$$

and its nonlocal normal derivative $\mathcal{N}_s \psi$ is given by

$$\mathcal{N}_s \psi(\cdot, t) = \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n(T-t)} \mathcal{N}_s \varphi_n(x).$$

Therefore, using the above representations of ψ and $\mathcal{N}_s \psi$, the orthonormality of the eigenfunctions in $L^2(-1, 1)$, and making the change of variable $T - t \to t$, we can deduce that the observability inequality (4.15) is equivalent to the following inequality:

$$\sum_{n=1}^{\infty} |\psi_{0,n}|^2 e^{-2\lambda_n T} \le C \int_0^T \int_{\mathcal{O}} \left| \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n t} \mathcal{N}_s \varphi_n(x) \right|^2 dx dt. \tag{4.16}$$

By means of the classical moment method (see, e.g., Fattorini and Russell, 1971, Sections 2 and 3), inequalities of the form (4.16) are well known to be true if and only if the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ and eigenfunctions $\{\varphi_n\}_{n \in \mathbb{N}}$ satisfy the following Müntz condition:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty, \tag{4.17}$$

and

$$\|\mathcal{N}_s \varphi_n\|_{L^2(\mathcal{O})} \ge \eta > 0, \ \forall \, n \in \mathbb{N}, \tag{4.18}$$

where the constant η is independent of n.

Lemma 5 implies that (4.18) holds.

As we have mentioned in the introduction, the eigenvalues $\{\lambda_n\}_{n \ge 1}$ satisfy (1.2). That is,

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-2s)\pi}{8} \right)^{2s} + O\left(\frac{1}{n} \right) \quad \text{as } n \to \infty. \tag{4.19}$$

Therefore, we easily see from (4.19) that the condition (4.17) is satisfied if and only if $\frac{1}{2} < s < 1$. Instead, if $0 < s \le \frac{1}{2}$, then the series diverges, since it will have the behavior of the harmonic series. In conclusion, the observability inequality (4.15) holds true when $\frac{1}{2} < s < 1$, and it is false when $0 < s \le \frac{1}{2}$. The proof of the theorem is finished. ∎

Acknowledgment

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