Opuscula Math. 34, no. 3 (2014), 523–560
http://dx.doi.org/10.7494/OpMath.2014.34.3.523 Opuscula Mathematica

CONJUGATE FUNCTIONS, L^p-NORM LIKE FUNCTIONALS, THE GENERALIZED HÖLDER INEQUALITY, MINKOWSKI INEQUALITY AND SUBHOMOGENEITY

Janusz Matkowski

Communicated by Karol Baron

Abstract. For $h : (0, \infty) \to \mathbb{R}$, the function $h^*(t) := th(\frac{1}{t})$ is called $(*)$-conjugate to h. This conjugacy is related to the Hölder and Minkowski inequalities. Several properties of $(*)$-conjugacy are proved. If φ and φ^* are bijections of $(0, \infty)$ then $(\varphi^{-1})^* = \left(\left[(\varphi^*)^{-1}\right]^*\right)^{-1}$. Under some natural rate of growth conditions at 0 and ∞, if φ is increasing, convex, geometrically convex, then $\left[(\varphi^{-1})^*\right]^{-1}$ has the same properties. We show that the Young conjugate functions do not have this property. For a measure space (Ω, Σ, μ) denote by $S = S(\Omega, \Sigma, \mu)$ the space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. Given a bijection $\varphi : (0, \infty) \to (0, \infty)$, define $\mathbf{P}_\varphi : S \to [0, \infty)$ by

$$\mathbf{P}_\varphi(x) := \varphi^{-1}\left(\int_{\Omega(x)} \varphi \circ |x| \, d\mu\right),$$

where $\Omega(x)$ is the support of x. Applying some properties of the $(*)$ operation, we prove that if $\int_\Omega xy \le \mathbf{P}_\varphi(x)\mathbf{P}_\psi(y)$ where φ^{-1} and ψ^{-1} are conjugate, then φ and ψ are conjugate power functions. The existence of nonpower bijections φ and ψ with conjugate inverse functions $\psi = \left[(\varphi^{-1})^*\right]^{-1}$ such that $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are subadditive and subhomogeneous is considered.

Keywords: L^p-norm like functional, homogeneity, subhomogeneity, subadditivity, the converses of Minkowski and Hölder inequalities, generalization of the Minkowski and Hölder inequalities, conjugate (complementary) functions, Young conjugate functions, convex function, geometrically convex function, Wright convex function, functional equation.

Mathematics Subject Classification: 26D15, 26A51, 39B22, 39B62, 46B25, 46E30.

1. INTRODUCTION

For a measure space (Ω, Σ, μ) denote by $S = S(\Omega, \Sigma, \mu)$ the linear real space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$ and put $S_+ := \{x \in S : x \ge 0\}$. For an

© AGH University of Science and Technology Press, Krakow 2014

arbitrary bijection $\varphi : (0,\infty) \to (0,\infty)$ define the functional $\mathbf{P}_\varphi : S \to [0,\infty)$ by the formula

$$\mathbf{P}_\varphi(x) := \begin{cases} \varphi^{-1}\left(\int\limits_{\Omega(x)} \varphi \circ x d\mu \right) & \text{if } \mu\left(\Omega(x)\right) > 0, \\ 0 & \text{if } \mu\left(\Omega(x)\right) = 0, \end{cases}$$

where $\Omega(x)$ is the support of $x \in S$. If $\varphi(t) = \varphi(1)t^p$ for some $p \geq 1$, this functional becomes the L^p-norm.

If (Ω, Σ, μ) is nontrivial then, under a weak regularity condition on φ, the functional $\mathbf{P}_\varphi$ is (positively) *homogeneous*, that is,

$$\mathbf{P}_\varphi(tx) = t\mathbf{P}_\varphi(x), \quad t > 0,\ x \in S_+,$$

if and only if φ is a power function, i.e. $\varphi(t) = \varphi(1)\, t^p$ for some real $p \neq 0$ (cf. [7]).

In [17] (cf. also [18]), under the assumption of the existence of two sets $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty, \tag{1.1}$$

and a weak regularity condition on φ (that is, sometime removable [11, 15]), it has been proved that if $\mathbf{P}_\varphi$ is subadditive, that is, if

$$\mathbf{P}_\varphi(x+y) \leq \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad x, y \in S_+,$$

then $\varphi(t) = \varphi(1)\, t^p$ for some $p \geq 1$, so $\mathbf{P}_\varphi$ is the L^p-norm (cf. also [8] where it is assumed that $\varphi(0) = 0$ and φ^{-1} is continuous at 0). This is a type of converse of the Minkowski inequality theorem.

In [14], under the same assumption (1.1) on the measure space, we proved that if φ and ψ are any bijections of $(0,\infty)$ such that

$$\int\limits_\Omega xy d\mu \leq \mathbf{P}_\varphi(x)\mathbf{P}_\psi(y), \quad x, y \in S_+, \tag{1.2}$$

then φ and ψ are the conjugate power functions, that is, there are $p, q > 1$ such that $\varphi(t) = \varphi(1)\, t^p$, $\psi(t) = \psi(1)\, t^q$ and $\frac{1}{p} + \frac{1}{q} = 1$ (cf. also [19]). This is a converse of the Hölder inequality theorem.

In a recent paper [23] we proved that if the measure space satisfies (1.1) for some $A, B \in \Sigma$ and φ is monotonic, then $\mathbf{P}_\varphi$ is *subhomogeneous*, i.e.

$$\mathbf{P}_\varphi(tx) \leq t\mathbf{P}_\varphi(x), \quad t > 1,\ x \in S_+,$$

if and only if φ is a power function, so if and only if $\mathbf{P}_\varphi$ is homogeneous.

A measure space (Ω, Σ, μ) does not satisfy condition (1.1) iff either

$$A \in \Sigma,\ \mu(A) \neq 0 \Longrightarrow \mu(A) \geq 1,$$

(i.e. (Ω, Σ, μ) is a generalized counting measure space) or

$$A \in \Sigma,\ \mu(A) < \infty \Longrightarrow \mu(A) \leq 1$$

(i.e. (Ω, Σ, μ) is a "defected" probability space).

It turns out that in each of these two cases there exist nonpower bijections φ such that $\mathbf{P}_\varphi$ is subadditive or subhomogeneous in S, and there are nonpower bijections φ and ψ satisfying inequality (1.2). Moreover, in the just quoted papers, under some natural regularity assumptions, these bijections have been characterized.

In Section 4 of the present paper we focus our attention on the Hölder type inequality where two arbitrary bijections φ and ψ have appeared. We show that the converse of the Hölder inequality theorem remains true if the assumption (1.1) on the underlying measure space is replaced by the equality

$$\psi^{-1} = (\varphi^{-1})^*$$

where, for an arbitrary function $h : (0,\infty) \to \mathbb{R}$, the function $h^* : (0,\infty) \to \mathbb{R}$ is defined by the formula

$$h^*(t) := th\Big(\frac{1}{t}\Big).$$

Since $(h^*)^* = h$, and for $h(t) := t^{1/p}$ we have $h^*(t) = t^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$, the functions h and h^* are said to be conjugate (or complementary). Some important properties of the conjugate functions are presented in Section 3. We give a simple argument showing that h is convex iff h^* is. The convexity of h is equivalent to the subadditivity of the two variable function $(s,t) \to th\left(\frac{s}{t}\right)$. This subadditivity generalizes elementary Minkowski and Hölder inequalities. Moreover, the conjugate function h^* appears here in a genuine way. We prove, what is important in the sequel, that if a function φ is increasing, convex, geometrically convex and such that

$$\lim_{t\to 0+} \frac{\varphi(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty,$$

then the function $\left[(\varphi^{-1})^*\right]^{-1}$ has the same properties. In Section 6 we show that the Young conjugate functions do not have this property. (For the classical properties of conjugate functions in the Young-sense see [28, 29].) If φ and φ^* are bijections of $(0,\infty)$ then

$$(\varphi^{-1})^* = \left(\left[(\varphi^*)^{-1}\right]^*\right)^{-1}.$$

Some connections between the involutory operations $T(\varphi) := \varphi^{-1}$ and $U(\varphi) := \varphi^*$ are considered. We show that there is a homeomorphism of $(0,\infty)$ such that

$$(\varphi^{-1})^* = (\varphi^*)^{-1},$$

i.e. that $U \circ T(\varphi) = T \circ U(\varphi)$. The existence of a bijection φ satisfying this equation is an open question.

The main result of Section 4 reads as follows. Suppose that the underlying measure space is not trivial, that is, there are at least two measurable disjoint sets A, B of finite positive measure such that $\frac{\log \mu(B)}{\log \mu(A)}$ is irrational. If the functions φ^{-1} and ψ^{-1} are conjugate and inequality (1.2) holds true, then φ^{-1} and ψ^{-1} are conjugate power functions. A suitable result in the case when inequality (1.2) is a reversed inequality is also given.

If $\varphi(t) = \varphi(1)\, t^p$ and $\psi(t) = \psi(1)\, t^q$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, that is, if φ and ψ are conjugate power functions then, of course,

$$\mathbf{P}_\varphi(x+y) \le \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad \mathbf{P}_\psi(x+y) \le \mathbf{P}_\psi(x) + \mathbf{P}_\psi(y), \qquad x, y \in S_+,$$

that is, $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ satisfy the Minkowski inequality. In this connection the following question arises. Assume that a measure space (Ω, Σ, μ) does not satisfy condition (1.1). Do there exist nonpower conjugate bijections φ^{-1} and ψ^{-1} (that is, $\psi^{-1} = (\varphi^{-1})^*$) satisfying simultaneously this system of Minkowski inequalities?

In Section 5 we show that the answer is affirmative. In the case when (Ω, Σ, μ) is a generalized counting measure space, we give sufficient conditions for the existence of a broad class of pairs of nonpower functions φ and ψ such that $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are subhomogeneous and subadditive (Theorem 5.2). Assuming additionally that $(1, \delta) \subset \operatorname{cl} \mu(\Sigma)$ for some $\delta > 1$, we prove that these conditions are also necessary (Theorem 5.3).

In the case when the measure space is a defected probability space, we construct a pair of nonpower bijections φ and ψ of $(0, \infty)$ satisfying the condition $\psi^{-1} = \left(\varphi^{-1}\right)^*$ and such that $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are subadditive and subhomogeneous. It is an open problem wether this construction gives a general criterion for the subhomogeneity and subadditivity of the functionals $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ with conjugate inverses φ^{-1} and ψ^{-1}. The simultaneous subadditivity of $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ with conjugate φ^{-1} and ψ^{-1} considered at the end of Section 5 leads to the following open problem.

Let φ and ψ be bijections of $(0, \infty)$ such that $\psi^{-1} = (\varphi^{-1})^*$. Suppose that the function $\Phi : (0, \infty)^2 \to (0, \infty)$ defined by

$$\Phi(s, t) := \varphi\left(\varphi^{-1}(s) + \varphi^{-1}(t)\right), \quad s, t > 0,$$

is concave. Is then the function $\Psi : (0, \infty)^2 \to (0, \infty)$ defined by

$$\Psi(s, t) := \psi\left(\psi^{-1}(s) + \psi^{-1}(t)\right), \quad s, t > 0,$$

concave?

In Section 6 we recall some basic properties of conjugate (complementary) functions in the sense of Young. Assuming that $\varphi : (0, \infty) \to (0, \infty)$ is differentiable with strictly increasing derivative and suitable rank of growth at 0 and ∞, we prove that if φ is geometrically convex then, contrary to the case of $(*)$-conjugacy, its Young conjugate function is geometrically concave. It follows that, independently on the type of the measure space, if φ and its Young conjugate function are geometrically convex, then they are conjugate power functions. Moreover, we prove that if (Ω, Σ, μ) is (or contains) a nontrivial probability measure space and φ and ψ are conjugate in the sense of Young, then the simultaneous subhomogeneity of the functionals $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ implies that φ and ψ are the conjugate power functions. The same holds true for the generalized counting measure if, for some $\delta > 1$, the closure of the range of the measure contains an interval $(1, \delta)$.

2. PRELIMINARIES AND AUXILIARY RESULTS

Let (Ω, Σ, μ) be a measure space. Denote by $S = S(\Omega, \Sigma, \mu)$ the linear real space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. Let $S_+ =: \{x \in S : x \geq 0\}$. For $x \in S_+$ put

$$\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}.$$

For a bijection $\varphi : (0, \infty) \to (0, \infty)$ the functional $\mathbf{P}_\varphi : S_+ \to \mathbb{R}_+$ by

$$\mathbf{P}_\varphi(x) := \begin{cases} \varphi^{-1}\Big(\int\limits_{\Omega(x)} \varphi \circ x d\mu \Big) & \text{if } \mu(\Omega(x)) > 0, \\ 0 & \text{if } \mu(\Omega(x)) = 0 \end{cases} \tag{2.1}$$

is correctly defined.

Denote by χ_C the characteristic function of a set C.

If $x \in S$ and $\mu(\Omega(x)) > 0$ then, clearly,

$$x = \sum_{j=1}^{n} x_j \chi_{A_j}$$

for some $n \in \mathbb{N}$, pairwise disjoint sets $A_j \in \Sigma$ such that $0 < \mu(A_j) < \infty$, and $x_j \in \mathbb{R}$, $x_j \neq 0$ for $j = 1, \ldots, n$. Moreover,

$$\mathbf{P}_\varphi(|x|) = \varphi^{-1}\Bigg(\sum_{j=1}^{n} \varphi(|x_j|) \mu(A_j) \Bigg).$$

Definition 2.1 ([12])**.** Let $I \subset (0, \infty)$ be an interval. A function $f : I \to (0, \infty)$ is said to be Jensen geometrically convex (Jensen geometrically concave) if

$$f\left(\sqrt{uv}\right) \leq \sqrt{f(u) f(v)}, \quad u, v \in I,$$

(the reversed inequality holds), and Jensen geometrically affine if the case of the equality is satisfied.

We say that $f : I \to (0, \infty)$ is geometrically convex (geometrically concave) if it is Jensen geometrically convex (Jensen geometrically concave) and continuous.

The above definition is a special case of the following more general one due to G. Aumann [1]. Let $J \subset \mathbb{R}$ be an interval and $M : J^2 \to J$ a mean, that is,

$$\min(u, v) \leq M(u, v) \leq \max(u, v), \quad u, v \in J.$$

A function $f : I \to J$, where I is a subinterval of J, is said to be convex with respect to the mean M (briefly, M-convex) if

$$f(M(u, v)) \leq M(f(u), f(v)), \quad u, v \in I.$$

Thus $f : I \to \mathbb{R}$ is Jensen convex in the interval $I \subset \mathbb{R}$ iff it is convex with respect to the arithmetic mean $M(u, v) = \frac{u+v}{2}$ $(u, v \in \mathbb{R})$; $f : I \to (0, \infty)$ is Jensen geometrically convex in the interval $I \subset (0, \infty)$ iff it is convex with respect to the geometric mean $M(u, v) = \sqrt{uv}$ $(u, v \in (0, \infty))$.

Remark 2.2. A function $f : I \to (0,\infty)$ is Jensen geometrically convex iff $\log \circ f \circ \exp$ is Jensen convex on the interval $\log(I)$.

A function $f : I \to (0,\infty)$ is geometrically convex if and only if

$$f\left(u^{\lambda}v^{1-\lambda}\right) \le f(u)^{\lambda} f(v)^{1-\lambda}, \quad u, v \in I;\ \lambda \in (0,1).$$

If f is Jensen geometrically convex and continuous at least at one point, then f is continuous (cf. Kuczma [6]). If f is differentiable, then it is geometrically convex (concave) iff the function

$$I \ni u \longmapsto \frac{f'(u)}{f(u)}u \ \text{ is increasing (decreasing).}$$

Note also that [5] if $f : (0,\infty) \to (0,\infty)$ is differentiable, f' is positive, geometrically convex, and $\lim_{x\to 0} f(x) = 0$, then f is geometrically convex.

Assume that f is continuous at least at one point. Then $f : I \to (0,\infty)$ is geometrically affine iff $f(u) = au^p$ for some $a > 0$ and $p \in \mathbb{R}$.

Example 2.3. Let $p, q \in \mathbb{R}$, $p < q$. Then the function

$$\varphi(t) := \begin{cases} t^p, & 0 < t \le 1, \\ t^q, & t > 1 \end{cases}$$

is geometrically convex on $(0,\infty)$. If moreover $p \ge 1$, then φ is convex.

This example shows that convex and geometrically convex functions need not be differentiable everywhere.

One can easily verify the following remark.

Remark 2.4. For $p, q, c \in \mathbb{R}$, $c \ge 0$, put

$$\varphi(t) = (t+c)^q\, t^p, \quad t > 0.$$

If $q \ge 0$ $(q \le 0)$, then φ is geometrically convex (geometrically concave).

If $p + q \ge 1$ and $pq \le 0$, or $p \ge 1$ and $q \ge 0$, then φ is convex.

Remark 2.5. Let ψ be an increasing homeomorphism of $(0,\infty)$. It is obvious that ψ and $\log \circ \psi \circ \exp$ are convex if and only if the function $\varphi := \psi^{-1}$ and $\log \circ \varphi \circ \exp$ are concave.

Lemma 2.6. *Assume that* $\varphi : (0,\infty) \to (0,\infty)$.

(i) *For every* $t > 1$, *the function*

$$(0,\infty) \ni r \to \frac{\varphi(tr)}{\varphi(r)}$$

is increasing (*decreasing*) *if and only if the function* $f : \mathbb{R} \to \mathbb{R}$,

$$f := \log \circ \varphi \circ \exp,$$

is Wright-convex, i.e. for all $u_1, u_2 \in \mathbb{R}$ *and* $\lambda \in (0,1)$,

$$f(\lambda u_1 + (1-\lambda) u_2) + f((1-\lambda) u_1 + \lambda u_2) \le f(u_1) + f(u_2), \qquad (2.2)$$

(*Wright-concave, i.e. the reversed inequality is satisfied*).

(ii) *Let* $\varphi : (0,\infty) \to (0,\infty)$ *be strictly increasing and onto. Then, for every* $t > 1$, *the function*

$$(0,\infty) \ni r \to \frac{\varphi\left(t\varphi^{-1}(r)\right)}{r}$$

is increasing (decreasing) if and only if φ *is geometrically convex (geometrically concave).*

Proof. (i) Note that for every $t > 1$ the function under consideration is increasing if and only if for every $v > 0$, the function $\mathbb{R} \ni u \longmapsto \log\varphi(\exp^{u+v}) - \log\varphi(\exp^{u})$ is increasing.

Thus, putting $f := \log \circ \varphi \circ \exp$, we have for all $u, w \in \mathbb{R}$ and all $v > 0$,

$$u < w \Longrightarrow f(u+v) - f(u) \le f(w+v) - f(w).$$

Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies this implication. Taking arbitrary $u_1, u_2 \in \mathbb{R}$, $u_1 < u_2$, $\lambda \in (0,1)$, and putting

$$u = u_1, \quad w = (1-\lambda) u_1 + \lambda u_2, \quad v = (1-\lambda)(u_2 - u_1),$$

we hence get (2.2), that is, f is convex in the sense of Wright.

Conversely, taking arbitrary $u, w, v \in \mathbb{R}$, $u < w$, $v > 0$, and setting

$$u_1 := u, \quad u_2 := w + v, \quad \lambda := \frac{w-u}{w+v-u}$$

in (2.2), we obtain $f(u+v) - f(u) \le f(w+v) - f(w)$. This completes the proof of (i).

Part (ii) follows from (i), the continuity of f ([26]) and Remark 2.2. □

Let us quote the following well known criterion of subadditivity (cf. [4, p. 239], also [30]).

Lemma 2.7. *Let* $f : (0,\infty) \to (0,\infty)$. *If the function*

$$(0,\infty) \ni r \longmapsto \frac{f(r)}{r}$$

is increasing (decreasing), then f *is superadditive, i.e.*

$$f(r+s) \ge f(r) + f(s), \qquad r, s > 0,$$

(*subadditive*).

A linear functional inequality generalizing both convexity as well as subadditivity is considered in [27]. Note that subadditive periodic functions were considered in [20].

We shall need the following (cf. [13, 23]).

Lemma 2.8. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure and $\varphi : (0,\infty) \to (0,\infty)$ be a monotonic bijection.*

(i) *If $\mathbf{P}_\varphi$ is subadditive, that is, if*

$$\mathbf{P}_\varphi(x+y) \le \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad x, y \in S_+, \tag{2.3}$$

then φ is increasing.

(ii) *If $\mathbf{P}_\varphi$ is subadditive and there are $A, B \in \Sigma$, $A \cap B = \emptyset$, such that either*

$$\mu(A) = 1 \quad and \quad 0 < \mu(B) < \infty,$$

or $\mu(A)$ and $\mu(B)$ are positive and

$$\mu(A) + \mu(B) = 1,$$

then φ is convex.

Proof. Ad (i). By the assumptions, there are $A, B \in \Sigma$ such that $A \cap B = \emptyset$ and $a := \mu(A)$, $b := \mu(B)$ are positive reals. Taking $x := x_1\chi_A + x_2\chi_B$, $y := y_1\chi_A + y_2\chi_B$ in (2.3) and making use of (2.1) we get

$$\begin{aligned} &\varphi^{-1}(a\varphi(x_1+y_1) + b\varphi(x_2+y_2)) \le \\ &\le \varphi^{-1}(a\varphi(x_1) + b\varphi(x_2)) + \varphi^{-1}(a\varphi(y_1) + b\varphi(y_2)) \end{aligned} \tag{2.4}$$

for all $x_1, x_2, y_1, y_2 > 0$. By assumptions, φ is strictly monotonic and, being bijective, it is continuous. To show that φ is increasing, assume for the contrary, that φ is decreasing. It follows that

$$\lim_{t\to 0} \varphi(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \varphi(t) = 0.$$

Since φ^{-1} is decreasing, letting y_2 tend to 0 in inequality (2.4), we get

$$\varphi^{-1}(a\varphi(x_1+y_1) + b\varphi(x_2)) \le \varphi^{-1}(a\varphi(x_1) + b\varphi(x_2)), \quad x_1, y_1, x_2 > 0.$$

Hence, letting here x_2 tend to ∞, we get

$$\varphi^{-1}(a\varphi(x_1+y_1)) \le \varphi^{-1}(a\varphi(x_1)), \quad x_1, y_1 > 0,$$

whence, letting x_1 tend to 0, we obtain

$$\varphi^{-1}(a\varphi(y_1)) \le 0, \quad y_1 > 0.$$

This contradiction completes the proof of (i).

Ad (ii). Assume that there are disjoint $A, B \in \Sigma$ such that $a := \mu(A) = 1$, and $b := \mu(B)$ is positive. From (2.4) we get the inequality

$$\begin{aligned} &\varphi^{-1}(\varphi(x_1+y_1) + b\varphi(x_2+y_2)) \le \\ &\le \varphi^{-1}(\varphi(x_1) + b\varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + b\varphi(y_2)) \end{aligned}$$

for all $x_1, x_2, y_1, y_2 > 0$. To show the convexity of φ take arbitrary $r, s > 0$, $r > s$. In view of part (i), the function φ is increasing. Thus φ^{-1} is increasing and, consequently, the numbers

$$x_1 := \varphi^{-1}(s), \quad x_2 := \varphi^{-1}\left(\frac{r-s}{2b}\right), \quad y_1 := \varphi^{-1}\left(\frac{r+s}{2}\right) - \varphi^{-1}(s)$$

are positive. Substituting them into the above inequality and letting y_2 tend to 0 we get

$$\frac{\varphi^{-1}(s) + \varphi^{-1}(r)}{2} \leq \varphi^{-1}\left(\frac{r+s}{2}\right),$$

which means that φ^{-1} is Jensen concave. The increasing monotonicity of φ implies that φ is convex.

Now assume that there are disjoint $A, B \in \Sigma$ such that $a := \mu(A)$, $b := \mu(B)$ are positive and $a + b = 1$. Take arbitrary $s, t > 0$. Setting

$$x_1 = y_2 := \varphi^{-1}(s), \quad x_2 = y_1 := \varphi^{-1}(t)$$

in (2.4), we get

$$\varphi^{-1}(s) + \varphi^{-1}(t) \leq \varphi^{-1}(as + (1-a)t) + \varphi^{-1}(at + (1-a)s), \quad s, t > 0,$$

which proves that φ^{-1} is a-Wright concave. Since, by the first part of this result, φ is increasing, it is convex ([26]). □

Lemma 2.9. *Let (Ω, Σ, μ) be a measure space such that*

$$A \in \Sigma \ \Rightarrow \mu(A) \leq 1 \quad \text{or} \quad \mu(A) = \infty,$$

and suppose that there are two sets $B, C \in \Sigma$ such that

$$0 < \mu(B) < \mu(C) = 1.$$

Let $\varphi : (0, \infty) \to (0, \infty)$ be one-to-one, onto and increasing.

(i) *The functional $\mathbf{P}_\varphi$ is subhomegeneous, that is,*

$$\mathbf{P}_\varphi(tx) \leq t\mathbf{P}_\varphi(x), \qquad x \in S_+, \ t > 1,$$

if and only if φ is differentiable and φ' is geometrically concave.

(ii) *The functional $\mathbf{P}_\varphi$ is superhomegeneous, that is,*

$$\mathbf{P}_\varphi(tx) \geq t\mathbf{P}_\varphi(x), \qquad x \in S_+, \ t > 1,$$

if and only if φ is differentiable and φ' is geometrically convex.

Proof. (i) Suppose that φ' is geometrically concave. Clearly, for every $t > 1$, the function $\varphi \circ (t\varphi^{-1})$ is differentiable and

$$\left(\varphi \circ (t\varphi^{-1})\right)'(u) = \frac{\varphi'(t\varphi^{-1}(u))}{\varphi'(\varphi^{-1}(u))}t, \quad u \in (0, \infty).$$

Consequently, for every $t > 1$,

$$\left(\varphi \circ (t\varphi^{-1})\right)' \circ \varphi\,(u) = \frac{\varphi'(tu)}{\varphi'(u)}t, \qquad u \in (0, \infty).$$

Hence, taking into account Lemma 2.6 (i) and the geometrical concavity of φ', we conclude that for every $t > 1$ the function $\left(\varphi \circ (t\varphi^{-1})\right)' \circ \varphi$ is decreasing. Since φ is increasing, it follows that, for every $t > 1$, the function $\varphi \circ (t\varphi^{-1})$ is concave.

Take an arbitrary $x \in S_+$ such that $\mu\,(\Omega(x)) > 0$. Then there exists $n \in \mathbb{N}$, the pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$ of positive measure and $x_1, \ldots, x_n > 0$ such that

$$x = \sum_{j=1}^{n} x_j \chi_{A_j}.$$

Put

$$a_j := \mu(A_j), \quad j = 1, \ldots, n; \qquad a := 1 - \sum_{j=1}^{n} a_j.$$

According to the assumptions,

$$0 < \sum_{j=1}^{n} a_j \leq 1,$$

whence $a \geq 0$. If $a > 0$ then, for every $t > 0$, by the concavity of $\varphi \circ (t\varphi^{-1})$, we have

$$\sum_{j=1}^{n} a_j \varphi(t\varphi^{-1}(x_j)) + a\varphi\left(t\varphi^{-1}(u)\right) \leq \varphi\left(t\varphi^{-1}\left(\sum_{j=1}^{n} a_j x_j + au\right)\right)$$

for all $x_1, \ldots, x_n > 0$ and $u > 0$. Since $\lim_{u \to 0+} \varphi\left(t\varphi^{-1}(u)\right) = 0$, letting u tend to 0, we hence obtain

$$\sum_{j=1}^{n} a_j \varphi\left(t\varphi^{-1}(x_j)\right) \leq \varphi\left(t\varphi^{-1}\left(\sum_{j=1}^{n} a_j x_j\right)\right), \quad x_1, \ldots, x_n > 0, \quad t > 1.$$

Of course, this inequality holds true also if $\sum_{j=1}^{n} a_j = 1$. Replacing x_j by $\varphi(x_j)$, $j = 1, \ldots, n$, and making use of the strict monotonicity of φ we hence get, for all $t > 1$ and $x_1, \ldots, x_n > 0$,

$$\varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(tx_j)\right) \leq t\varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(x_j)\right),$$

which means that $\mathbf{P}_\varphi(tx) \leq t\mathbf{P}_\varphi(x)$ for all $t > 1$.

Now we prove the converse implication. Note that $\mu(B \cup C) = 1$. Putting $D := (B \cup C) \backslash B$ we have $b := \mu(B) \in (0,1)$ and $\mu(D) = 1 - b$. For $x = x_1 \chi_B + x_2 \chi_D$, with arbitrary $x_1, x_2 > 0$, by the subhomogeneity of $\mathbf{P}_\varphi$, we have, for all $t > 1$,

$$\varphi^{-1}\left(b\varphi(tx_1) + (1-b)\varphi(tx_2)\right) \leq t\varphi^{-1}\left(b\varphi(x_1) + (1-b)\varphi(x_2)\right).$$

Replacing x_1 by $\varphi^{-1}(x_1)$ and x_2 by $\varphi^{-1}(x_2)$ and making use of the increasing monotonicity of φ, we hence get

$$b\varphi(t\varphi^{-1}(x_1)) + (1-b)\varphi(t\varphi^{-1}(x_2)) \leq \varphi(t\varphi^{-1}\left(bx_1 + (1-b)x_2\right))$$

for all $t > 1$ and $x_1, x_2 > 0$, which proves that for every $t > 1$ the function $\varphi \circ (t\varphi^{-1})$ is b-concave. The Daróczy-Páles identity ([2]):

$$b\left(b\frac{x+y}{2} + (1-b)\,x\right) + (1-b)\left(by + (1-b)\,\frac{x+y}{2}\right) = \frac{x+y}{2}, \quad x, y \in \mathbb{R},$$

implies that $\varphi \circ (t\varphi^{-1})$ is $\frac{1}{2}$-concave, that is, Jensen concave. Since $\varphi \circ (t\varphi^{-1})$ is continuous, it follows that for every $t > 1$ this function is concave and, consequently,

$$a\varphi(t\varphi^{-1}(x_1)) + (1-a)\varphi(t\varphi^{-1}(x_2)) \leq \varphi(t\varphi^{-1}\left(ax_1 + (1-a)x_2\right)),$$
$$t > 1,\ a \in (0,1),\ x_1, x_2 > 0.$$

Letting x_2 tend to 0, we hence get

$$a\varphi(t\varphi^{-1}(r)) \leq \varphi(t\varphi^{-1}\left(ar\right)), \quad t > 1,\ a \in (0,1),\ r > 0,$$

or, equivalently,

$$\frac{\varphi(t\varphi^{-1}(r))}{r} \leq \frac{\varphi(t\varphi^{-1}(ar))}{ar}, \qquad t > 1,\ a \in (0,1),\ r > 0.$$

Thus the function $(0,\infty) \ni r \longmapsto \frac{\varphi(t\varphi^{-1}(r))}{r}$ is decreasing and, by Lemma 2.6 (ii), the function φ is geometrically concave. It follows that the one-sided derivatives φ'_-, φ'_+ of φ exist, they are monotonic, equal up to at most a countable set.

On the other hand the concavity of the function $\varphi \circ (t\varphi^{-1})$ for $t > 1$ implies that, for every $t > 1$, its one sided derivatives $\left(\varphi \circ (t\varphi^{-1})\right)'_-$, $\left(\varphi \circ (t\varphi^{-1})\right)'_+$ exist and are decreasing. Since φ is increasing we have, for every $t > 1$,

$$\left(\varphi \circ (t\varphi^{-1})\right)'_+(r) = \frac{\varphi'_+(t\varphi^{-1}(r))}{\varphi'_+\left(\varphi^{-1}(r)\right)}t, \qquad r > 0,$$

and the function

$$(0,\infty) \ni r \longmapsto \frac{\varphi'_+(t\varphi^{-1}(r))}{\varphi'_+\left(\varphi^{-1}(r)\right)}$$

is decreasing. It follows that, for every $t > 1$, the function

$$(0,\infty) \ni r \longmapsto \frac{\varphi'_+(tr)}{\varphi'_+(r)}$$

is decreasing. By Lemma 2.6 (i), the function $\log \circ \varphi'_+ \circ \exp$ is Wright-concave. Since this function is measurable, applying the result of [26], we conclude that φ'_+ is continuous. Consequently, φ is differentiable in $(0, \infty)$, and $\varphi' = \varphi'_+$ is geometrically concave. This completes the proof of result (i).

Since the proof of (ii) is analogous, we omit it. □

In the first part of the proof we have shown that, for every $t > 1$, the function $\varphi \circ \left(t\varphi^{-1}\right)$ is concave and $\lim_{u \to 0+} \varphi\left(t\varphi^{-1}(u)\right) = 0$. It follows that, for every $t > 1$, the function

$$(0, \infty) \ni u \to \frac{\varphi\left(t\varphi^{-1}(u)\right)}{u}$$

is decreasing. Composing this function with the increasing function φ we conclude that, for every $t > 1$, the function

$$(0, \infty) \ni r \to \frac{\varphi(tr)}{\varphi(r)}$$

is decreasing. In view of Lemma 2.6, the function φ is geometrically concave. This proves the following remark.

Remark 2.10. Let $\varphi : (0, \infty) \to (0, \infty)$ be one-to-one, increasing and onto. If φ is differentiable and φ' is geometrically concave (convex), the function φ is geometrically concave (convex) (see also [5, 23]).

From Theorem 5 and Theorem 6 in [23] we immediately get the following lemma.

Lemma 2.11. *Let (Ω, Σ, μ) be a measure space such that*

$$A \in \Sigma \ \Rightarrow \mu(A) = 0 \quad \text{or} \quad \mu(A) \geq 1. \tag{2.5}$$

Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is strictly increasing and onto.

(i) *If φ is geometrically convex then $\mathbf{P}_\varphi$ is subhomegeneous, that is,*

$$\mathbf{P}_\varphi(tx) \leq t\mathbf{P}_\varphi(x), \qquad x \in S_+, \ t > 1.$$

(ii) *If φ is geometrically concave then $\mathbf{P}_\varphi$ is superhomegeneous, that is,*

$$\mathbf{P}_\varphi(tx) \geq t\mathbf{P}_\varphi(x), \qquad x \in S_+, \ t > 1.$$

Moreover, if for some $\delta > 1$,

$$(1, \delta) \subset \operatorname{cl}(\mu(\Sigma)),$$

then the converses of the implications of (i) *and* (ii) *hold true.*

We shall need also the following generalization of Mulholland's inequality ([25], see also [21]).

Theorem 2.12 ([13])**.** *Let (Ω, Σ, μ) be a measure space such that condition* (2.5) *is satisfied. If $\varphi : (0, \infty) \to (0, \infty)$ is strictly increasing, onto, convex and geometrically convex, then*

$$\mathbf{P}_\varphi(x + y) \leq \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad x, y \in S_+.$$

3. CONJUGATE FUNCTIONS

Definition 3.1. Let $h : (0, \infty) \to \mathbb{R}$ be an arbitrary function. The function $h^* : (0, \infty) \to \mathbb{R}$,

$$h^*(t) := th\left(\frac{1}{t}\right), \quad t > 0,$$

is said to be $(*)$-conjugate of h (briefly, conjugate of h). If $h^* = h$, then h is called $(*)$-selfconjugate (briefly, selfconjugate).

Remark 3.2. Let $\mathbb{R}^{(0,\infty)}$ denote the real linear space of all functions $h : (0, \infty) \to \mathbb{R}$. Obviously, the operator $* : \mathbb{R}^{(0,\infty)} \to \mathbb{R}^{(0,\infty)}$, given by $\mathbb{R}^{(0,\infty)} \ni h \longmapsto h^*$, is homogeneous, additive (so linear) and bijective.

Moreover, this operator is strictly related to two variable homogeneous functions.

Properties of conjugate functions. Let $h : (0, \infty) \to \mathbb{R}$ be an arbitrary function. Then:

1. $(h^*)^* = h$; $h(1) = h^*(1)$.
2. If $p \in \mathbb{R}$, $p \neq 0$, and $h(t) = h(1)t^{1/p}$, then $h^*(t) = h(1)\, t^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.
3. ([9, 10]) h is convex (concave) iff so is h^*.
4. Let $g(t) := \frac{h(t)}{t}$ for $t > 0$. Then h is convex iff the function $(0, \infty) \ni t \to g(\frac{1}{t})$ is convex. If follows that either g is monotonic or unimodal ([22]). If h^* is increasing, then g is decreasing; if moreover h is convex and $h(0+) = 0$, then $h(t) = h(1)t$ for all $t > 0$.
5. a) If h is a concave bijection of $(0, \infty)$, then h and h^* are increasing, h^* is concave and maps $(0, \infty)$ onto the interval (α, β), where

   $$\alpha := \lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} h'_-(t), \qquad \beta := \lim_{t\to 0+} \frac{h(t)}{t} = \lim_{t\to 0+} h'_+(t),$$

   where $h'_-(t)$ and $h'_+(t)$ denote, respectively, the left and right derivative of h at t.

   b) If h is a convex and increasing bijection of $(0, \infty)$, then h^* is decreasing and maps $(0, \infty)$ onto the interval (α, β), where

   $$\alpha := \lim_{t\to 0+} \frac{h(t)}{t} = \lim_{t\to 0+} h'_+(t), \quad \beta := \lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} h'_-(t).$$

   c) If h is a decreasing and convex bijection of $(0, \infty)$, then h^* is an increasing, convex bijection of $(0, \infty)$;
   (of course, h cannot be a decreasing and concave bijection of $(0, \infty)$).
6. If h is convex and

   $$\lim_{t\to 0} \frac{h(t)}{t} = 0 \text{ and } \lim_{t\to\infty} \frac{h(t)}{t} = \infty,$$

   or

   $$\lim_{t\to 0} \frac{h(t)}{t} = \infty \text{ and } \lim_{t\to\infty} \frac{h(t)}{t} = 0,$$

then, respectively,
$$\lim_{t\to 0} h^*(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} h^*(t) = 0,$$
or
$$\lim_{t\to 0} h^*(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} h^*(t) = \infty.$$

7. A function h is self-conjugate, that is, $h = h^*$, if and only if the function of two variables
$$(0,\infty)^2 \ni (s,t) \longrightarrow th\left(\frac{s}{t}\right)$$
is symmetric.
8. Let $H : (0,\infty)^2 \to \mathbb{R}$ be positively homogeneous, i.e.
$$H(\tau s, \tau t) = \tau H(s,t), \quad \tau, s, t > 0.$$
Then H is subadditive if and only if the functions $h_1, h_2 : (0,\infty) \to \mathbb{R}$ given by
$$h_1(s) := H(s,1), \quad s > 0; \qquad \text{and} \qquad h_2(t) := H(1,t), \quad t > 0,$$
are convex. Moreover,
$$H(s,t) = th_1\left(\frac{s}{t}\right) = sh_2\left(\frac{t}{s}\right), \quad s,t > 0,$$
and
$$h_2 = (h_1)^*.$$
9. ([9, 10]) (a generalization of the Minkowski and Hölder inequalities for sums) The function h is convex (concave) iff the function $H : (0,\infty)^2 \to \mathbb{R}$,
$$H(s,t) := th\left(\frac{s}{t}\right), \quad s,t > 0,$$
is subadditive (superadditive), i.e.
$$(y_1+y_2)\, h\left(\frac{x_1+x_2}{y_1+y_2}\right) \le y_1 h\left(\frac{x_1}{y_1}\right) + y_2 h\left(\frac{x_2}{y_2}\right), \quad x_1, x_2, y_1, y_2 > 0,$$
(the reversed inequality is satisfied).
10. ([9, 10]) (a generalization of the Minkowski and Hölder inequalities for integrals) Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) > 0$. If a function h is convex (concave), then
$$\int_\Omega y d\mu h\left(\frac{\int_\Omega x d\mu}{\int_\Omega y d\mu}\right) \le \int_\Omega yh \circ \left(\frac{x}{y}\right) d\mu$$
for all positive $x, y \in L^1_+(\Omega, \Sigma, \mu)$ (the reversed inequality is satisfied).
11. The function h is geometrically convex (geometrically concave) iff so is h^*.
12. If $h : (0,\infty) \to (0,\infty)$ is geometrically convex and $\alpha, \beta : (0,\infty) \to (0,\infty)$ are geometrically affine, then $\alpha \cdot h \circ \beta$ is geometrically convex.

We shall prove, for instance Property 3, applying a simpler argument than that in [9] and [10]. Suppose that h^* is convex. Hence, by the definition of h^*, we have

$$(tr + (1-t)\,s)\,h\left(\frac{1}{tr + (1-t)\,s}\right) \le trh\left(\frac{1}{r}\right) + (1-t)\,sh\left(\frac{1}{s}\right), \quad t \in (0,1)\,,\ r,s > 0,$$

and, of course, it follows that h is continuous. Taking here

$$r := \frac{1}{u}, \quad s := \frac{1}{v}, \quad t := \frac{u}{u+v}$$

for arbitrary $u, v \in (0, \infty)$, we obtain

$$h\left(\frac{u+v}{2}\right) \le \frac{h\,(u) + h\,(v)}{2}, \quad u, v > 0.$$

The continuity of h implies its convexity (cf. [6]). Since, by Property 1, $h = (h^*)^*$, to prove the converse implication, it is enough to replace in the above inequalities the function h by h^*.

Now we prove the following result.

Theorem 3.3. *Suppose that $\gamma : (0,\infty) \to (0,\infty)$ is bijective, increasing, concave, geometrically concave and*

$$\lim_{t\to 0+} \frac{\gamma(t)}{t} = \infty, \qquad \lim_{t\to\infty} \frac{\gamma(t)}{t} = 0.$$

Then $\gamma^* : (0,\infty) \to (0,\infty)$, the conjugate of γ,

$$\gamma^*\,(t) := t\gamma\Big(\frac{1}{t}\Big), \qquad t > 0,$$

has the same properties, i.e. γ^* is bijective, increasing, concave, geometrically concave and

$$\lim_{t\to 0+} \frac{\gamma^*(t)}{t} = \infty, \qquad \lim_{t\to\infty} \frac{\gamma^*(t)}{t} = 0.$$

Proof. By the Properties 3 and 11, the function γ^* is concave and geometrically concave. Since γ is an increasing homeomorphism of $(0,\infty)$, we have

$$\gamma\,(0+) := \lim_{t\to 0+} \gamma\,(t) = 0, \quad \gamma\,(\infty) := \lim_{t\to\infty} \gamma\,(t) = \infty.$$

The first of these relations and the concavity of γ imply that the function $(0,\infty) \ni t \to \gamma(t)/t$ is strictly decreasing. Now, for arbitrary $s, t > 0$ such that $s < t$ we have

$$\gamma^*\,(s) = \frac{\gamma\left(s^{-1}\right)}{s^{-1}} < \frac{\gamma\left(t^{-1}\right)}{t^{-1}} = \gamma^*\,(t)\,,$$

which shows that γ^* is increasing. Since

$$\lim_{t\to 0+} \gamma^*\,(t) = \lim_{t\to 0+} \frac{\gamma\left(t^{-1}\right)}{t^{-1}} = \lim_{t\to\infty} \frac{\gamma\,(t)}{t} = 0$$

and

$$\lim_{t\to\infty}\gamma^{*}(t)=\lim_{t\to\infty}\frac{\gamma\left(t^{-1}\right)}{t^{-1}}=\lim_{t\to 0+}\frac{\gamma(t)}{t}=\infty,$$

the function γ^* is a bijection of $(0,\infty)$. Since γ is an increasing bijection of $(0,\infty)$, we have

$$\lim_{t\to 0+}\frac{\gamma^{*}(t)}{t}=\lim_{t\to 0+}\gamma\left(t^{-1}\right)=\lim_{t\to\infty}\gamma(t)=\infty,$$

and

$$\lim_{t\to\infty}\frac{\gamma^{*}(t)}{t}=\lim_{t\to\infty}\gamma\left(t^{-1}\right)=\lim_{t\to 0+}\gamma(t)=0.$$

This completes the proof. □

Theorem 3.4. *Suppose that $\varphi:(0,\infty)\to(0,\infty)$ is bijective, increasing, convex, geometrically convex and*

$$\lim_{t\to 0+}\frac{\varphi(t)}{t}=0,\qquad \lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty. \tag{3.1}$$

Then the function

$$\psi:=\left[\left(\varphi^{-1}\right)^{*}\right]^{-1}$$

has the same properties, i.e. $\psi:(0,\infty)\to(0,\infty)$ is bijective, increasing, convex, geometrically convex and

$$\lim_{t\to 0+}\frac{\psi(t)}{t}=0,\qquad \lim_{t\to\infty}\frac{\psi(t)}{t}=\infty.$$

Proof. It is easy to verify that the function $\gamma:=\varphi^{-1}$ maps $(0,\infty)$ onto itself, is bijective, increasing, concave, such that

$$\lim_{t\to 0+}\frac{\gamma(t)}{t}=\infty,\qquad \lim_{t\to\infty}\frac{\gamma(t)}{t}=0.$$

In view of Theorem 2, the function $\psi^{-1}=\gamma^*$ inherits the same properties. It follows that ψ has the same properties as φ. This completes the proof. □

Remark 3.5. Let $\varphi:(0,\infty)\to(0,\infty)$. Note that:

(i) φ^* is one-to-one iff the function $t\longmapsto\frac{\varphi(t)}{t}$ is one-to-one,
(ii) φ^* is onto iff the function $t\longmapsto\frac{\varphi(t)}{t}$ is onto,
(iii) φ^* is bijective iff the function $t\longmapsto\frac{\varphi(t)}{t}$ is bijective.

It is natural to ask what is the relation between φ^* and $(\varphi^{-1})^*$. The answer is given by the following theorem.

Theorem 3.6. *Suppose that $\varphi:(0,\infty)\to(0,\infty)$ and φ^* are bijective. Then*

$$\left(\varphi^{-1}\right)^{*}=\left(\left[\left(\varphi^{*}\right)^{-1}\right]^{*}\right)^{-1}.$$

Proof. Since

$$\varphi(t) = t\varphi^*\left(\frac{1}{t}\right), \quad \varphi^{-1}(t) = t(\varphi^{-1})^*\left(\frac{1}{t}\right) \tag{3.2}$$

and $\varphi^{-1}(\varphi(t)) = t$, we have

$$t\varphi^*\left(\frac{1}{t}\right)\left[(\varphi^{-1})^*\left(\frac{1}{t\varphi^*\left(\frac{1}{t}\right)}\right)\right] = t, \quad t > 0,$$

that is,

$$\varphi^*\left(\frac{1}{t}\right)\left[(\varphi^{-1})^*\left(\frac{1}{t\varphi^*(\frac{1}{t})}\right)\right] = 1, \quad t > 0.$$

By the first of the formulas (3.2), this equality can be written in the form

$$\varphi^*\left(\frac{1}{t}\right)\left[(\varphi^{-1})^*\left(\frac{1}{\varphi(t)}\right)\right] = 1, \quad t > 0,$$

whence

$$\varphi^*\left(\frac{1}{\varphi^{-1}(t)}\right)\left[(\varphi^{-1})^*\left(\frac{1}{t}\right)\right] = 1, \quad t > 0.$$

Replacing t by $1/t$ we hence get

$$\varphi^*\left(\frac{1}{\varphi^{-1}(\frac{1}{t})}\right)\left[(\varphi^{-1})^*(t)\right] = 1, \quad t > 0,$$

that, by (3.2), is equivalent to the equality

$$\varphi^*\left(\frac{t}{(\varphi^{-1})^*(t)}\right)\left[(\varphi^{-1})^*(t)\right] = 1, \quad t > 0.$$

Writing this equation in the form

$$\frac{1}{(\varphi^{-1})^*(t)} = \varphi^*\left(\frac{t}{(\varphi^{-1})^*(t)}\right), \quad t > 0,$$

and taking $(\varphi^*)^{-1}$ of both sides gives

$$(\varphi^*)^{-1}\left(\frac{1}{(\varphi^{-1})^*(t)}\right) = \frac{t}{(\varphi^{-1})^*(t)}, \quad t > 0,$$

whence

$$(\varphi^{-1})^*(t)\left[(\varphi^*)^{-1}\left(\frac{1}{(\varphi^{-1})^*(t)}\right)\right] = t, \quad t > 0.$$

Since φ and φ^* are bijective, Remark 3.5 implies that $(\varphi^{-1})^*$ is a bijection of $(0,\infty)$. Now the last equality says that $(\varphi^{-1})^*$ and the function

$$(0,\infty) \ni t \longmapsto t(\varphi^*)^{-1}\left(\frac{1}{t}\right)$$

are inverses of one another. This completes the proof. □

Denote by $\mathcal{W}(0,\infty)$, briefly $\mathcal{W}$, the set of all bijective functions $\varphi:(0,\infty)\to(0,\infty)$ such that φ^* is bijective. Let $T,U:\mathcal{W}\to\mathcal{W}$ be the operators defined by

$$T(\varphi):=\varphi^{-1},\quad U(\varphi):=\varphi^*,\qquad \varphi\in\mathcal{W}.$$

(Thus U is the restriction of the operator $*$ to the set $\mathcal{W}$.)

For a selfmapping F of a set, by F^n denote the n-th iterate of F.

From Theorem 3.6 we obtain the following corollary.

Corollary 3.7. *T and U are involutions, i.e.*

$$T^2=T\circ T=\mathrm{id}\,|_{\mathcal{W}}=U\circ U=U^2,$$

and

$$(U\circ T)^3=\mathrm{id}\,|_{\mathcal{W}}=(T\circ U)^3,$$

$$T\circ U\circ T\circ U\circ T=U,\quad U\circ T\circ U\circ T\circ U=T,$$

$$(T\circ U)^2=U\circ T,\quad (U\circ T)^2=T\circ U,$$

$$U\circ T\circ U=T\circ U\circ T,\quad (U\circ T\circ U)^2=\mathrm{id}\,|_{\mathcal{W}}=(T\circ U\circ T)^2,$$

(i.e. $U\circ T\circ U$ and $T\circ U\circ T$ are the involutions of $\mathcal{W}$). Moreover U and T do not commute, i.e. $U\circ T\neq T\circ U$.

Remark 3.8. Taking $\varphi\in\mathcal{W}$, $\varphi(t)=t^p$ $(t>0)$ for some real $p\neq 0$, by the definitions of U and T, we have

$$U(\varphi)(t)=t^{1-p},\quad T\circ U(\varphi)(t)=t^{1/(1-p)},\quad U\circ T\circ U(\varphi)(t)=t^{p/(p-1)},$$

$$T\circ U\circ T\circ U(\varphi)(t)=t^{(p-1)/p},\quad U\circ T\circ U\circ T\circ U(\varphi)(t)=t^{1/p},$$

$$T\circ U\circ T\circ U\circ T\circ U(\varphi)(t)=t^p.$$

For $p\in\mathbb{R}\backslash\{-1,0,\frac{1}{2},1,2\}$, all functions in these sequence are different. This shows that $n=3$ is the smallest positive integer such that $(T\circ U)^n=\mathrm{id}\,|_{\mathcal{W}}$.

In view of Corollary 3.7, the operators U and T do not commute. We prove even more.

Proposition 3.9. *There is no a continuous function $\varphi\in\mathcal{W}$ such that*

$$(\varphi^{-1})^*=(\varphi^*)^{-1}. \tag{3.3}$$

Proof. Assume that there is $\varphi\in\mathcal{W}$ such that (3.3) is satisfied. Taking first the $(*)$-complementary functions, and then the inverse functions of both sides we get

$$\varphi=\left(\left[(\varphi^*)^{-1}\right]^*\right)^{-1}.$$

Hence, applying Theorem 3.6, we obtain $\varphi=\left(\varphi^{-1}\right)^*$, whence

$$\varphi^{-1}=\varphi^*.$$

Thus $\varphi \circ \varphi^* = \mathrm{id}\left|_{(0,\infty)}\right.$, i.e.

$$\varphi\Big(t\varphi\Big(\frac{1}{t}\Big)\Big) = t, \quad t > 0.$$

Replacing here t by $1/t$ we conclude that φ satisfies the functional equation

$$\varphi\left(\frac{\varphi(t)}{t}\right) = \frac{1}{t}, \quad t > 0. \tag{3.4}$$

Now the following lemma concludes the proof. □

Lemma 3.10. *There is no a continuous function $\varphi : (0,\infty) \to (0,\infty)$ satisfying equation* (3.4).

Proof. It is easy to verify that $a = \frac{1}{\varphi(1)}$ is the only fixed point of any function $\varphi : (0,\infty) \to (0,\infty)$ satisfying equation (3.4). Suppose that φ is continuous. Since φ composed with the continuous function $(0,\infty) \ni t \to \varphi(t)/t$ is a strictly decreasing reciprocal function, φ must be strictly increasing and $(0,\infty) \ni t \to \varphi(t)/t$ strictly decreasing.

Setting $t = a$ into (3.4) gives $\varphi(1) = \frac{1}{a}$. Setting $t = \frac{1}{a}$ into (3.4) we get $\varphi(a\varphi(1/a)) = a$. Since $\varphi(a) = a$ and φ is one-to-one, it follows that $a\varphi(1/a) = a$, whence $\varphi(1/a) = 1$. Now the relations $\varphi(1) = \frac{1}{a}$, $\varphi(1/a) = 1$ and the increasing monotonicity of φ imply that $a = 1$.

Since φ is strictly increasing and $(0,\infty) \ni t \to \varphi(t)/t$ is strictly decreasing, we have

$$\varphi(t) > t \text{ for all } t \in (0,1), \quad \varphi(t) < t \text{ for all } t > 1.$$

Hence

$$\frac{\varphi(t)}{t} > 1 \text{ for all } t \in (0,1).$$

Since $\varphi(t) < t$ for all $t > 1$, applying equation (3.4), we obtain

$$\frac{1}{t} = \varphi\left(\frac{\varphi(t)}{t}\right) < \frac{\varphi(t)}{t}, \quad t \in (0,1),$$

whence $\varphi(t) > 1$ for all $t \in (0,1)$. This contradiction completes the proof. □

Problem 3.11. Does there exists a (discontinuous) solution of (3.4)?

Remark 3.12. If $\varphi : (0,\infty) \to (0,\infty)$ satisfies equation (3.4) then, obviously, the number $\frac{1}{\varphi(1)}$ is the only fixed point of φ. It is an open question if it is necessary that $\varphi(1) = 1$. We show that there is no function $\varphi : (0,\infty) \to (0,\infty)$ satisfying (3.4), differentiable at 1 such that $\varphi(1) = 1$.

Proof. Assume, for the contrary, that φ is such a solution. Differentiating both sides of (3.4) at point 1 leads to $[\varphi'(1)]^2 - \varphi'(1) + 1 = 0$. □

Remark 3.13. In the first part of the proof of Proposition 3.9 we have applied Theorem 3.6 to show the implication

$$(\varphi^{-1})^* = (\varphi^*)^{-1} \Longrightarrow \varphi^{-1} = \varphi^*, \quad (\varphi \in \mathcal{W}).$$

A direct argument reads as follows:

If a function $\varphi \in \mathcal{W}$ satisfies (3.3), then

$$\varphi^* \left[(\varphi^{-1})^* (t) \right] = t, \quad t > 0,$$

and, by the definition of φ^*,

$$\left[(\varphi^{-1})^* (t) \right] \cdot \varphi \left(\frac{1}{(\varphi^{-1})^* (t)} \right) = t, \quad t > 0.$$

Applying again the definition of the operator $*$, we obtain

$$\left[t\varphi^{-1} \left(\frac{1}{t} \right) \right] \cdot \varphi \left(\frac{1}{t\varphi^{-1}(\frac{1}{t})} \right) = t, \qquad t > 0.$$

Dividing both sides by t and then replacing t by $1/t$ we obtain

$$\varphi^{-1}(t) \cdot \varphi \left(\frac{t}{\varphi^{-1}(t)} \right) = 1, \quad t > 0.$$

Replacing t by $\varphi(t)$ yields

$$t\varphi \left(\frac{\varphi(t)}{t} \right) = 1, \quad t > 0.$$

Replacing here t by $1/t$ gives

$$\varphi \left(t\varphi(\frac{1}{t}) \right) = t, \qquad t > 0,$$

that is,

$$\varphi(\varphi^*(t)) = t, \quad t > 0.$$

Thus we have shown that $\varphi^{-1} = \varphi^*$.

Remark 3.14. For a set $I \subset (0, \infty)$ and a function $h : I \to (0, \infty)$ put $I^* := \left\{ \frac{1}{t} : t \in I \right\}$ and define a conjugate function h^* ([22]) by

$$h^*(x) = xh\left(\frac{1}{x}\right), \quad x \in I^*.$$

It is an open question if Theorem 4 remains true on replacing $\mathcal{W}$ by the set of all one-to-one functions $\varphi : (0, \infty) \to (0, \infty)$ such that φ^* is one-to-one.

Remark 3.15. Define $V : \mathcal{W} \to \mathcal{W}$ by $V(\varphi) := \frac{1}{\varphi}$. Then, for all $\varphi \in \mathcal{W}$ and $t > 0$, we have

$$T \circ V(\varphi)(t) = \varphi^{-1}\Big(\frac{1}{t}\Big),\ V \circ T \circ V(\varphi)(t) = \frac{1}{\varphi^{-1}(\frac{1}{t})},\ T \circ V \circ T \circ V(\varphi)(t) = \frac{1}{\varphi(\frac{1}{t})},$$

$$V \circ T \circ V \circ T \circ V(\varphi)(t) = \varphi\Big(\frac{1}{t}\Big), \quad T \circ V \circ T \circ V \circ T \circ V(\varphi)(t) = \frac{1}{\varphi^{-1}(t)},$$

and

$$V \circ T \circ V \circ T \circ V \circ T \circ V(\varphi) = \varphi^{-1}, \quad T \circ V \circ T \circ V \circ T \circ V \circ T \circ V(\varphi) = \varphi.$$

This shows that $n = 4$ is the smallest possible positive integer such that $(T \circ V)^n = \mathrm{id}|_{\mathcal{W}}$.

A substitute of the concept of $(*)$-conjugacy defined on an arbitrary interval $I \subset (0, \infty)$ allows us to give a simple analytical proof that if $f : I \to \mathbb{R}$ is convex, then the function $I \ni x \longrightarrow \frac{f(x)}{x}$ is either monotonic or unimodal ([22]).

Let us mention that the converse of the Hölder inequality theorem remains true (cf. [14,16]) if the assumption (1.1) is replaced by the following one: the functions φ^{-1} and ψ^{-1} are *multiplicatively conjugate*, i.e. there exists a constant $c > 0$ such that

$$\varphi^{-1}(t)\,\psi^{-1}(t) = ct, \quad t > 0.$$

These facts raise some questions concerning a characterization of the concept of conjugacy.

4. CONJUGATE FUNCTIONS AND A GENERALIZED HÖLDER INEQUALITY

In this section we prove the following result.

Theorem 4.1. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets $A, B \in \Sigma$ of finite and positive measures such that*

$$\frac{\log \mu(B)}{\log \mu(A)} \quad \text{is irrational number.}$$

Suppose that $\varphi, \psi : (0, \infty) \to (0, \infty)$ are bijective, φ is continuous, and the functions φ^{-1} and ψ^{-1} are conjugate, i.e.

$$\psi^{-1} = (\varphi^{-1})^*. \tag{4.1}$$

Then

(i) *the inequality* (1.2)

$$\int_\Omega xy\,d\mu \le \mathbf{P}_\varphi(x)\mathbf{P}_\psi(y), \quad x, y \in S_+,$$

holds true if and only if there is $p > 1$ *such that*

$$\varphi(t) = \varphi(1) t^p, \quad \psi(t) = \psi(1) t^q, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1;$$

(ii) *the inequality*

$$\int_{\Omega} xyd\mu \geq \mathbf{P}_{\varphi}(x)\mathbf{P}_{\psi}(y), \quad x, y \in S_+,$$

holds true if and only if there is $p < 1$, $p \neq 0$, *such that*

$$\varphi(t) = \varphi(1) t^p, \quad \psi(t) = \psi(1) t^q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Put $a := \mu(A)$ and $b := \mu(B)$. By assumption we have $a, b \in \mathbb{R} \setminus \{0\}$ and $\frac{\log b}{\log a}$ is irrational. Taking arbitrary $x_1, y_1 > 0$ and setting $x := x_1 \chi_A$ and $y := y_1 \chi_A$ in (1.2), we get

$$ax_1y_1 \leq \varphi^{-1}(a\varphi(x_1)) \psi^{-1}(a\psi(y_1)), \quad x_1, y_1 > 0,$$

whence, replacing x_1 and y_1 by $\varphi^{-1}(x_1)$ and $\psi^{-1}(y_1)$, respectively,

$$a\varphi^{-1}(x_1) \psi^{-1}(y_1) \leq \varphi^{-1}(ax_1) \psi^{-1}(ay_1), \quad x_1, y_1 > 0.$$

Hence, from (4.1) and the definition of the conjugate function, we get

$$a\varphi^{-1}(x_1) y_1 \varphi^{-1}\left(\frac{1}{y_1}\right) \leq \varphi^{-1}(ax_1) ay_1 \varphi^{-1}\left(\frac{1}{ay_1}\right), \quad x_1, y_1 > 0,$$

that is,

$$\varphi^{-1}(x_1) \varphi^{-1}\left(\frac{1}{y_1}\right) \leq \varphi^{-1}(ax_1) \varphi^{-1}\left(\frac{1}{ay_1}\right), \quad x_1, y_1 > 0.$$

Putting

$$f := \varphi^{-1}$$

and replacing here y_1 by $\frac{1}{ay_1}$ we obtain

$$f(x_1) f(ay_1) \leq f(ax_1) f(y_1), \quad x_1, y_1 > 0,$$

or, equivalently,

$$\frac{f(ay_1)}{f(y_1)} \leq \frac{f(ax_1)}{f(x_1)}, \quad x_1, y_1 > 0.$$

Obviously it follows that there is a constant $\alpha > 0$ such that

$$\frac{f(ax_1)}{f(x_1)} = \alpha, \quad x_1 > 0,$$

which can be written in the following form

$$f(at) = \alpha f(t), \quad t > 0.$$

By obvious induction, we hence get

$$f\left(a^m t\right) = \alpha^m f\left(t\right), \quad m \in \mathbb{N},\ t > 0.$$

Replacing here t by t/a^m we get

$$f\left(a^{-m} t\right) = \alpha^{-m} f\left(t\right), \quad m \in \mathbb{N},\ t > 0.$$

It follows that

$$f\left(a^m t\right) = \alpha^m f\left(t\right), \quad m \in \mathbb{Z},\ t > 0,$$

where $\mathbb{Z}$ denotes the set of all integer numbers.

In the same way we get, for some constant $\beta > 0$,

$$f\left(b^n t\right) = \beta^n f\left(t\right), \quad n \in \mathbb{Z},\ t > 0.$$

From the last two equations we get

$$f\left(a^m b^n t\right) = \alpha^m \beta^n f\left(t\right), \qquad m, n \in \mathbb{Z},\ t > 0. \tag{4.2}$$

Since $\frac{\log b}{\log a}$ is irrational, by the Kronecker theorem, the set

$$D := \{a^m b^n : m, n \in \mathbb{Z}\}$$

is dense in $(0, \infty)$. It follows that there exist two sequences (m_k) and (n_k) of integers such that

$$\lim_{k\to\infty} a^{m_k} b^{n_k} = 1, \tag{4.3}$$

and, by the irrationality of $\frac{\log b}{\log a}$,

$$\lim_{k\to\infty} |m_k| = \lim_{k\to\infty} |n_k| = \infty.$$

The continuity of the function log at 1 and (4.3) imply that

$$\lim_{k\to\infty} \left(m_k \log a + n_k \log b\right) = 0.$$

If follows that

$$\lim_{k\to\infty} \left(\frac{m_k}{n_k} \log a + \log b\right) = \lim_{k\to\infty} \frac{m_k \log a + n_k \log b}{n_k} = 0,$$

whence

$$\lim_{k\to\infty} \frac{m_k}{n_k} = -\frac{\log b}{\log a}. \tag{4.4}$$

Applying (4.2), (4.3), and the continuity of f at point 1, we get

$$f\left(1\right) = \lim_{k\to\infty} f\left(a^{m_k} b^{n_k}\right) = f(1) \lim_{k\to\infty} \alpha^{m_k} \beta^{n_k},$$

and

$$\lim_{k\to\infty} \alpha^{m_k}\beta^{n_k} = 1.$$

The same reasoning as in the case of (4.4) shows that

$$\lim_{k\to\infty} \frac{m_k}{n_k} = -\frac{\log\beta}{\log\alpha}. \tag{4.5}$$

From (4.4) and (4.5) we get

$$\frac{\log\beta}{\log\alpha} = \frac{\log b}{\log a},$$

whence

$$\frac{\log\alpha}{\log a} = \frac{\log\beta}{\log b}.$$

Putting

$$\frac{1}{p} := \frac{\log\alpha}{\log a} = \frac{\log\beta}{\log b},$$

we have, of course, $p \neq 0$, and

$$\alpha = a^{1/p}, \quad \beta = b^{1/p}.$$

Hence, applying (4.2) with $t = 1$, we get

$$f(a^m b^n) = f(1)(a^m b^n)^{1/p}, \quad m, n \in \mathbb{Z}.$$

The continuity of f and the density of the set D imply that

$$f(t) = f(1)\, t^{1/p}, \quad t > 0,$$

that is,

$$\varphi^{-1}(t) = \varphi^{-1}(1)\, t^{1/p}, \quad t > 0.$$

From (4.1) and the definition of conjugate functions we obtain

$$\psi^{-1}(t) = t\varphi^{-1}(1)\left(\frac{1}{t}\right)^{1/p} = \varphi^{-1}(1)\, t^{1-\frac{1}{p}}, \quad t > 0.$$

Since φ and ψ are bijective, we have $1 - \frac{1}{p} \neq 0, 1$. Thus

$$\psi^{-1}(t) = \psi^{-1}(1)\, t^{1/q}, \quad \text{where} \quad \frac{1}{q} := 1 - \frac{1}{p},$$

which shows that φ^{-1} and ψ^{-1} are the conjugate power functions.

Hence, as for arbitrary $x_1, x_2, y_1, y_2 > 0$, the functions

$$x := x_1\chi_A + x_2\chi_B, \quad y := y_1\chi_A + y_2\chi_B$$

belong to S_+, setting them into (1.2) gives

$$x_1y_1 + x_2y_2 \le (x_1^p + x_2^p)^{1/p}(y_1^q + y_2^q)^{1/q}, \quad x_1, x_2, y_1, y_2 > 0,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, that is, the simplest version of the Hölder inequality. This implies that $p > 1$ and $q > 1$.

Since in the case of the reversed inequality the proof is analogous, we omit it. $\square$

5. CONJUGATE FUNCTIONS, SUBADDITIVITY AND SUBHOMOGENEITY

Let (Ω, Σ, μ) be a measure space. Suppose that $\varphi, \psi : (0, \infty) \to (0, \infty)$ are bijective, one of them is continuous, and the functions φ^{-1} and ψ^{-1} are conjugate, i.e.

$$\psi^{-1} = (\varphi^{-1})^*.$$

(Clearly, this equality and the continuity of φ or ψ imply that both these functions are continuous.) Is it then true that, for all $x, y \in S_+$,

$$\mathbf{P}_\varphi(x + y) \le \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad \mathbf{P}_\psi(x + y) \le \mathbf{P}_\psi(x) + \mathbf{P}_\psi(y)?$$

Note that in the case when the measure space satisfies the following condition: there are $A, B \in \Sigma$ such that (1.1) holds, the answer is affirmative. Indeed, in this case, according to the converse theorem for the Minkowski inequality (depending on the measure space, without any regularity conditions on φ and ψ [15]) applied separately for $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$, there are $p \ge 1$ and $q \ge 1$ such that $\varphi(t) = \varphi(1)t^p$, $\psi(t) = \psi(1)t^q$. The conjugacy condition implies that φ and ψ are the conjugate power functions. Moreover, in this case, we have

$$\mathbf{P}_\varphi(tx) = t\mathbf{P}_\varphi(x), \quad \mathbf{P}_\psi(tx) = t\mathbf{P}_\psi(x), \qquad t > 0, x \in S_+,$$

that is, the functionals $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are positively homogeneous.

Therefore to answer the question we have to consider the following two cases:

- the measure space (Ω, Σ, μ) is such that condition (2.5) is satisfied, that is, (Ω, Σ, μ) is a generalized counting measure space;
- the measure space (Ω, Σ, μ) is such that

$$A \in \Sigma \Longrightarrow \mu(A) = \infty \text{ or } \mu(A) \le 1.$$

Remark 5.1. The Minkowski type inequality

$$\mathbf{P}_\varphi(x + y) \le \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y), \quad x, y \in S_+,$$

that is, the subadditivity of $\mathbf{P}_\varphi$, implies that

$$\mathbf{P}_\varphi(x_1 + \ldots + x_n) \le \mathbf{P}_\varphi(x_1) + \ldots + \mathbf{P}_\varphi(x_n), \quad n \in \mathbb{N}, x_1, \ldots, x_n \in S_+,$$

whence

$$\mathbf{P}_\varphi(nx) \le n\mathbf{P}_\varphi(x), \quad n \in \mathbb{N}, x \in S_+,$$

which shows that $\mathbf{P}_\varphi$ is n-subhomogeneous for every positive integer n. In the sequel we shall be interested in the functionals $\mathbf{P}_\varphi$ that satisfy (2.3) and are subhomogeneous, that is, such that

$$\mathbf{P}_\varphi(tx) \le t\mathbf{P}_\varphi(x), \quad x, y \in S_+, \ t > 1. \tag{5.1}$$

In the case of a generalized counting measure space we have the following result.

Theorem 5.2. *Let the measure space* (Ω, Σ, μ) *be such that condition* (2.5) *is satisfied. Suppose that* $\varphi : (0, \infty) \to (0, \infty)$ *is bijective, increasing, convex, geometrically convex and satisfies* (3.1). *Then* $\mathbf{P}_\varphi$ *satisfies* (2.3) *and* (5.1).

Proof. Since $\varphi : (0, \infty) \to (0, \infty)$ is bijective, increasing, convex and geometrically convex, by Theorem 1 of [13], the functional $\mathbf{P}_\varphi$ satisfies (2.3).

To show the subhomogeneity of $\mathbf{P}_\varphi$ take an arbitrary $x \in S_+$, such that $\mu(\Omega(x)) > 0$. Then there exists an $n \in \mathbb{N}$, the pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$ of positive measure and $x_1, \ldots, x_n > 0$ such that

$$x = \sum_{j=1}^{n} x_j \chi_{A_j}.$$

Put

$$a_j := \mu(A_j), \quad j = 1, \ldots, n.$$

According to the assumption on the measure space,

$$a_j \geq 1, \qquad j = 1, \ldots, n.$$

By Lemmas 2.6 and 2.7, for every $t > 1$, the function $\varphi \circ (t\varphi^{-1})$ is superadditive. Therefore, for all $t > 1$,

$$\sum_{j=1}^{n} \varphi(t\varphi^{-1}(a_j x_j)) \leq \varphi\left(t\varphi^{-1}\left(\sum_{j=1}^{n} a_j x_j\right)\right).$$

Since, by Lemma 2.6, the function

$$(0, \infty) \ni r \longmapsto \frac{\varphi\left(t\varphi^{-1}(r)\right)}{r}$$

is increasing and $a_j \geq 1, j = 1, \ldots, n$, we have, for all $t > 1$,

$$\frac{\varphi\left(t\varphi^{-1}(x_j)\right)}{x_j} \leq \frac{\varphi\left(t\varphi^{-1}(a_j x_j)\right)}{a_j x_j}, \qquad j = 1, \ldots, n,$$

whence

$$a_j \varphi\left(t\varphi^{-1}(x_j)\right) \leq \varphi\left(t\varphi^{-1}(a_j x_j)\right), \qquad j = 1, \ldots, n.$$

Adding these inequalities by sides we get

$$\sum_{j=1}^{n} a_j \varphi\left(t\varphi^{-1}(x_j)\right) \leq \sum_{j=1}^{n} \varphi\left(t\varphi^{-1}(a_j x_j)\right),$$

and, consequently, for all $t > 1$,

$$\sum_{j=1}^{n} a_j \varphi\left(t\varphi^{-1}(x_j)\right) \leq \varphi\left(t\varphi^{-1}\left(\sum_{j=1}^{n} a_j x_j\right)\right).$$

Since for all positive reals $y_1, \dots, y_n$ the function $\sum_{j=1}^{n} y_j \chi_{A_j}$ is in S_+, we may replace here x_j by $\varphi(x_j)$ and making use of the increasing monotonicity of φ^{-1}, we hence get

$$\varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(tx_j)\right) \leq t\varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(x_j)\right),$$

which, by the definition of the functional $\mathbf{P}_\varphi$, means that $\mathbf{P}_\varphi(tx) \leq t\mathbf{P}_\varphi(x)$ for all $t > 1$ and $x \in S$. This completes the proof. □

Taking

$$\varphi(t) = (t + c)^q t^p, \quad t > 0,$$

where $c > 0$, $p > 1$ and $q > 1$ are arbitrarily fixed (cf. Remark 2.4), and applying this result, we obtain a broad class of pairs of nonpower functions φ and ψ such that $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are subhomogeneous and subadditive.

Theorem 5.3. *Let a measure space (Ω, Σ, μ) be such that $(1, \delta) \subset \operatorname{cl} \mu(\Sigma)$ for some $\delta > 1$ and condition (2.5) is satisfied. Suppose that a bijective function $\varphi : (0, \infty) \to (0, \infty)$ satisfies (3.1), $\varphi^{-1}(0+) = 0$. Then $\mathbf{P}_\varphi$ is subadditive and subhomogeneous in S_+ if and only if φ is increasing, convex and geometrically convex.*

Proof. Assume that $\mathbf{P}_\varphi$ is subadditive. Take arbitrary $A, B \in \Sigma$, $A \cap B = \emptyset$ of positive and finite measure. Setting $x = \varphi^{-1}\left(\frac{s}{\mu(A)}\right)$, $y = \varphi^{-1}\left(\frac{t}{\mu(B)}\right)$ in (2.3) we get

$$\varphi^{-1}(s + t) \leq \varphi^{-1}(s) + \varphi^{-1}(t), \quad s, t > 0.$$

Since $\varphi^{-1}(0+) = 0$, applying the main result of [24], we infer that φ is an increasing homeomorphism of $(0, \infty)$.

The assumption $(1, \delta) \subset \operatorname{cl} \mu(\Sigma)$ implies the existence of sets $A_n \in \Sigma$, $n \in \mathbb{N}$, and $B \in \Sigma$ such $A_n \cap B = \emptyset$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \mu(A_n) = 1$ and $0 < \mu(B) < \infty$. Put $a_n := \mu(A_n)$ for $n \in \mathbf{N}$ and $b := \mu(B)$. Setting $x := x_1\chi_{A_n} + x_2\chi_B$, $y := y_1\chi_{A_n} + y_2\chi_B$ in (2.3) we get

$$\begin{aligned}&\varphi^{-1}(a_n\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) \leq \\ &\leq \varphi^{-1}(a_n\varphi(x_1) + b\varphi(x_2)) + \varphi^{-1}(a_n\varphi(y_1) + b\varphi(y_2))\end{aligned}$$

for all $x_1, x_2, y_1, y_2 > 0$.

Making use of the just proved continuity of φ and φ^{-1}, and letting $n \to \infty$, we hence get, for all $x_1, x_2, y_1, y_2 > 0$,

$$\begin{aligned}&\varphi^{-1}(\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) \leq \\ &\leq \varphi^{-1}(\varphi(x_1) + b\varphi(x_2)) + \varphi^{-1}(\varphi(y_1) + b\varphi(y_2)),\end{aligned}$$

and, to get convexity of φ, we can argue as in the proof of the second part of Lemma 2.8 (or apply Lemma 2.8 (ii) to a simple measure space). Now the "only if" part is a consequence of Lemma 2.11.

The "if" part follows from the previous Theorem 5.2. □

In connection with this result we propose the following problem.

Problem 5.4. Suppose that $\varphi : (0,\infty) \to (0,\infty)$ is bijective, φ^{-1} and $(\varphi^{-1})^*$ are subadditive and condition (3.1) is satisfied. Is it then true that

$$\varphi^{-1}(0+) = 0?$$

Now we consider the problem in the case when (Ω, Σ, μ) is a defected probability space.

We shall need the following theorem.

Theorem 5.5. *Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) = 1$ and there is $A \in \Sigma$ such that $0 < \mu(A) < 1$. Suppose that $\varphi : (0,\infty) \to (0,\infty)$ is bijective and monotonic. Then inequality (2.3) holds true if and only if φ is increasing and the function $\Phi : (0,\infty)^2 \to (0,\infty)$ defined by*

$$\Phi(s,t) := \varphi\left(\varphi^{-1}(s) + \varphi^{-1}(t)\right), \quad s,t > 0, \tag{5.2}$$

is concave.

Proof. Suppose that φ satisfies (2.3). By Lemma 2.8, the function φ is increasing and, consequently, φ is an increasing homeomorphism of $(0,\infty)$. Now the result follows from the result in [3, p. 85–88] (see also [8]). $\square$

Remark 5.6. If the measure space (Ω, Σ, μ) is such that for every $A \in \Sigma$, we have $\mu(A) \le 1$ or $\mu(A) = \infty$, and $\varphi : (0,\infty) \to (0,\infty)$ is increasing, bijective such that the function Φ defined by (5.2) is concave, then inequality (2.3) is satisfied.

Indeed, from the concavity of function (5.2),

$$\sum_{j=1}^{n} a_j \varphi\left(\varphi^{-1}(x_j) + \varphi^{-1}(y_j)\right) \le \varphi\left(\varphi^{-1}\left(\sum_{j=1}^{n} a_j x_j\right) + \varphi^{-1}\left(\sum_{j=1}^{n} a_j y_j\right)\right)$$

for all $n \in \mathbb{N}$, $a_j, x_j, y_j > 0$, $j = 1, \ldots, n$, such that $\sum_{j=1}^{n} a_j = 1$. Obviously, this inequality remains true if $\sum_{j=1}^{n} a_j \le 1$. Replacing here x_j by $\varphi(x_j)$, y_j by $\varphi(y_j)$, $j = 1, \ldots, n$, and making use of the monotonicity of φ, we obtain

$$\varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(x_j + y_j)\right) \le \varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(x_j)\right) + \varphi^{-1}\left(\sum_{j=1}^{n} a_j \varphi(y_j)\right)$$

for all $n \in \mathbb{N}$, $a_j, x_j, y_j > 0$, $j = 1, \ldots, n$, such that $\sum_{j=1}^{n} a_j \le 1$.

Take $x, y \in S_+$. Then there exist $n \in \mathbb{N}$, the pairwise disjoint sets $A_j \in \Sigma$ of positive measure and $x_j, y_j > 0$, $j = 1, \ldots, n$, such that

$$x = \sum_{j=1}^{n} x_j \chi_{A_j}, \quad y = \sum_{j=1}^{n} y_j \chi_{A_j}.$$

Putting $a_j := \mu(A_j)$ for $j = 1, \ldots, n$, we have $\sum_{j=1}^{n} a_j \le 1$, and the last inequality says that $\mathbf{P}_\varphi(x+y) \le \mathbf{P}_\varphi(x) + \mathbf{P}_\varphi(y)$.

The following lemma will be helpful in establishing a criterion for the concavity of the function Φ defined by (5.2).

Lemma 5.7. *Let $\varphi : (0,\infty) \to (0,\infty)$ is bijective, twice differentiable, $\varphi' > 0$ and $\varphi'' > 0$ in $(0,\infty)$. Then the function $\Phi : (0,\infty)^2 \to (0,\infty)$ defined by (5.2) is concave, if and only if the function $\frac{\varphi''}{\varphi'}$ is decreasing, and $\frac{\varphi'}{\varphi''}$ is superadditive, i.e.*

$$\frac{\varphi'(s+t)}{\varphi''(s+t)} \geq \frac{\varphi'(s)}{\varphi''(s)} + \frac{\varphi'(t)}{\varphi''(t)}, \qquad s,t>0. \tag{5.3}$$

Proof. For arbitrarily fixed $s,t>0$ and all admissible $u,v\in\mathbb{R}$ define the function of a single variable

$$F(\tau) := \Phi(s+u\tau, t+v\tau) = \varphi\left(\varphi^{-1}(s+u\tau)+\varphi^{-1}(t+v\tau)\right).$$

The function Φ is concave if and only if for all $s,t>0$ and all admissible $u,v\in\mathbb{R}$ we have $F''(0)\leq 0$ (cf. [3, p. 79–81]). Put $f := \varphi^{-1}$. From the definition of F we have

$$f(F(\tau)) = f(s+u\tau) + f(t+v\tau)$$

for τ in a neighborhood of 0. Hence we obtain

$$f'(F(0))\,F'(0) = f'(s)u + f'(t)v$$

and

$$f''(F(0))\,[F'(0)]^2 + f'(F(0))\,F''(0) = f''(s)u^2 + f''(t)v^2.$$

Eliminating $F'(0)$ from these two equalities, we get

$$[f'(F(0))]^3\,F''(0) = Au^2 - 2Buv + Cv^2, \tag{5.4}$$

where

$$\begin{gathered} A := f''(s)\,[f'(F(0))]^2 - f''(F(0))\,[f'(s)]^2, \\ B := f''(F(0))\,f'(s)\,f'(t), \\ C := f''(t)\,[f'(F(0))]^2 - f''(F(0))\,[f'(t)]^2. \end{gathered}$$

Since φ' is positive, we have $f'(F(0))>0$. From (5.4) we conclude that $F''(0)\leq 0$ if and only if

$$A\leq 0, \quad C\leq 0, \quad AC\geq B^2.$$

Note that

$$F(0) = \varphi\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right), \quad f'\circ\varphi = \frac{1}{\varphi'}, \quad f''\circ\varphi = -\frac{\varphi''}{(\varphi')^3}.$$

It follows that, for all $s,t>0$,

$$f'(F(0)) = f'\left(\varphi\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right)\right) = \frac{1}{\varphi'\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right)},$$

$$f''(F(0)) = f''\left(\varphi\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right)\right) \quad = -\frac{\varphi''\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right)}{\left[\varphi'\left(\varphi^{-1}(s)+\varphi^{-1}(t)\right)\right]^3},$$

$$f'(t) = \frac{1}{\varphi'\left(\varphi^{-1}(t)\right)}, \quad f''(t) = -\frac{\varphi''\left(\varphi^{-1}(t)\right)}{\left[\varphi'\left(\left(\varphi^{-1}(t)\right)\right)\right]^3}.$$

Hence, by the definitions of the functions A, B, C, and replacing, for the simplicity of notations, s by $\varphi(s)$, t by $\varphi(t)$, we obtain

$$A \le 0 \Longleftrightarrow \frac{\varphi''(s+t)}{\varphi'(s+t)} \le \frac{\varphi''(s)}{\varphi'(s)}, \quad C \le 0 \Longleftrightarrow \frac{\varphi''(s+t)}{\varphi'(s+t)} \le \frac{\varphi''(t)}{\varphi'(t)},$$

that is, $A \le 0$ and $C \le 0$ if and only if $\frac{\varphi''}{\varphi'}$ is decreasing, and

$$AC \ge B^2 \Longleftrightarrow \frac{\varphi''(s+t)}{[\varphi'(s+t)]^3}\frac{\varphi''(s)}{[\varphi'(s)]^3}\frac{\varphi''(t)}{[\varphi'(t)]^3}\left(\frac{\varphi'(s+t)}{\varphi''(s+t)} - \frac{\varphi'(s)}{\varphi''(s)} - \frac{\varphi'(t)}{\varphi''(t)}\right) \ge 0$$

for all $s, t > 0$. □

From this lemma we obtain the following proposition (cf. [8]).

Proposition 5.8. *Let $\varphi : (0,\infty) \to (0,\infty)$ be bijective, twice differentiable with positive first and second derivatives in $(0,\infty)$. If $\frac{\varphi'}{\varphi''}$ is superdadditive, i.e. inequality* (5.3) *is satisfied, then Φ defined by* (5.2) *is concave.*

Example 5.9. Assume that a measure space (Ω, Σ, μ) is such that $\mu(\Omega) = 1$ and there is $A \in \Sigma$ *such that* $0 < \mu(A) < 1$. Consider the function $\varphi : (0,\infty) \to (0,\infty)$ defined by

$$\varphi(t) := \frac{t^3}{t+1}, \quad t > 0.$$

It is easy to see that φ is bijective and satisfies (3.1). Making simple calculations we have

$$\varphi'(t) = \frac{t^2(2t+3)}{(t+1)^2}, \quad \varphi''(t) = \frac{2t\left(t^2+3t+3\right)}{(t+1)^3}, \quad \frac{\varphi'(t)}{\varphi''(t)} = \frac{t(t+1)(2t+3)}{2(t^2+3t+3)}$$

for $t > 0$. Thus φ' and φ'' are positive and, as the function

$$(0,\infty) \ni t \longmapsto \frac{\frac{\varphi'}{\varphi''}(t)}{t} = \frac{(t+1)(2t+3)}{2(t^2+3t+3)}$$

is increasing, by Lemma 2.7, the function $\frac{\varphi'}{\varphi''}$ is superadditive (that is, inequality (5.3) is satisfied). Thus, in view of Proposition 2 and Theorem 8, the functional $\mathbf{P}_\varphi$ is subadditive. Since

$$\frac{\varphi''(t)}{\varphi'(t)}t = 1 + \frac{t+3}{2t^2+5t+3}, \quad t > 0,$$

the function φ' is geometrically concave (cf. Remark 2.2). In view of Lemma 2.9, the functional $\mathbf{P}_\varphi$ is also subhomogeneous.

From the definition of φ we have

$$t = \frac{\left[\varphi^{-1}(t)\right]^3}{\varphi^{-1}(t)+1}, \quad t>0,$$

whence

$$\frac{1}{t} = \frac{\left[\varphi^{-1}(\frac{1}{t})\right]^3}{\varphi^{-1}(\frac{1}{t})+1}, \quad t>0,$$

or, equivalently,

$$\frac{1}{t} = \frac{\frac{1}{t^3}\left[t\varphi^{-1}\left(\frac{1}{t}\right)\right]^3}{\frac{1}{t}\left[t\varphi^{-1}\left(\frac{1}{t}\right)\right]+1}, \quad t>0.$$

By the definition of $(\varphi^{-1})^*$, we hence get

$$\frac{1}{t} = \frac{\frac{1}{t^3}\left[(\varphi^{-1})^*(t)\right]^3}{\frac{1}{t}\left[(\varphi^{-1})^*(t)\right]+1}, \quad t>0,$$

that is,

$$\left[(\varphi^{-1})^*(t)\right]^3 - t(\varphi^{-1})^*(t) - t^2 = 0, \quad t>0.$$

Putting here $\psi := \left((\varphi^{-1})^*\right)^{-1}$ we get

$$\left[\psi^{-1}(t)\right]^3 - t\psi^{-1}(t) - t^2 = 0, \quad t>0,$$

whence, replacing t by $\psi(t)$, we obtain

$$t^3 - \psi(t)\,t - [\psi(t)]^2 = 0, \quad t>0.$$

It follows that, for every $t>0$, either $\psi(t) = \frac{1}{2}t\left(\sqrt{4t+1}-1\right)$ or $\psi(t) = -\frac{1}{2}t\left(\sqrt{4t+1}+1\right)$. Since ψ is continuous and increasing, we conclude that

$$\psi(t) = \frac{1}{2}t\left(\sqrt{4t+1}-1\right), \quad t>0.$$

Of course, $\psi:(0,\infty)\to(0,\infty)$ is bijective, twice differentiable and, it is easy to see that, ψ' and ψ'' are positive (the bijectivity, increasing monotonicity and convexity of ψ follow from Property 5.a). Moreover, we have

$$\frac{\psi'(t)}{\psi''(t)} = \frac{(4t+1)\left(6t+1-\sqrt{4t+1}\right)}{4(3t+1)}, \quad t>0.$$

Since

$$\left(\frac{\frac{\psi'}{\psi''}(t)}{t}\right)' = \frac{\sqrt{4t+1}\left(6t^2+4t+1\right)-6t^2-6t-1}{4t^2(3t+1)^2} > 0, \quad t>0, \tag{5.5}$$

the function

$$(0,\infty)\ni t\longmapsto \frac{\frac{\psi'}{\psi''}(t)}{t}=\frac{(4t+1)\left(6t+1-\sqrt{4t+1}\right)}{4t\,(3t+1)}$$

is increasing. Hence, by Lemma 2.7, the function $\frac{\psi'}{\psi''}$ is superadditive. Thus, in view of Proposition 5.8 and Theorem 5.5, the functional $\mathbf{P}_\psi$ is subadditive.

Since according to (5.5)

$$\left(\frac{\psi''(t)}{\psi'(t)}t\right)'<0,\qquad t>0,$$

it follows from Remark 2.2 that the derivative ψ' is geometrically concave. In view of Lemma 2.9, the functional $\mathbf{P}_\psi$ is subhomogeneous.

This example proves the following proposition.

Proposition 5.10. *If the measure space* (Ω,Σ,μ) *is such that* $\mu(\Omega)=1$ *and* $0<\mu(A)<1$ *for some* $A\in\Sigma$, *then there are conjugate nonpower bijections* φ^{-1},ψ^{-1} *of* $(0,\infty)$ *such that* $\mathbf{P}_\varphi$ *and* $\mathbf{P}_\psi$ *are subadditive and subhomogeneous.*

This proposition and, first of all, our consideration related to Example 5.9 raise the following problem.

Problem 5.11. Let $\varphi:(0,\infty)\to(0,\infty)$ be bijective, twice differentiable with positive first and second derivatives such that conditions (3.1) are satisfied. Then, in view of Property 5a and Remark 5.13, the function $\psi=\left((\varphi^{-1})^*\right)^{-1}$ has the same properties. Suppose that $\frac{\varphi'}{\varphi''}$ is superdadditive. Is then $\frac{\psi'}{\psi''}$ superdadditive?

In connection with this problem we prove the following remark.

Remark 5.12. Let a bijection $\varphi:(0,\infty)\to(0,\infty)$ satisfying (3.1) be twice differentiable with $\varphi'>0$, $\varphi''>0$. Putting $\psi=\left((\varphi^{-1})^*\right)^{-1}$ we have $(\varphi^{-1})^*\left((0,\infty)\right)=(0,\infty)$ and

$$\frac{\varphi'\left(\frac{t}{\psi(t)}\right)}{\varphi''\left(\frac{t}{\psi(t)}\right)}\frac{[\psi(t)]^3}{[\psi(t)-t\psi'(t)]^2}=\frac{\psi'(t)}{\psi''(t)},\quad t>0.$$

Proof. We have $\psi^{-1}=(\varphi^{-1})^*$, that is,

$$\psi^{-1}(t)=t\varphi^{-1}\left(\frac{1}{t}\right),\qquad t>0.$$

Since

$$\varphi\left(\frac{t}{\psi(t)}\right)=\frac{1}{\psi(t)},\quad t>0,$$

differentiating the both sides gives

$$\varphi'\left(\frac{t}{\psi(t)}\right)\frac{\psi(t)-t\psi'(t)}{[\psi(t)]^2}=-\frac{\psi'(t)}{[\psi(t)]^2},\quad t>0,$$

that is,

$$\varphi'\left(\frac{t}{\psi(t)}\right)(\psi(t)-t\psi'(t))=-\psi'(t), \quad t>0, \tag{5.6}$$

whence

$$\varphi'\left(\frac{t}{\psi(t)}\right)\psi(t)=\psi'(t)\left(t\varphi'\left(\frac{t}{\psi(t)}\right)-1\right), \quad t>0. \tag{5.7}$$

Differentiating both sides of (5.6) we get

$$\varphi''\left(\frac{t}{\psi(t)}\right)\frac{[\psi(t)-t\psi'(t)]^2}{[\psi(t)]^2}-t\varphi'\left(\frac{t}{\psi(t)}\right)\psi''(t)=-\psi''(t), \quad t>0,$$

whence

$$\varphi''\left(\frac{t}{\psi(t)}\right)\frac{[\psi(t)-t\psi'(t)]^2}{[\psi(t)]^2}=\psi''(t)\left(t\varphi'\left(\frac{t}{\psi(t)}\right)-1\right), \quad t>0.$$

Dividing the respective sides of (5.7) and this equation we get

$$\frac{\varphi'\left(\frac{t}{\psi(t)}\right)}{\varphi''\left(\frac{t}{\psi(t)}\right)}\frac{[\psi(t)]^3}{[\psi(t)-t\psi'(t)]^2}=\frac{\psi'(t)}{\psi''(t)}, \quad t>0.$$

$\square$

Remark 5.13. From the equality $\psi^{-1}=(\varphi^{-1})^*$ we easily get

$$\frac{1}{\psi'\left(\psi^{-1}(t)\right)}=\varphi^{-1}\Big(\frac{1}{t}\Big)-\frac{1}{t}\frac{1}{\varphi'\left(\varphi^{-1}(\frac{1}{t})\right)}, \quad t>0,$$

and

$$\frac{\psi''\left(\psi^{-1}(t)\right)}{\left[\psi'\left(\psi^{-1}(t)\right)\right]^3}=\frac{1}{t^3}\frac{\varphi''\left(\varphi^{-1}\left(\frac{1}{t}\right)\right)}{\left[\varphi'\left(\varphi^{-1}\left(\frac{1}{t}\right)\right)\right]^3}, \quad t>0.$$

Since

$$\varphi^{-1}\left(\frac{1}{\psi(t)}\right)=\frac{t}{\psi(t)}, \quad t>0,$$

replacing here t by $\psi(t)$, we get

$$\frac{\psi''(t)}{[\psi'(t)]^3}=\frac{1}{[\psi(t)]^3}\frac{\varphi''\left(\frac{t}{\psi(t)}\right)}{\left[\varphi'\left(\frac{t}{\psi(t)}\right)\right]^3}, \quad t>0.$$

Problem 5.14. Let $\varphi:(0,\infty)\to(0,\infty)$ be bijective, increasing and such that (3.1) is fulfilled. Suppose that the function $\Phi:(0,\infty)^2\to(0,\infty)$ defined by (5.2) is concave. For $\psi:=\left((\varphi^{-1})^*\right)^{-1}$ define $\Psi:(0,\infty)^2\to(0,\infty)$ by

$$\Psi(s,t):=\psi\left(\psi^{-1}(s)+\psi^{-1}(t)\right), \quad s,t>0.$$

Is the function Ψ concave?

6. YOUNG CONJUGATE FUNCTIONS

Let us quote (cf. [28, 29]) some well known properties of Young conjugate functions:

Lemma 6.1. *Suppose that $\varphi : (0,\infty) \to (0,\infty)$ is continuously differentiable with strictly increasing derivative and conditions* (3.1) *are fulfilled. Then:*

(i) *φ and φ' are strictly increasing mapping $(0,\infty)$ onto $(0,\infty)$;*
(ii) *the function $\psi : (0,\infty) \to (0,\infty)$, Young conjugate (complementary) function to φ, defined by*

$$\psi(t) := \int_0^t (\varphi')^{-1}(s)\,ds, \quad t > 0,$$

(where $(\varphi')^{-1}$ is the inverse function of φ') is continuously differentiable, ψ' is strictly increasing and

$$\lim_{t\to 0+} \frac{\psi(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\psi(t)}{t} = \infty;$$

(iii) $\psi' = (\varphi')^{-1}$;
(iv) *for every $t > 0$,*

$$\psi(t) = \sup\{st - \varphi(s) : s > 0\}, \quad \varphi(t) = \sup\{st - \psi(s) : s > 0\};$$

(v) *(the Young inequality) for all $u, v > 0$,*

$$uv \le \varphi(u) + \psi(v);$$

(vi) *if φ and ψ are Young conjugate and one of them is a power function, then there is $p > 1$ such that*

$$\varphi(t) = \varphi(1)\,t^p, \quad \psi(t) = \psi(1)\,t^q, \qquad t > 0, \tag{6.1}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{6.2}$$

We shall prove the following theorem.

Theorem 6.2. *Suppose that $\varphi : (0,\infty) \to (0,\infty)$ is differentiable with strictly increasing derivative and conditions* (3.1) *are fulfilled. If φ is geometrically convex (resp. geometrically concave), then its Young complementary function is geometrically concave (resp. geometrical convex).*

Proof. The Darboux property of the derivative implies that φ is continuously differentiable. Let $\psi : (0,\infty) \to (0,\infty)$ be the Young complementary function for φ, i.e.

$$\psi(t) = \sup\{st - \varphi(s) : s > 0\}, \quad t > 0.$$

Hence, by the differentiability and convexity of φ, we get

$$\psi\left(\varphi'\left(t\right)\right) = t\varphi'\left(t\right) - \varphi\left(t\right), \quad t > 0. \tag{6.3}$$

By Lemma 6.1 (iii), we have

$$\psi'\left(\varphi'\left(t\right)\right) = t, \quad t \in (0, \infty).$$

Hence, making use of (6.3), we get

$$\frac{\psi'\left(\varphi'\left(t\right)\right)}{\psi\left(\varphi'\left(t\right)\right)}\varphi'\left(t\right) = \frac{t\varphi'\left(t\right)}{t\varphi'\left(t\right) - \varphi\left(t\right)}, \quad t > 0. \tag{6.4}$$

Now suppose that φ is geometrically convex. Then (cf. Remark 2.2) the function

$$(0, \infty) \ni t \longmapsto \frac{t\varphi'\left(t\right)}{\varphi\left(t\right)} \quad \text{is increasing.}$$

It follows that

$$(0, \infty) \ni t \longmapsto \frac{t\varphi'\left(t\right) - \varphi(t)}{t\varphi'\left(t\right)} = 1 - \frac{\varphi\left(t\right)}{t\varphi'\left(t\right)} \quad \text{is increasing,}$$

and, consequently, the function

$$(0, \infty) \ni t \longmapsto \frac{t\varphi'\left(t\right)}{t\varphi'\left(t\right) - \varphi\left(t\right)} \quad \text{is decreasing.}$$

Since φ' is increasing, from (6.4) we conclude that the function

$$(0, \infty) \ni t \longmapsto \frac{t\psi'\left(t\right)}{\psi\left(t\right)} \quad \text{is decreasing,}$$

which proves that ψ is geometrically concave. □

Corollary 6.3. *Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is bijective, increasing, differentiable, strictly convex and conditions* (3.1) *are fulfilled. If φ and its Young complementary function ψ are geometrically convex, then there exists $p > 1$ such that* (6.1) *and* (6.2) *hold true.*

Proof. By assumption, the function ψ is geometrically convex and, by Theorem 6.2, it is geometrically concave. Thus ψ is geometrically affine and, in view of Remark 2.2, it is a power function. Now the result follows from Lemma 6.1. □

Theorem 6.4. *Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is twice differentiable, strictly convex and conditions* (3.1) *are fulfilled. If φ' is geometrically convex* (*resp. geometrically concave*)*, and ψ is the Young complementary function for φ, then ψ' is geometrically concave* (*resp. geometrically convex*)*.*

Proof. Assume that φ' is geometrically convex. As in the proof of Theorem 6.2, we have $\psi'(\varphi'(t)) = t$ for $t \in (0,\infty)$, whence

$$\psi''(\varphi'(t))\,\varphi''(t) = 1, \quad t > 0.$$

These relations imply that

$$\frac{\psi''(\varphi'(t))}{\psi'(\varphi'(t))}\varphi'(t) = \left(\frac{\varphi''(t)}{\varphi'(t)}t\right)^{-1}, \quad t > 0.$$

Since φ' is strictly increasing and, by geometrical convexity of φ', the function

$$(0,\infty) \ni t \longmapsto \frac{\varphi''(t)}{\varphi'(t)}t \quad \text{is increasing,}$$

we infer that the function

$$(0,\infty) \ni t \longmapsto \frac{\psi''(t)}{\psi'(t)}t \quad \text{is decreasing,}$$

and, consequently, ψ' is geometrically concave. This completes the proof. □

Theorem 6.5. *Let (Ω, Σ, μ) be a measure space such that*

$$A \in \Sigma \ \Rightarrow \mu(A) \le 1 \quad \textit{or} \quad \mu(A) = \infty,$$

and suppose that there are two sets $B, C \in \Sigma$ such that

$$0 < \mu(B) < \mu(C) = 1.$$

Suppose that $\varphi : (0,\infty) \to (0,\infty)$ is twice differentiable with strictly increasing derivative, conditions (3.1) *are fulfilled, and ψ is the Young conjugate function to φ. If $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are subhomogeneous, then φ and ψ are the conjugate power functions.*

Proof. In view of Lemma 2.9, Lemma 6.1, the subhomogeneity of $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$implies that φ' and ψ' are geometrically concave. By Theorem 10 the function ψ' is geometrically convex. Thus ψ' is geometrically affine and, by Remark 2.2 it is a power function. Now Lemma 6.1 implies that both φ and ψ are conjugate power functions. □

Similarly, applying Lemma 2.11 and Theorem 6.2, we obtain the following result.

Theorem 6.6. *Let (Ω, Σ, μ) be a measure space such that* (2.5) *holds and, for some $\delta > 1$,*

$$(1, \delta) \subset \mathrm{cl}\,(\mu(\Sigma)).$$

Suppose that $\varphi : (0,\infty) \to (0,\infty)$ is strictly increasing, differentiable, φ' is strictly increasing, (3.1) *holds and ψ is the Young conjugate to φ. If $\mathbf{P}_\varphi$ and $\mathbf{P}_\psi$ are subhomogeneous, then φ and ψ are the conjugate power functions.*

Acknowledgments

The author would to thank an anonymous referee for very careful reading the manuscript and several constructive remarks.

REFERENCES

[1] G. Aumann, *Konvexe Funktionen und Induktion bei Ungleichungen zwischen Mittelwerten*, S.-B. Math.-Naturw. Abt. Bayer. Akad. Wiss. München, (1933), 405–413.

[2] Z. Dároczy, Zs. Páles, *Convexity with given infinite weight sequences*, Stochastica **11** (1987), 5–12.

[3] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 2nd ed., 1952.

[4] E. Hille, R.S. Phillips, *Functional Analysis and Semigrops*, American Mathematical Society Publications 31, A.M.S., Providence, R.I., 1957.

[5] W. Jarczyk, J. Matkowski, *On Mulholland's inequality*, Proc. Amer. Math. Soc. **130** (2002), 3243–3247.

[6] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, P.W.N, Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.

[7] J. Matkowski, *On a characterization of L^p-norm*, Ann. Polon. Math. **50** (1989), 137–144.

[8] J. Matkowski, *The converse of the Minkowski's inequality theorem and its generalization*, Proc. Amer. Math. Soc. **109** (1990), 663–675.

[9] J. Matkowski, *A functional inequality characterizing convex functions, conjugacy and a generalization of Hölder's and Minkowski's inequalities*, Aequationes Math. **40** (1990), 168–180.

[10] J. Matkowski, *A generalization of Hölder's and Minkowski's inequalities and conjugate fubctions*, Constantin Carathéodory: An International Tribute, Th.M. Rassias (Ed.), 1991 World Scientific Publ. Co., 819-827.

[11] J. Matkowski, *Functional inequality characterizing concave functions in* $(0,\infty)^k$, Aequationes Math. **43** (1992), 219–224.

[12] J. Matkowski, *L^p-like paranorms, in Selected Topics in Functional Equations and Iteration Theory*, Proceedings of the Austrian-Polish Seminar, Universität Graz, October 24–26, 1991, D. Gronau and L. Reich (Eds.), Grazer Math. Ber. **316** (1992), 103–138.

[13] J. Matkowski, *On a generalization of Mulholland's inequality*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 97–103.

[14] J. Matkowski, *The converse of the Hölder inequality and its generalizations*, Studia Math. **109** (1994) 2, 171–182.

[15] J. Matkowski, *On a characterization of L^p-norm and a converse of Minkowski's inequality*, Hiroshima Math. J. **26** (1996), 277–287.

[16] J. Matkowski, *The converse of a generalized Hölder inequality*, Publicationes Math. Debrecen **50** (1997), 135–143.

[17] J. Matkowski, *The converse theorem for Minkowski's inequality*, Indag. Math. (N.S.) **15** (2004) 1, 73–84.

[18] J. Matkowski, *The converse theorem for the Minkowski inequality*, J. Math. Anal. Appl. **348** (2008), 315–323.

[19] J. Matkowski, *A converse of the Hölder inequality theorem*, Math. Inequal. Appl. **12** (2009) 1, 21–32.

[20] J. Matkowski, *Subadditive periodic functions*, Opuscula Math. **31** (2011) 1, 75–96.

[21] J. Matkowski, *The Pexider type generalization of the Minkowski inequality*, J. Math. Anal. Appl. **393** (2012), 298–310.

[22] J. Matkowski, *Conjugate functions and a short proof of a property of convex functions*, (to appear).

[23] J. Matkowski, *Subhomogeneity and subadditivity of the L^p-norm like functionals*, J. Math. Anal. Appl. **404** (2013), 172–184.

[24] J.Matkowski, T. Świątkowski, *Quasi-monotonicity, subadditive bijections of $\mathbb{R}_+$ and characterization of L^p-norm*, J. Math. Anal. Appl. **154** (1991), 493–506.

[25] H.P. Mulholland, *On generalizations of Minkowski's inequality in the form of a triangle inequality*, Proc. London Math. Soc. **51** (1950), 294–307.

[26] C.T. Ng, *Functions generating Schur-convex sums*, General Inequalities 5 (Oberwolfach, 1986), 433–438, Internat. Schriftenreihe Numer. Math., 80, Birkhäuser, Basel, 1987.

[27] M. Pycia, *Linear functional inequalities – a general theory and new special cases*, Dissertationes Math. **438** (2006), 62pp.

[28] M.M. Rao, Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker, Inc., New York-Basel--Hong Kong, 1991.

[29] A.W. Roberts, D.L. Varberg, *Convex Functions*, Academic Press, New York and London, 1973.

[30] R.A. Rosenbaum, *Sub-additive functions*, Duke Math. J. **17** (1950), 227–247.

Janusz Matkowski
J.Matkowski@wmie.uz.zgora.pl

University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
ul. Szafrana 4a, 65-516 Zielona Góra, Poland

Silesian University
Institute of Mathematics
ul. Bankowa 14, 40-007 Katowice, Poland

Received: March 11, 2013.
Revised: January 19, 2014.
Accepted: February 4, 2014.