Analysis, adaptive control and synchronization of a novel 4-D hyperchaotic hyperjerk system via backstepping control method

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A hyperjerk system is a dynamical system, which is modelled by an nth order ordinary differential equation with $n \geqslant 4$ describing the time evolution of a single scalar variable. Equivalently, using a chain of integrators, a hyperjerk system can be modelled as a system of n first order ordinary differential equations with $n \geqslant 4$. In this research work, a 4-D novel hyperchaotic hyperjerk system with two nonlinearities has been proposed, and its qualitative properties have been detailed. The novel hyperjerk system has a unique equilibrium at the origin, which is a saddle-focus and unstable. The Lyapunov exponents of the novel hyperjerk system are obtained as $L_1 = 0.14219$, $L_2 = 0.04605$, $L_3 = 0$ and $L_4 = -1.39267$. The Kaplan-Yorke dimension of the novel hyperjerk system is obtained as $D_{KY} = 3.1348$. Next, an adaptive controller is designed via backstepping control method to stabilize the novel hyperjerk chaotic system with three unknown parameters. Moreover, an adaptive controller is designed via backstepping control method to achieve global synchronization of the identical novel hyperjerk systems with three unknown parameters. MATLAB simulations are shown to illustrate all the main results derived in this research work on a novel hyperjerk system.

Key words: hyperchaos, hyperjerk system, adaptive control, backstepping control, synchronization.

1. Introduction

Chaos theory describes the qualitative study of unstable aperiodic behaviour in deterministic nonlinear dynamical systems. For the motion of a dynamical system to be chaotic, the system variables should contain nonlinear terms and it must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions [1, 2].

The first famous chaotic system was accidentally discovered by Lorenz, when he was designing a 3-D model for atmospheric convection in 1963 [3]. Subsequently, Rössler discovered a 3-D chaotic system in 1976 [4], which is algebraically simpler than the Lorenz system.

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Some well-known paradigms of 3-D chaotic systems are Arneodo system [5], Sprott systems [6], Chen system [7], Lü-Chen system [8], Liu system [9], Cai system [10], T-system [11], etc. Many new chaotic systems have been also discovered like Li system [12], Sundarapandian systems [13, 14], Vaidyanathan systems [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], Pehlivan system [35], Jafari system [36], Pham systems [37, 38, 39, 40], Sampath system [41], Akgul system [42], etc.

A hyperchaotic system is generally defined as a chaotic system with at least two positive Lyapunov exponents [43]. Thus, the dynamics of a hyperchaotic system are expended in several different directions simultaneously. Thus, the hyperchaotic systems have more complex dynamical behaviour and hence they have miscellaneous applications in engineering such as secure communications [44, 45], cryptosystems [46, 47], encryption [48, 49], electrical circuits [50, 51], etc.

The minimum dimension for an autonomous, continuous-time, hyperchaotic system is four. Since the discovery of a first 4-D hyperchaotic system by Rössler in 1979 [52], many 4-D hyperchaotic systems have been found in the literature such as hyperchaotic Lorenz system [53], hyperchaotic Lü system [54], hyperchaotic Chen system [55], hyperchaotic Wang system [56], hyperchaotic Newton-Leipnik system [57], hyperchaotic Jia system [58], hyperchaotic Vaidyanathan systems [59, 60, 61, 62, 63, 64, 65, 66, 67, 68], hyperchaotic Pham system [69], hyperchaotic Sampath system [70], etc.

In this paper, we propose a 4-D novel hyperchaotic hyperjerk system by adding a hyperbolic sinusoidal nonlinearity to the Chlouverakis-Sprott hyperjerk system [71].

First, we detail the fundamental qualitative properties of the novel hyperchaotic hyperjerk system. We show that the Lyapunov exponents of the novel hyperjerk system are given by $L_1 = 0.14219$, $L_2 = 0.04605$, $L_3 = 0$ and $L_4 = -1.39267$. Since the sum of the Lyapunov exponents is negative, we deduce that the novel hyperjerk system is dissipative. Also, we show that the Kaplan-Yorke dimension of the novel hyperjerk system is obtained as $D_{KY} = 3.1348$.

The study of control of a chaotic system investigates methods for designing feedback control laws that globally or locally asymptotically stabilize or regulate the outputs of a chaotic system.

Next, this paper derives an adaptive backstepping control law that stabilizes the novel hyperjerk system, when the system parameters are unknown. The backstepping control method is a recursive procedure that links the choice of a Lyapunov function with the design of a controller and guarantees global asymptotic stability of strict feedback systems [72, 73, 74, 75].

This paper also derives an adaptive backstepping control law that achieves global chaos synchronization of the identical 4-D novel hyperchaotic hyperjerk systems with unknown parameters.

Chaos synchronization problem deals with the synchronization of a couple of systems called the master or drive system and the slave or response system. To solve this problem, control laws are designed so that the output of the slave system tracks the output of the master system asymptotically with time.

In the chaos literature, an impressive variety of techniques have been proposed to solve the problem of chaos synchronization such as active control method [76, 77], adaptive control method [78, 79], backstepping control method [80, 81, 82, 83], sliding mode control method [84, 85, 86, 87, 88, 89], etc.

All the main adaptive results in this paper are derived using backstepping control method and proved using Lyapunov stability theory [90]. MATLAB simulations are depicted to illustrate the phase portraits of the novel hyperchaotic hyperjerk system with two positive Lyapunov exponents, adaptive stabilization and synchronization results for the novel 4-D hyperchaotic hyperjerk system.

2. A 4-D novel hyperchaotic hyperjerk system

In mechanics, a *jerk system* is described by an explicit third order ordinary differential equation describing the time evolution of a single scalar variable x according to the dynamics

$$\frac{d^3x}{dt^3} = f\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right) \tag{1}$$

The differential equation (1) is called a *jerk system* because the successive derivatives of the displacement in a mechanical system are the velocity, acceleration, and jerk.

A generalization of the jerk dynamics is given by the dynamics

$$\frac{d^{(n)}x}{dt^n} = f\left(\frac{d^{(n-1)}x}{dt^{n-1}}, \ldots, \frac{dx}{dt}, x\right), \quad (n \geqslant 4) \tag{2}$$

An ordinary differential equation of the form (2) is called a *hyperjerk* system since it involves time derivatives of a jerk function.

In [71], Chlouverakis and Sprott discovered a simple hyperchaotic hyperjerk system given by the dynamics

$$\frac{d^4x}{dt^4} + \frac{d^3x}{dt^3}x^4 + A\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0 \tag{3}$$

In system form, the differential equation (3) can be expressed as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 - x_2 - Ax_3 - x_1^4 x_4 \end{cases} \tag{4}$$

When $A = 3.6$, the hyperjerk system (4) exhibits *hyperchaos* with Lyapunov exponent spectrum $(0.132, 0.035, 0, -1.25)$. Thus, the maximum Lyapunov exponent (MLE)

of the Chlouverakis-Sprott hyperchaotic hyperjerk system (4) is $L_1 = 0.132$ and the Kaplan-Yorke dimension of this hyperjerk system is easily calculated a $D_{KY} = 3.13$.

In this work, we propose a novel hyperjerk system by adding a hyperbolic sinusoidal nonlinearity to the Chlouverakis-Sprott hyperjerk system (4) and with a different set of values for the system parameters.

Our novel hyperjerk system is given in system form as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 - x_2 - ax_3 - b\sinh(x_2) - cx_1^4 x_4 \end{cases} \tag{5}$$

where a, b and c are positive parameters.

In this paper, we shall show that the system (5) is *hyperchaotic* when the parameters a, b and c take the values

$$a = 3.7, \quad b = 0.05, \quad c = 1.3 \tag{6}$$

For the parameter values in (6), the Lyapunov exponents of the novel hyperjerk system (5) are obtained as

$$L_1 = 0.14219, \quad L_2 = 0.04605, \quad L_3 = 0, \quad L_4 = -1.39267 \tag{7}$$

From the LE spectrum given in (7), it is easily seen that the maximal Lyapunov exponent (MLE) of our novel hyperchaotic hyperjerk system (5) is $L_1 = 0.14219$, which is greater than the MLE of the Chlouverakis-Sprott hyperchaotic hyperjerk system (4). Also, the Kaplan-Yorke dimension of the novel hyperjerk system (5) is calculated as $D_{KY} = 3.1348$, which is greater than the Kaplan-Yorke dimension of the Chlouverakis-Sprott hyperjerk system (4). This shows that the novel hyperchaotic hyperjerk system (5) exhibits more complex behaviour than the Chlouverakis-Sprott hyperchaotic hyperjerk system (4).

For numerical simulations, we take the initial values of the novel hyperjerk system (5) as $x_1(0) = 0.5, x_2(0) = 0.5, x_3(0) = 0.5$ and $x_4(0) = 0.5$.

Figs. 1-4 depict the 3-D projections of the 4-D novel hyperjerk system (5) on $(x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_4)$ and (x_2, x_3, x_4) spaces respectively.
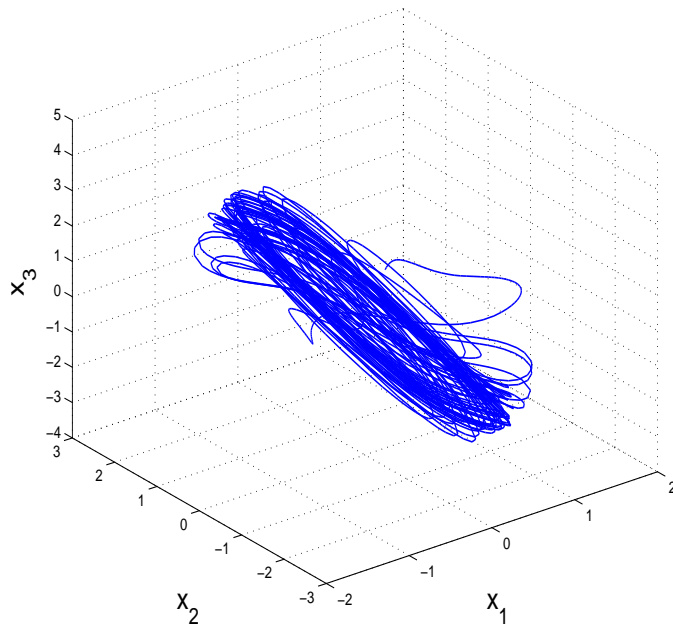
Figure 1: 3-D projection of the 4-D novel hyperjerk system on (x_1, x_2, x_3) space
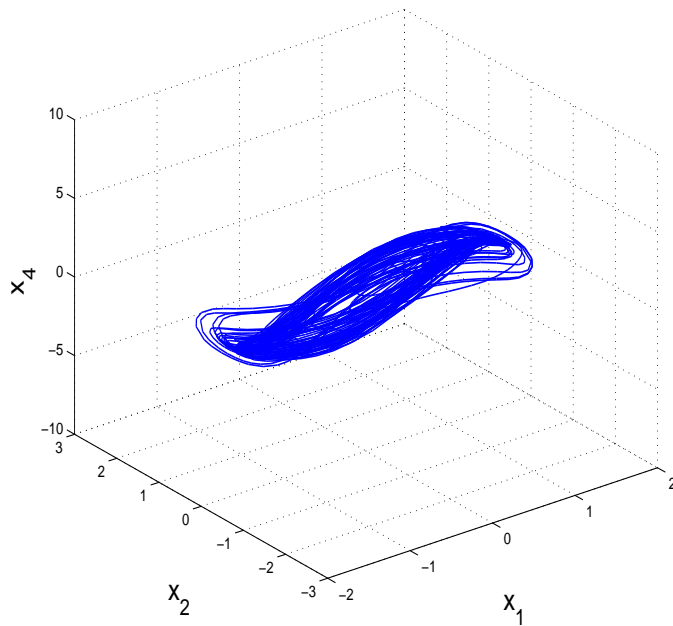

Figure 2: 3-D projection of the 4-D novel hyperjerk system on (x_1, x_2, x_4) space
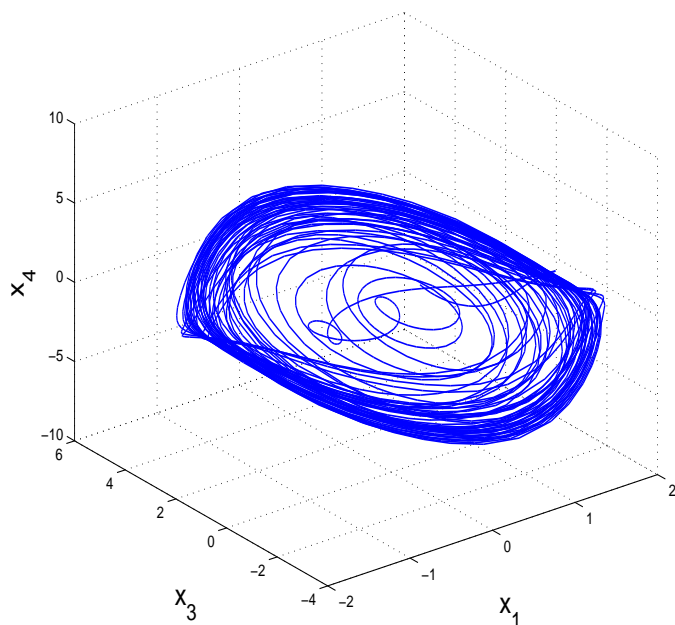
Figure 3: 3-D projection of the 4-D novel hyperjerk system on (x_1, x_3, x_4) space
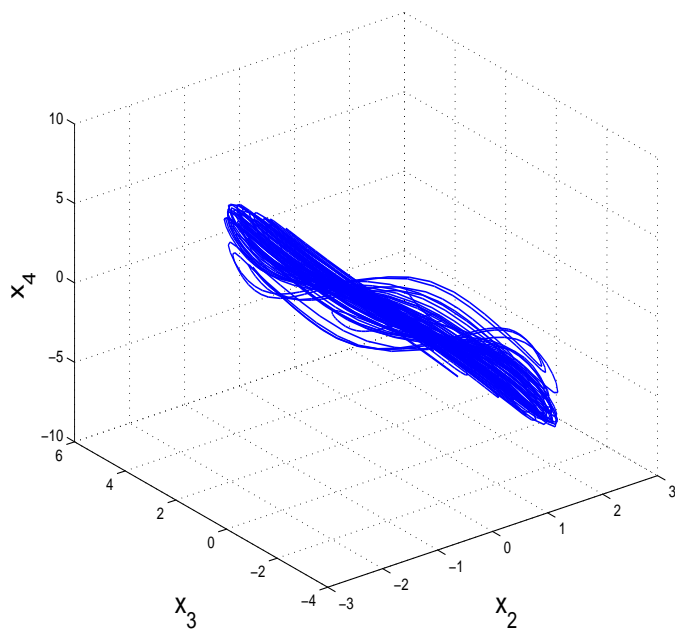

Figure 4: 3-D projection of the 4-D novel hyperjerk system on (x_2, x_3, x_4) space

3.   Analysis of the 4-D novel hyperjerk system

3.1.   Equilibrium points

The equilibrium points of the 4-D novel hyperjerk system (5) are obtained by solving the equations

$$\left.\begin{array}{rclcl}
f_1(x_1,x_2,x_3,x_4) & = & x_2 & = & 0 \\
f_2(x_1,x_2,x_3,x_4) & = & x_3 & = & 0 \\
f_3(x_1,x_2,x_3,x_4) & = & x_4 & = & 0 \\
f_4(x_1,x_2,x_3,x_4) & = & -x_1 - x_2 - ax_3 - b\sinh(x_2) - cx_1^4 x_4 & = & 0
\end{array}\right\} \tag{8}$$

We take the parameter values as in the hyperchaotic case (6).
Thus, the equilibrium points of the system (5) are characterized by the equations

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 \tag{9}$$

Solving the system (9), we note that the 4-D novel hyperjerk system (5) has a unique equilibrium at the origin, *i.e.*

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

To test the stability type of the equilibrium points E_0, we calculate the Jacobian matrix of the novel hyperjerk system (5) at E_0 as

$$J_0 = J(E_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1-b & -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1.05 & -3.7 & 0 \end{bmatrix} \tag{11}$$

The matrix J_0 has the eigenvalues

$$\lambda_{1,2} = 0.1622 \pm 1.8695i, \quad \lambda_{3,4} = -0.1662 \pm 0.5076i \tag{12}$$

This shows that the equilibrium point E_0 is a saddle-focus point, which is unstable.

3.2. Lyapunov exponents and Kaplan-Yorke dimension

For the parameter values $a = 3.7$, $b = 0.05$ and $c = 1.3$, the Lyapunov exponents of the novel hyperjerk system (5) are numerically obtained using MATLAB as

$$L_1 = 0.14219, \ \ L_2 = 0.04605, \ \ L_3 = 0 \ \text{ and } \ L_4 = -1.39627 \tag{13}$$

Since the LE spectrum in (13) has two positive Lyapunov exponents, the novel hyperjerk system (5) is hyperchaotic.

Since $L_1 + L_2 + L_3 + L_4 = -1.2080 < 0$, it follows that the novel hyperjerk system (5) is dissipative.

Also, the Kaplan-Yorke dimension of the novel hyperchaotic hyperjerk system (5)is obtained as

$$D_{KY} = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.1348, \tag{14}$$

which is fractional.

4. Adaptive control of the 4-D novel hyperjerk system with unknown parameters

In this section, we use backstepping control method to derive an adaptive feedback control law for globally stabilizing the 4-D novel hyperjerk system with unknown parameters.

Thus, we consider the 4-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 & = \ x_2 \\ \dot{x}_2 & = \ x_3 \\ \dot{x}_3 & = \ x_4 \\ \dot{x}_4 & = \ -x_1 - x_2 - ax_3 - b\sinh(x_2) - cx_1^4 x_4 + u \end{cases} \tag{15}$$

where a, b and c are unknown constant parameters, and u is a backstepping control law to be determined using estimates $\hat{a}(t), \hat{b}(t)$ and $\hat{c}(t)$ for a, b and c, respectively.

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) & = \ a - \hat{a}(t) \\ e_b(t) & = \ b - \hat{b}(t) \\ e_c(t) & = \ c - \hat{c}(t) \end{cases} \tag{16}$$

Differentiating (16) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) & = & -\dot{\hat{a}}(t) \\ \dot{e}_b(t) & = & -\dot{\hat{b}}(t) \\ \dot{e}_c(t) & = & -\dot{\hat{c}}(t) \end{cases} \tag{17}$$

Next, we shall state and prove the main result of this section.

Theorem 1 *The 4-D novel hyperjerk system* (15)*, with unknown parameters a, b and c, is globally and exponentially stabilized by the adaptive feedback control law,*

$$u(t) = -4x_1 - 9x_2 - [9 - \hat{a}(t)]x_3 - 4x_4 + \hat{b}(t)\sinh(x_2) + \hat{c}(t)x_1^4 x_4 - kz_4, \tag{18}$$

where $k > 0$ is a gain constant,

$$z_4 = 3x_1 + 5x_2 + 3x_3 + x_4 \tag{19}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) & = & -x_3 z_4 \\ \dot{\hat{b}}(t) & = & -\sinh(x_2)z_4 \\ \dot{\hat{c}}(t) & = & -x_1^4 x_4 z_4 \end{cases} \tag{20}$$

Proof We prove this result via backstepping control and Lyapunov stability theory [90].

First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2 \tag{21}$$

where

$$z_1 = x_1 \tag{22}$$

Differentiating V_1 along the dynamics (15), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = x_1 x_2 = -z_1^2 + z_1(x_1 + x_2) \tag{23}$$

Now, we define

$$z_2 = x_1 + x_2 \tag{24}$$

Using (24), we can simplify the equation (23) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{25}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2} \left(z_1^2 + z_2^2 \right) \tag{26}$$

Differentiating V_2 along the dynamics (15), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2x_1 + 2x_2 + x_3) \tag{27}$$

Now, we define

$$z_3 = 2x_1 + 2x_2 + x_3 \tag{28}$$

Using (28), we can simplify the equation (27) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{29}$$

Thirdly, we define a quadratic Lyapunov function

$$V_3(z_1, z_2, x_3) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 = \frac{1}{2} \left(z_1^2 + z_2^2 + z_3^2 \right) \tag{30}$$

Differentiating V_3 along the dynamics (15), we get

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(3x_1 + 5x_2 + 3x_3 + x_4) \tag{31}$$

Now, we define

$$z_4 = 3x_1 + 5x_2 + 3x_3 + x_4 \tag{32}$$

Using (32), we can simplify the equation (31) as

$$\dot{V}_2 = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4 \tag{33}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, z_4, e_a, e_b, e_c) = V_3(z_1, z_2, z_3) + \frac{1}{2} z_4^2 + \frac{1}{2} e_a^2 + \frac{1}{2} e_b^2 + \frac{1}{2} e_c^2 \tag{34}$$

which is a positive definite function on $\Re^7$.

Differentiating V along the dynamics (15), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_4 + z_3 + \dot{z}_4) - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} - e_c \dot{\hat{c}} \tag{35}$$

Eq. (35) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4 S - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} - e_c \dot{\hat{c}} \tag{36}$$

where

$$S = z_4 + z_3 + \dot{z}_4 = z_4 + z_3 + 3\dot{x}_1 + 5\dot{x}_2 + 3\dot{x}_3 + \dot{x}_4 \tag{37}$$

A simple calculation gives

$$S = 4x_1 + 9x_2 + (9-a)x_3 + 4x_4 - b\sinh(x_2) - cx_1^4 x_4 + u \tag{38}$$

Substituting the adaptive control law (18) into (38), we obtain

$$S = -[a - \hat{a}(t)]x_3 - [b - \hat{b}(t)]\sinh(x_2) - [c - \hat{c}(t)]x_1^4 x_4 - kz_4 \tag{39}$$

Using the definitions (17), we can simplify (39) as

$$S = -e_a x_3 - e_b \sinh(x_2) - e_c x_1^4 x_4 - kz_4 \tag{40}$$

Substituting the value of S from (40) into (36), we obtain

$$\begin{cases} \dot{V} &= -z_1^2 - z_2^2 - z_3^2 - (1+k)z_4^2 + e_a(-x_3 z_4 - \dot{\hat{a}}) \\ &\quad + e_b(-\sinh(x_2)z_4 - \dot{\hat{b}}) + e_c(-x_1^4 x_4 z_4 - \dot{\hat{c}}) \end{cases} \tag{41}$$

Substituting the update law (20) into (41), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+k)z_4^2, \tag{42}$$

which is a negative semi-definite function on $\mathfrak{R}^7$.

From (42), it follows that the vector $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & z_4(t) & e_a(t) & e_b(t) & e_c(t) \end{bmatrix} \in \mathbf{L}_\infty \tag{43}$$

Also, it follows from (42) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 - z_4^2 = -\|\mathbf{z}\|^2 \tag{44}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V} \tag{45}$$

Integrating the inequality (45) from 0 to t, we get

$$\int_0^t |\mathbf{z}(\tau)|^2 \, d\tau \leqslant V(0) - V(t) \tag{46}$$

From (46), it follows that $\mathbf{z}(t) \in \mathbf{L}_2$.
From Eq. (15), it can be deduced that $\dot{\mathbf{z}}(t) \in \mathbf{L}_\infty$.
Thus, using Barbalat's lemma [90], we conclude that $\mathbf{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{z}(0) \in \mathfrak{R}^4$.

Hence, it is immediate that $\boldsymbol{x}(t) \to \boldsymbol{0}$ exponentially as $t \to \infty$ for all initial conditions $\boldsymbol{x}(0) \in \mathfrak{R}^4$.

This completes the proof.

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (15) and (20), when the adaptive control law (18) is applied.

The parameter values of the novel hyperjerk system (15) are taken as in the hyper-chaotic case, *viz.* $a = 3.7$, $b = 0.05$, $c = 1.3$, and the positive gain constant as $k = 10$.

Furthermore, as initial conditions of the novel hyperjerk system (15), we take

$$x_1(0) = 2.3, \quad x_2(0) = -8.5, \quad x_3(0) = 6.2, \quad x_4(0) = 3.6 \tag{47}$$

Also, as initial conditions of the parameter estimates, we take

$$\hat{a}(0) = 5.3, \quad \hat{b}(0) = 12.4, \quad \hat{c}(0) = 7.5 \tag{48}$$

In Fig. 5, the exponential convergence of the controlled states (x_1, x_2, x_3, x_4) is depicted, when the adaptive control law (18) and parameter update law (20) are implemented.
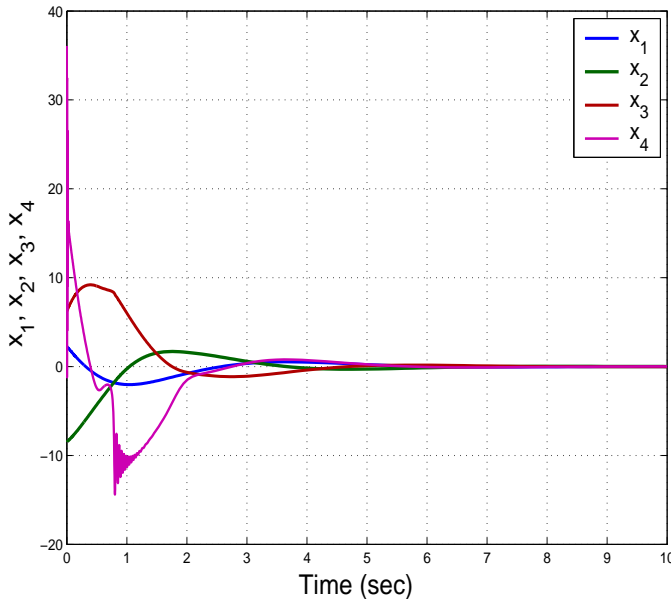

Figure 5: Time-history of the controlled states $x_1(t), x_2(t), x_3(t), x_4(t)$

5. Adaptive synchronization of the identical 4-D novel hyperjerk systems with unknown parameters

In this section, we use backstepping control method to derive an adaptive control law for globally and exponentially synchronizing the identical 4-D novel hyperjerk systems with unknown parameters.

As the master system, we consider the 4-D novel hyperjerk system given by

$$
\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_1 - x_2 - ax_3 - b\sinh(x_2) - cx_1^4 x_4
\end{cases}
\tag{49}
$$

where x_1, x_2, x_3, x_4 are the states of the system, and a, b, c are unknown constant parameters.

As the slave system, we consider the 4-D novel hyperjerk system given by

$$
\begin{cases}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= y_4 \\
\dot{y}_4 &= -y_1 - y_2 - ay_3 - b\sinh(y_2) - cy_1^4 y_4 + u
\end{cases}
\tag{50}
$$

where y_1, y_2, y_3, y_4 are the states of the system, and u is a backstepping control to be determined using estimates $\hat{a}(t), \hat{b}(t)$ and $\hat{c}(t)$ for a, b and c, respectively.

We define the synchronization errors between the states of the master system (49) and the slave system (50) as

$$
\begin{cases}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3 \\
e_4 &= y_4 - x_4
\end{cases}
\tag{51}
$$

Then the error dynamics is easily obtained as

$$
\begin{cases}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= -e_1 - e_2 - ae_3 - b[\sinh(y_2) - \sinh(x_2)] - c(y_1^4 y_4 - x_1^4 x_4) + u
\end{cases}
\tag{52}
$$

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \\ e_c(t) &= c - \hat{c}(t) \end{cases} \tag{53}$$

Differentiating (53) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) &= -\dot{\hat{a}}(t) \\ \dot{e}_b(t) &= -\dot{\hat{b}}(t) \\ \dot{e}_c(t) &= -\dot{\hat{c}}(t) \end{cases} \tag{54}$$

Next, we shall state and prove the main result of this section.

Theorem 2 *The identical 4-D novel hyperjerk systems* (49) *and* (50) *with unknown parameters a, b and c are globally and exponentially synchronized by the adaptive control law*

$$\begin{cases} u(t) &= -4e_1 - 9e_2 - [9 - \hat{a}(t)]e_3 - 4e_4 + \hat{b}(t)\left[\sinh(y_2) - \sinh(x_2)\right] \\ &\quad + \hat{c}(t)\left(y_1^4 y_4 - x_1^4 x_4\right) - kz_4 \end{cases} \tag{55}$$

where $k > 0$ is a gain constant,

$$z_4 = 3e_1 + 5e_2 + 3e_3 + e_4, \tag{56}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) &= -e_3 z_4 \\ \dot{\hat{b}}(t) &= -\left[\sinh(y_2) - \sinh(x_2)\right] z_4 \\ \dot{\hat{c}}(t) &= -\left(y_1^4 y_4 - x_1^4 x_4\right) z_4 \end{cases} \tag{57}$$

Proof We prove this result via backstepping control and Lyapunov stability theory [90].

First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2 \tag{58}$$

where

$$z_1 = e_1 \tag{59}$$

Differentiating V_1 along the error dynamics (52), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = e_1 e_2 = -z_1^2 + z_1(e_1 + e_2) \tag{60}$$

Now, we define

$$z_2 = e_1 + e_2 \tag{61}$$

Using (61), we can simplify the equation (60) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{62}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2} \left(z_1^2 + z_2^2 \right) \tag{63}$$

Differentiating V_2 along the error dynamics (52), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 + e_3) \tag{64}$$

Now, we define

$$z_3 = 2e_1 + 2e_2 + e_3 \tag{65}$$

Using (65), we can simplify the equation (64) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{66}$$

Thirdly, we define a quadratic Lyapunov function

$$V_3(z_1, z_2, x_3) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 = \frac{1}{2} \left(z_1^2 + z_2^2 + z_3^2 \right) \tag{67}$$

Differentiating V_3 along the error dynamics (52), we get

$$\dot{V}_3 = -z_1^2 - z_2^2 - z_3^2 + z_3(3e_1 + 5e_2 + 3e_3 + e_4) \tag{68}$$

Now, we define

$$z_4 = 3e_1 + 5e_2 + 3e_3 + e_4 \tag{69}$$

Using (69), we can simplify the equation (68) as

$$\dot{V}_2 = -z_1^2 - z_2^2 - z_3^2 + z_3 z_4 \tag{70}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, z_4, e_a, e_b, e_c) = V_3(z_1, z_2, z_3) + \frac{1}{2} z_4^2 + \frac{1}{2} e_a^2 + \frac{1}{2} e_b^2 + \frac{1}{2} e_c^2 \tag{71}$$

Clearly, V is a positive definite function on $\Re^7$.

Differentiating V along the error dynamics (52), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4(z_4 + z_3 + \dot{z}_4) - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} - e_c \dot{\hat{c}} \tag{72}$$

Eq. (72) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - z_4^2 + z_4 S - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} - e_c \dot{\hat{c}} \tag{73}$$

where

$$S = z_4 + z_3 + \dot{z}_4 = z_4 + z_3 + 3\dot{e}_1 + 5\dot{e}_2 + 3\dot{e}_3 + \dot{e}_4 \tag{74}$$

A simple calculation gives

$$S = 4e_1 + 9e_2 + (9-a)e_3 + 4e_4 - b\left[\sinh(y_2) - \sinh(x_2)\right] - c\left(y_1^4 y_4 - x_1^4 x_4\right) + u \tag{75}$$

Substituting the adaptive control law (55) into (75), we obtain

$$S = -\left[a - \hat{a}(t)\right]e_3 - \left[b - \hat{b}(t)\right]\left[\sinh(y_2) - \sinh(x_2)\right] - \left[c - \hat{c}(t)\right]\left(y_1^4 y_4 - x_1^4 x_4\right] - kz_4 \tag{76}$$

Using the definitions (54), we can simplify (76) as

$$S = -e_a e_3 - e_b\left[\sinh(y_2) - \sinh(x_2)\right] - e_c\left(y_1^4 y_4 - x_1^4 x_4\right) - kz_4 \tag{77}$$

Substituting the value of S from (77) into (73), we obtain

$$\begin{cases} \dot{V} &= -z_1^2 - z_2^2 - z_3^2 - (1+k)z_4^2 + e_a(-e_3 z_4 - \dot{\hat{a}}) \\ & \quad + e_b\left[-\left[\sinh(y_2) - \sinh(x_2)\right]z_4 - \dot{\hat{b}}\right] + e_c\left[-\left(y_1^4 y_4 - x_1^4 x_4\right)z_4 - \dot{\hat{c}}\right] \end{cases} \tag{78}$$

Substituting the update law (57) into (78), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 - (1+k)z_4^2, \tag{79}$$

which is a negative semi-definite function on $\mathfrak{R}^7$.

From (79), it follows that the vector $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\left[\begin{array}{ccccccc} z_1(t) & z_2(t) & z_3(t) & z_4(t) & e_a(t) & e_b(t) & e_c(t) \end{array}\right] \in \boldsymbol{L}_\infty \tag{80}$$

Also, it follows from (79) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 - z_4^2 = -\|\mathbf{z}\|^2 \tag{81}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V} \tag{82}$$

Integrating the inequality (82) from 0 to t, we get

$$\int_0^t |\mathbf{z}(\tau)|^2 \, d\tau \leqslant V(0) - V(t) \tag{83}$$

From (83), it follows that $\mathbf{z}(t) \in \mathbf{L}_2$.

From Eq. (52), it can be deduced that $\dot{\mathbf{z}}(t) \in \mathbf{L}_\infty$.

Thus, using Barbalat's lemma [90], we conclude that $\mathbf{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{z}(0) \in \mathfrak{R}^4$.

Hence, it is immediate that $\mathbf{e}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{e}(0) \in \mathfrak{R}^4$.

This completes the proof.   □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (49) and (50).

The parameter values of the novel hyperjerk systems are taken as in the hyperchaotic case, *viz.* $a = 3.7, b = 0.2, c = 1.5$ and the positive gain constant as $k = 10$.

Also, as initial conditions of the master system (49), we take

$$x_1(0) = 1.5, \quad x_2(0) = 0.4, \quad x_3(0) = -0.8, \quad x_4(0) = 1.5 \tag{84}$$

As initial conditions of the slave system (50), we take

$$y_1(0) = 3.7, \quad y_2(0) = 1.2, \quad y_3(0) = 1.6, \quad y_4(0) = 2.3 \tag{85}$$

Furthermore, as initial conditions of the parameter estimates $\hat{a}(t), \hat{b}(t)$ and $\hat{c}(t)$, we take

$$\hat{a}(0) = 6.1, \quad \hat{b}(0) = 3.8, \quad \hat{c}(0) = 2.9 \tag{86}$$

In Figs. 6-9, the complete synchronization of the identical 4-D novel hyperchaotic hyperjerk systems (49) and (50) is shown, when the adaptive control law and the parameter update law are implemented.

Also, in Fig. 10, the time-history of the synchronization errors $(e_1(t), e_2(t), e_3(t), e_4(t))$ is shown.

6. Conclusion

In this research work, a 4-D novel hyperchaotic hyperjerk system with two non-linearities has been proposed, and its qualitative properties have been detailed. The novel hyperjerk system has a unique equilibrium at the origin, which is a saddle-focus and unstable. The Lyapunov exponents of the novel hyperjerk system were obtained as $L_1 = 0.14219$, $L_2 = 0.04605$, $L_3 = 0$ and $L_4 = -1.39267$. The Kaplan-Yorke dimension of the novel hyperjerk system was obtained as $D_{KY} = 3.1348$. Next, an adaptive
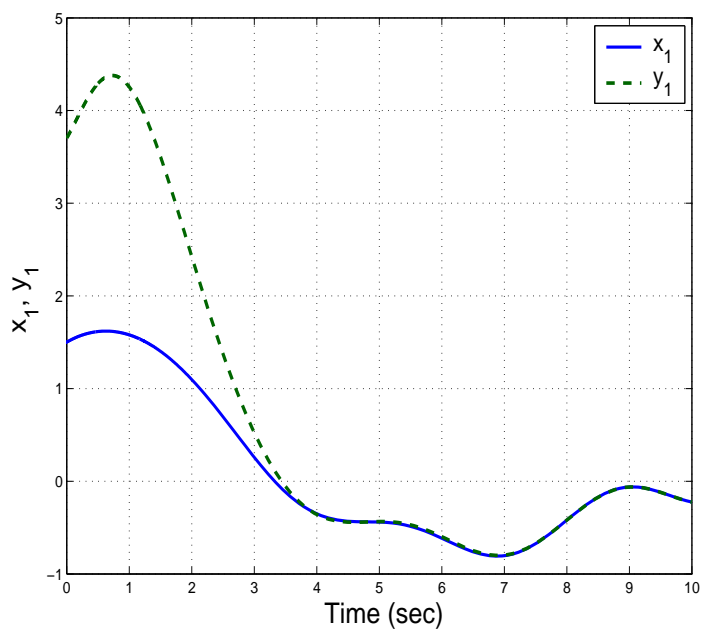
Figure 6: Synchronization of the states $x_1(t)$ and $y_1(t)$
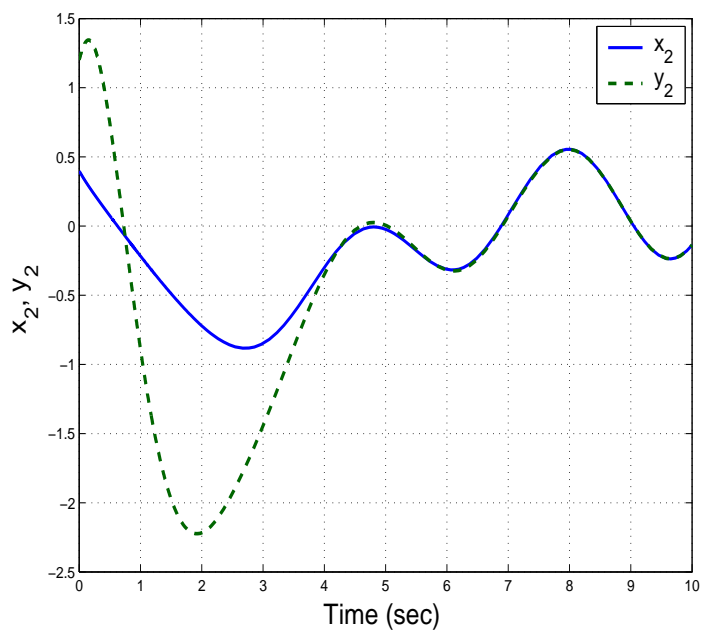
Figure 7: Synchronization of the states $x_2(t)$ and $y_2(t)$
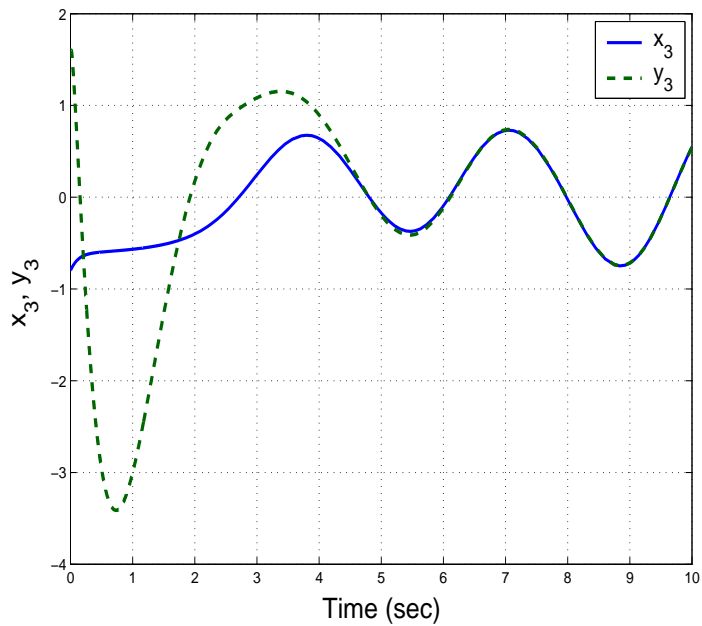
Figure 8: Synchronization of the states $x_3(t)$ and $y_3(t)$
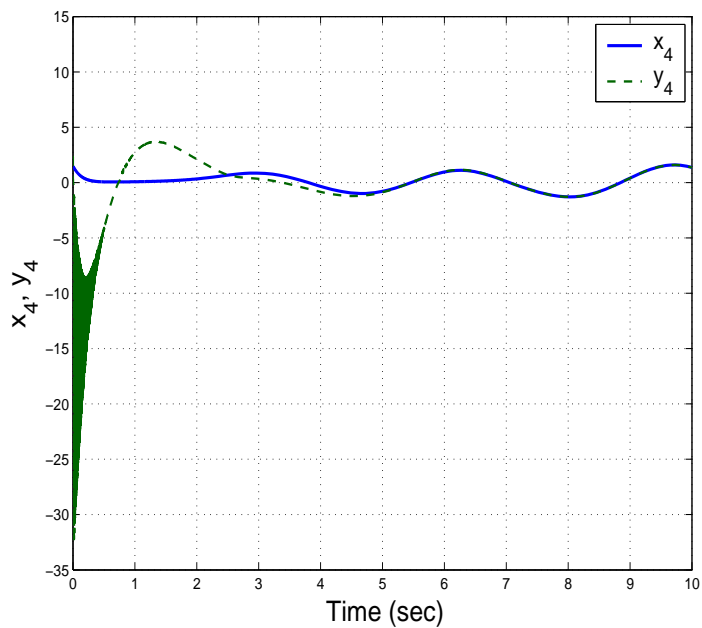
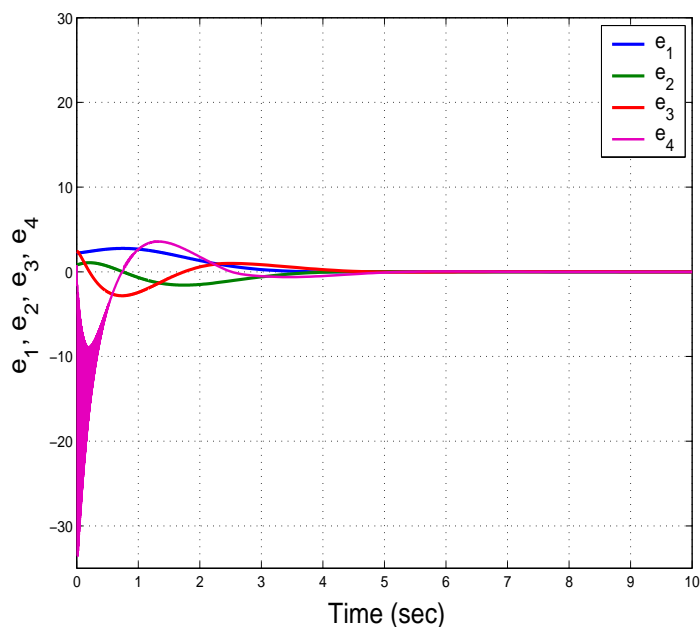Figure 9: Synchronization of the states $x_4(t)$ and $y_4(t)$

Figure 10: Time-history of the synchronization errors e_1, e_2, e_3, e_4

controller was designed via backstepping control method to stabilize the novel hyper-jerk chaotic system with three unknown parameters. Moreover, an adaptive controller was designed via backstepping control method to achieve global synchronization of the identical novel hyperjerk systems with three unknown parameters. MATLAB simulations have been provided to illustrate all the main results derived in this research work on a novel hyperjerk system.

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