

# Stability Conditions for the Leaky LMS Algorithm Based on Control Theory Analysis

Dariusz BISMOR, Marek PAWELCZYK

*Institute of Automatic Control  
Silesian University of Technology  
Akademicka 16, 44-100 Gliwice, Poland; e-mail: Dariusz.Bismor@polsl.pl*

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The Least Mean Square (LMS) algorithm and its variants are currently the most frequently used adaptation algorithms; therefore, it is desirable to understand them thoroughly from both theoretical and practical points of view. One of the main aspects studied in the literature is the influence of the step size on stability or convergence of LMS-based algorithms. Different publications provide different stability upper bounds, but a lower bound is always set to zero. However, they are mostly based on statistical analysis. In this paper we show, by means of control theoretic analysis confirmed by simulations, that for the leaky LMS algorithm, a small negative step size is allowed. Moreover, the control theoretic approach allows to minimize the number of assumptions necessary to prove the new condition. Thus, although a positive step size is fully justified for practical applications since it reduces the mean-square error, knowledge about an allowed small negative step size is important from a cognitive point of view.

**Keywords:** adaptive filtering; leaky LMS; stability; negative step size; identification; adaptive line enhancer; active noise control.

## 1. Introduction

Studies on the Least Mean Square (LMS) algorithm stability, using different sets of assumptions, have resulted in many different upper bounds on the step size that we know so far. There are two general attitudes towards the LMS algorithm analysis: one uses the independence theory (i.e., the sequence of input vectors is i.i.d. – independent, identically distributed) (GARDNER, 1984), the other uses the small-step-size-assumption (HAYKIN, 2002). Both approaches result in step-size upper bounds given by some positive values depending on the input signal. However, there is no doubt that the step size should be positive to guarantee stable operation of the LMS algorithm.

The two probably most frequently used LMS-based algorithms are the normalized LMS (NLMS) and the leaky LMS (LLMS) (BISMOR *et al.*, 2016). The stability and convergence of the LLMS algorithm has been studied in (MAYYAS, ABOULNASR, 1997; NASCIMENTO, SAYED, 1999). In this paper, we show that the equal-to-zero lower bound for the step size resulting from those studies defines only a sufficient stability condition, contrary to common understanding. We also

show that the LLMS algorithm is capable of stable operation with a negative step size, and we will define proper stability bounds with a very minimal set of assumptions, which do not restrict areas of application. Particularly, we avoid both the i.i.d. and small-step-size assumptions.

The reason for studying the case of a negative step size is twofold. First of all, showing that the LLMS algorithm can use a negative step size and remain stable brings new insight into this algorithm. Second, stable operation with a negative step size may be misleading in some situations, and useful in others. Misleading, for example, when the LLMS algorithm is used for active noise control without secondary path modeling, but with the step-size sign being self-tuned to maintain the stability and convergence. Such novel algorithms are already under development (KURCZYK, PAWELCZYK, 2014a; 2014b), and use increasing or decreasing error values to decide about the change of the step-size sign. Awareness that a small negative step size may not make the control system output diverge can bring new insight and affect parameterisation of soft computing algorithms employed for tuning parameters of the active control filter without a secondary path model.

A usefulness of the negative-step-size stable operation lies in the fact that it can let the filter coefficients to leave a local optimum in cases when the error surface is not convex, e.g., during bilinear model identification (MALIŃSKI, 2012), or in other sophisticated scenarios (ŁAWRYŃCZUK, 2009). To the best of our knowledge, these important issues have not yet been addressed in the literature.

The adaptive filtering problem considered in this paper is depicted in Fig. 1. We use the usual notation, with  $\mathbf{u}(n)$  being the input signal vector,  $d(n)$  the desired signal, and  $e(n)$  the error signals, sampled at time instant  $n$ . Usually, the input signal vector consists of samples of the same signal delayed in time:  $\mathbf{u}(n) = [u(n), u(n-1), \dots, u(n-L+1)]^T$  (HAYKIN, 2002). The adaptive filter is of finite impulse response type; therefore, the filter output can be calculated as  $y(n) = \mathbf{u}^T(n)\mathbf{w}(n)$ , where  $\mathbf{w}(n) = [w_0(n), w_1(n), \dots, w_{L-1}(n)]^T$  is the vector of adapted filter coefficients.

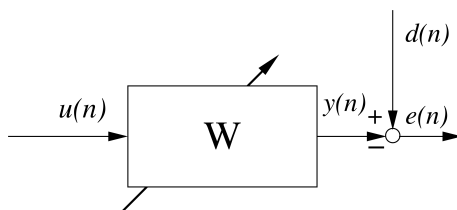


Fig. 1. The adaptive filtering problem.

For simplicity of presentation, we will express the LLMS algorithm in its original form (SETHARES *et al.*, 1986):

$$\mathbf{w}(n+1) = \gamma\mathbf{w}(n) + \mu\mathbf{u}(n)e(n), \quad (1)$$

where  $0 \ll \gamma \leq 1$  is the leakage, and  $\mu$  is the step size. Another representation of the LLMS algorithm, used, e.g., by MAYYAS and ABOULNASR (1997), is given by:

$$\mathbf{w}(n+1) = (1 - \beta\mu)\mathbf{w}(n) + \mu\mathbf{u}(n)e(n), \quad (2)$$

where  $\beta \geq 0$  is a parameter that relates the step size to the leakage. This representation, which comes as a direct result of the optimization problem resulting in the LLMS algorithm, allows one to understand that the leakage and the step size are related, i.e., when one is increased the other should be decreased to keep the same cost function value.

As in (BISMOR, 2015), we will assume the input and the desired signals are real-valued and bounded, and that the adaptive filter is a linear (for frozen time), transversal filter of finite impulse response (FIR).

In this publication we intentionally use *bounded-input bounded-output* (BIBO) stability (referred to as simply “stability”), which ensures that a discrete system is “well behaved”, i.e., it produces bounded outputs for bounded inputs (ZAMES, 1966). For the adaptive filtering problem, such understanding of stability

assures that the adaptive filter coefficients do not diverge, which is important from a practical point of view. The formal definition of BIBO stability can be found e.g., in (LIN, VARAIYA, 1967).

## 2. Leaky LMS stability sufficient condition

A sufficient stability condition for the LMS algorithm, derived based on control system theory, without using either the independence, or small-step-size assumptions, has been recently presented in (BISMOR, 2015). The same approach will be applied here to derive a sufficient stability condition for the LLMS algorithm. Substituting

$$e(n) = d(n) - y(n) = d(n) - \mathbf{u}^T(n)\mathbf{w}(n) \quad (3)$$

for the error signal into the LLMS algorithm update Eq. (1) and rearranging the terms, yields:

$$\mathbf{w}(n+1) = [\gamma\mathbf{I} - \mu\mathbf{u}(n)\mathbf{u}^T(n)]\mathbf{w}(n) + \mu\mathbf{u}(n)d(n), \quad (4)$$

where  $\mathbf{I} \in \mathcal{R}^{L \times L}$  is the identity matrix.

Treating the above equation as a state equation of a nonstationary discrete-time system (BISMOR, 2015), we recognize the state transition matrix as:

$$\mathbf{A}(n) = \gamma\mathbf{I} - \mu\mathbf{u}(n)\mathbf{u}^T(n). \quad (5)$$

This matrix is responsible for stability of the LLMS algorithm, and therefore will be referred to as the *stability matrix*. Observe that the matrix is symmetric. For this matrix, the following theorem holds.

**Theorem 1** (Leaky LMS Stability Matrix Eigenvalues and Eigenvector). *Assume matrix  $\mathbf{A}(n) \in \mathcal{R}^{L \times L}$  is the LLMS stability matrix, (5) at any discrete time  $n$ . Then, the matrix has an eigenvalue:*

$$\lambda_1(n) = \gamma - \mu \sum_{i=0}^{L-1} u^2(n-i), \quad (6)$$

*with the corresponding eigenvector  $\mathbf{u}(n)$ . The remaining eigenvalues are all equal to  $\gamma$ .*

The proof of this theorem is given in Appendix.

In (BUBNICKI, 2005), a theorem has been proven that relates the eigenvalues of a symmetric state transition matrix with the stability of a nonstationary system (Theorem 10.5 on page 272). The theorem states that if all the eigenvalues of a symmetric state transition matrix describing a nonstationary system have magnitudes less than 1, the system is asymptotically stable. Considering that theorem, Theorem 1, and control-theory results extensively discussed in (BISMOR, 2015), we conclude that for the stability of the LLMS algorithm, it suffices that the absolute value of the only non- $\gamma$  eigenvalue of (5) at any discrete time  $n$  has magnitude less than 1:

$$\forall_n \quad |\lambda_1(n)| = \left| \gamma - \mu \sum_{i=0}^{L-1} u^2(n-i) \right| < 1. \quad (7)$$

The solution of the above inequality is:

$$\forall_n \frac{\gamma - 1}{\sum_{i=0}^{L-1} u^2(n-i)} < \mu < \frac{\gamma + 1}{\sum_{i=0}^{L-1} u^2(n-i)}, \quad (8)$$

for  $\sum_{i=0}^{L-1} u^2(n-i) = \|\mathbf{u}(n)\|^2 \neq 0$ . Therefore, for  $\gamma < 1$ , the lower bound for the step size, defined by the left-hand side of the above inequality, is negative. For example, with  $\gamma = 0.98$ , the step size should have a value greater than  $-0.02/\|\mathbf{u}(n)\|^2$ , and less than  $1.98/\|\mathbf{u}(n)\|^2$ .

MAYYAS, ABOULNASR (1997) derive a condition that guarantees convergence in the mean as:

$$0 < \mu < \frac{2}{\beta + \lambda_{\max}}, \quad (9)$$

where  $\lambda_{\max}$  is the largest eigenvalue of the input signal autocorrelation matrix. Please observe that the representation of the LLMS algorithm in Eq. (2) assumes a positive step size, because  $1 - \mu\beta > 1$  for  $\mu < 0$ , which would make the filter weights divergent. However, we can compare the upper bound on the step size from (MAYYAS, ABOULNASR, 1997) with the condition in (8). To do this, we will substitute  $1 - \mu\beta$  for  $\gamma$  in the right-hand side of (8), and solve the inequality for  $\mu$ . This results in:

$$\mu < \frac{2}{\beta + \sum_{i=0}^{L-1} u^2(n-i)} = \frac{2}{\beta + \|\mathbf{u}(n)\|^2}. \quad (10)$$

Thus, the only difference in the upper bound for the step size is in the second component of the denominator, which is equal to the largest eigenvalue of the input signal autocorrelation matrix for the condition provided in (MAYYAS, ABOULNASR, 1997), and is equal to the norm of the input vector for the new condition developed in this paper. Clearly, the condition in MAYYAS, ABOULNASR, (1997) relates the stability to the statistical properties of the input signal, whereas the new condition uses instantaneous properties of the input signal instead. Observe that the latter may be easier to calculate in practical, real-time applications because the estimation of the input signal autocorrelation matrix is time-consuming. The calculation of the input signal squared norm is commonly used, e.g., in the NLMS algorithm.

To summarize, the main difference between the lower bound for the step size known from the literature and the one derived above is that the latter allows for small negative values of the step size, which will be confirmed in Sec. 5 by means of a number of simulation examples.

### 3. Leaky LMS instability sufficient condition

The condition defined by (8) is a stability sufficient condition, which means that if the step size is limited by the upper and lower bounds, the LLMS algorithm is stable. However, if the step size is not within these bounds, we still do not know whether the algorithm is stable or not. Therefore, we use the theorem below to define the instability range. We again refer to control theory, whereby for stationary systems, the following lemma can be found (KACZOREK, 1993).

**Lemma 1.** *The discrete-time, stationary system given by the state equation:*

$$\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n) + \mathbf{B}\tilde{\mathbf{u}}(n), \quad (11)$$

where  $\mathbf{A} \in \mathcal{R}^{L \times L}$  is a state transition matrix, is unstable, if:

$$\sum_{i=1}^L |a_{ii}| = \sum_{i=1}^L |\lambda_i| > L, \quad (12)$$

where  $\lambda_i$  are the eigenvalues of the state matrix  $\mathbf{A}$ .

To apply this lemma to the nonstationary system (5) we shall assume the condition holds at any discrete time  $n$ ; which constitutes the instability sufficient condition.

From Theorem 1 we know that all the eigenvalues but one are equal to  $\gamma$ , and therefore they are all positive. The remaining eigenvalue may be non-negative or negative. We will discuss both cases separately.

#### 3.1. Eigenvalue $\lambda_1 \geq 0$

If the non- $\gamma$  eigenvalue is non-negative, it follows from (6) that:

$$\lambda_1 = \gamma - \mu \sum_{i=0}^{L-1} u^2(n-i) \geq 0. \quad (13)$$

Therefore, again assuming  $\sum_{i=0}^{L-1} u^2(n-i) \neq 0$ , the step size must obey:

$$\mu \leq \frac{\gamma}{\sum_{i=0}^{L-1} u^2(n-i)}. \quad (14)$$

Thus, in this subsection we are dealing with small positive or negative step sizes.

Assuming (14) holds and substituting all positive eigenvalues into (12), yields:

$$\sum_{i=1}^L |\lambda_i| = (L-1)\gamma + \gamma - \mu \sum_{i=0}^{L-1} u^2(n-i) > L. \quad (15)$$

Solving this inequality for  $\mu$ , gives:

$$\mu < \frac{L(\gamma-1)}{\sum_{i=0}^{L-1} u^2(n-i)}. \quad (16)$$

This solution is within the domain, because the right-hand side of (16) is negative or equal to zero, and therefore is smaller than the right-hand side of (14).

The above condition is the LLMS algorithm instability sufficient condition. Comparing this condition with the left-hand side of inequality (8), we notice that to be sure the leaky LMS algorithm is unstable, the negative value of the step size must be  $L$  times larger than in the stability condition. In-between these two limits, we are unable to say whether the algorithm is stable or not.

### 3.2. Eigenvalue $\lambda_1 < 0$

For completeness, we now check the instability sufficient condition for large, positive  $\mu$ , that is for

$$\mu > \frac{\gamma}{\sum_{i=0}^{L-1} u^2(n-i)}. \quad (17)$$

In this case, the eigenvalue  $\lambda_1 < 0$ , and therefore  $|\lambda_1| = -\lambda_1$ . Considering this and substituting the eigenvalues into (12), yields:

$$\sum_{i=1}^L |\lambda_i| = (L-1)\gamma - \gamma + \mu \sum_{i=0}^{L-1} u^2(n-i) > L. \quad (18)$$

Solving the above inequality for  $\mu$  gives:

$$\mu > \frac{L(1-\gamma) + 2\gamma}{\sum_{i=0}^{L-1} u^2(n-i)}. \quad (19)$$

This solution is within the domain, because the right-hand side of (19) is always greater than the right-hand side of (17).

## 4. Leaky NLMS algorithm stability and instability conditions

The results presented in Sec. 2 take particularly simple forms if applied to the leaky normalized LMS algorithm. The leaky NLMS algorithm is defined as:

$$\mathbf{w}(n+1) = \gamma \mathbf{w}(n) + \mu(n) \mathbf{u}(n) e(n), \quad (20)$$

where

$$\mu(n) = \frac{\bar{\mu}}{\sum_{i=0}^{L-1} u^2(n-i)}. \quad (21)$$

Combining (8) and (21), we conclude that for this algorithm to be stable it suffices that:

$$\gamma - 1 < \bar{\mu} < \gamma + 1. \quad (22)$$

For example, with  $\gamma = 0.98$ , the normalized step size should be greater than  $-0.02$ , and should be less than  $1.98$ .

Condition (22) is only a stability sufficient condition. Using (16) and (19), the instability sufficient condition is:

$$\bar{\mu} < L(\gamma - 1) \quad \text{or} \quad \bar{\mu} > L(1 - \gamma) + 2\gamma. \quad (23)$$

One should, however, be aware that following the derivation of the LLMS algorithm in (MAYYAS, ABOULNASR, 1997), a negative step size results in a negative weight of the filter parameters term in the minimised cost function. The authors do not intend to justify such kind of operation of the algorithm for a long period of time, but want to indicate that stability of such solution can be cleverly used for temporary operations, e.g., when retuning the algorithm in case of phase modelling errors in active control.

## 5. Validation by simulations

Unfortunately, it is impossible to perform simulations using standard NLMS implementations provided in Matlab or Simulink because they require the step size to be positive. Simulations with negative step sizes need custom implementation of the LLMS or the leaky NLMS algorithm.

The simulations presented below use the leaky NLMS algorithm, defined by Eqs. (20) and (21). This algorithm has been chosen, because of the very clear and simple stability condition given by (22). Note that if the LLMS algorithm were used, the condition given by (8) should be applied, which connects the stability bound with the squared norm of the input vector. In that case, to operate near the stability bound, one would need to modify the step size in each iteration, based on this norm. Effectively, that would be equivalent to normalization of the step size, as in the NLMS algorithm.

### 5.1. Identification of a 2-parameter FIR filter

To shed some light on the effect of a negative step size on stable operation of the leaky NLMS algorithm, we will initially consider the very basic and simple case of system identification of a two-parameter FIR filter:

$$W(z^{-1}) = w_1 z^{-1} + w_2 z^{-2}. \quad (24)$$

The values of the parameters were selected as  $w_1 = 0.5$  and  $w_2 = 0.8$ . The experiments were performed with non-zero initial conditions:  $w_1(0) = -0.2$ ,  $w_2(0) = 0.2$ .

The filter was excited using a white noise sequence of unit variance. The output was disturbed by another independent white noise sequence, of variance  $10^{-4}$ . The identified model was also in the form of a 2-parameter FIR filter. The leakage factor was  $\gamma = 0.98$ . The simulations were repeated 100 times and averaged (using different noise sequences). The NLMS algorithm without leakage was also simulated for comparison.

For the selected leakage factor, the leaky NLMS algorithm stability sufficient condition is  $-0.02 \leq \bar{\mu} \leq 1.98$ . Figure 2 presents the mean squared error (MSE)<sup>1</sup> evolution for different values of the step size. The NLMS algorithm (without leakage), with the step size  $\bar{\mu} = 0.1$ , converges very fast, giving a MSE value close to the variance of the additive noise. The leaky NLMS algorithm with  $\bar{\mu} = 0.03$  and  $\bar{\mu} = 0.1$  converges, giving MSE values lower than the initial MSE. However, they are considerably higher than for the case without leakage. This is common for the LLMS algorithm, which is a solution to the optimisation problem being a trade-off between minimisation of the MSE and sum of squares of the filter parameters. The leaky NLMS algorithm with a negative step size  $\bar{\mu} = -0.02$  remains stable (i.e. the MSE does not grow unboundedly), although the final value of the MSE is higher than the initial value.

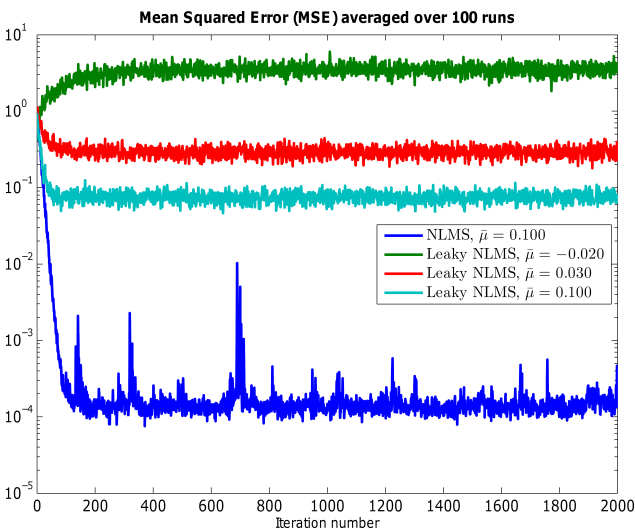


Fig. 2. Identification experiment showing MSE plots for 2-parameter FIR filter, with  $\gamma = 0.98$  for different normalized step sizes.

The best way to understand this behavior is by analysing the parameter trajectories depicted in Fig. 3a. The parameter trajectories represent plots of  $w_1(n)$  vs.  $w_2(n)$ , and contain also the points resulting from the operation  $\gamma \mathbf{w}(n)$  (before the application of the correction term  $\mu(n) \mathbf{u}(n) e(n)$ ). All the trajectories start at the point  $(-0.2, 0.2)$  defined by the initial condition. The trajectory of the NLMS algorithm goes to the optimum point  $(0.5, 0.8)$  in a straight line. This point is the optimum Wiener filter solution, obtained as the product of the inverse input signal autocorrelation matrix and the input-desired signal cross-correlation vector. The filter satisfies the orthogonality principle, and cancels the correlation between the minimised error signal and the input signal over the

<sup>1</sup>By the mean squared error we understand the mean-square value of the difference between the desired signal and the filter output.

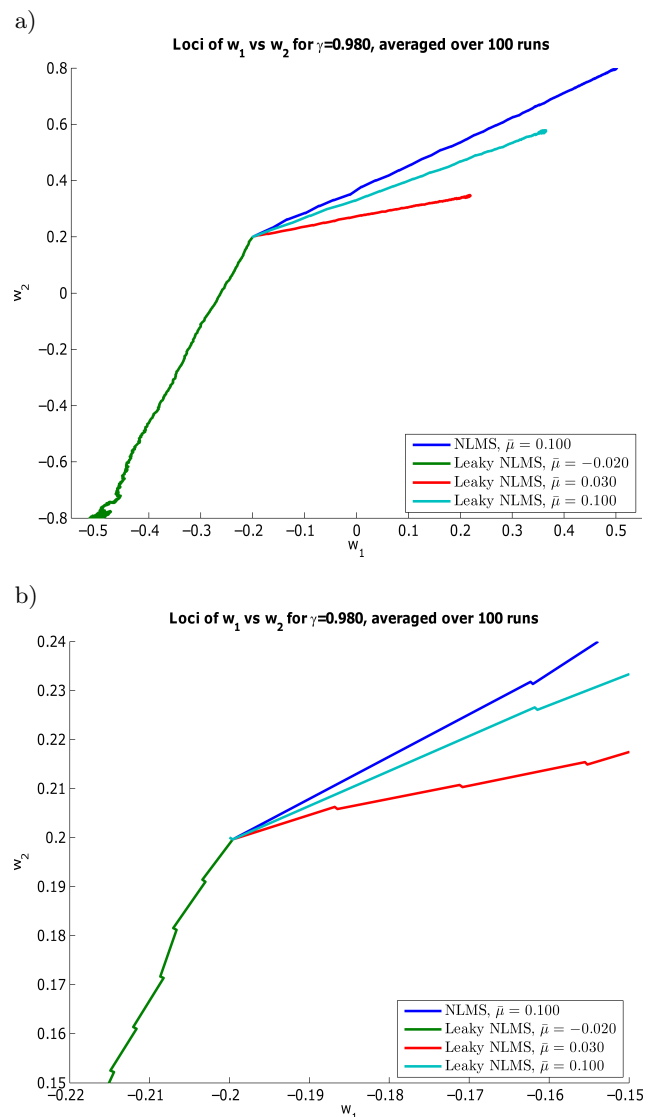


Fig. 3. Identification experiment showing parameter trajectories for a 2-parameter FIR filter, with  $\gamma = 0.98$  for different normalized step sizes: a) the first 100 iterations, b) zoomed plot from (a).

length of the filter. Hence, the MSE is equal to that of the additive noise. As also expected, the trajectories of the leaky NLMS algorithm with positive step sizes go to some vicinity of the optimum point, but do not reach it. This is due to the fact that the leaky NLMS exhibits a bias in the estimate – the inverted matrix in the optimal Wiener filter estimate is then the input signal autocorrelation with the main diagonal extended by the control filter weighting factor in the cost function being minimised (HAYKIN, 2002). Nevertheless, it can be observed that these trajectories tend in the right direction, although they end rather far from the optimum Wiener solution point. The trajectory of the leaky NLMS algorithm with the negative step size, on the other hand, goes in a direction opposite to the optimum point.

Further conclusions may be drawn if the plot from Fig. 3a is zoomed, as in Fig. 3b. From the zoomed plot, we conclude that the leakage pulls the trajectory towards the origin  $(0, 0)$ . The higher the absolute value of the parameter, the stronger the parameters are attracted to zero values. Finally, a balance is established between the two update terms of the filter vector, and the trajectory stops. Therefore, the algorithm, although not convergent to the optimum point, remains stable (i.e., the MSE does not grow unboundedly).

### 5.2. Identification of an IIR filter

As another system identification experiment we now consider an IIR filter with transfer function:

$$K(z^{-1}) = \frac{1}{(z - 0.8)(z - 0.9)}. \quad (25)$$

First, the system was excited using a white noise sequence of unit variance. The output was disturbed with another independent white noise sequence, of variance  $10^{-2}$ . The identified model was in the form of a FIR filter with 10 parameters. The leakage factor was  $\gamma = 0.98$ . The simulations were repeated 100 times with different noise sequences, and the mean squared error (MSE) was averaged. The results of these experiments, for different values of the step size, are presented in Fig. 4a.

For the leakage factor  $\gamma = 0.98$  and the assumed model length equal to 10, the stability sufficient condition is  $-0.02 < \bar{\mu} < 1.98$  and the instability sufficient condition is:  $\bar{\mu} < -0.2$  or  $\bar{\mu} > 2.16$ . From Fig. 4a, it is clear that the leaky NLMS algorithm remains stable (although not convergent) for  $\bar{\mu} \geq -0.02$ ; moreover, it is even stable for  $\bar{\mu} = -0.03$ . On the other hand, the algorithm is unstable for  $\bar{\mu} < -0.2$ .

The fact that the leaky NLMS algorithm remains stable with  $\bar{\mu} = -0.03$  can be explained by considering that the condition in (22) is the stability *sufficient* condition only. To make it clearer, the results from another experiment are presented in Fig. 4b. The experimental conditions were the same, except for the input signal (excitation), which was a square wave. Now the algorithm remains stable for  $\bar{\mu} \geq -0.02$ , and the value  $\bar{\mu} = -0.021$  makes it unstable.

Figure 5 presents the MSE for IIR identification experiments with white noise input and the step size equal to  $-0.51$ , for different values of the leakage factor. The cyan curve is for  $\gamma = 0.95$ ; in this case the stability sufficient condition is  $-0.05 \leq \bar{\mu} \leq 1.95$ , and the instability sufficient condition is  $\bar{\mu} < -0.5$  or  $\bar{\mu} > 2.4$ . The step size has slightly lower value than the instability sufficient condition, and therefore the leaky NLMS algorithm is unstable. The red curve is for  $\gamma = 0.9$  and the green curve is for  $\gamma = 0.8$ ; in both cases the step size falls somewhere between the instability sufficient condition and the stability sufficient condition,

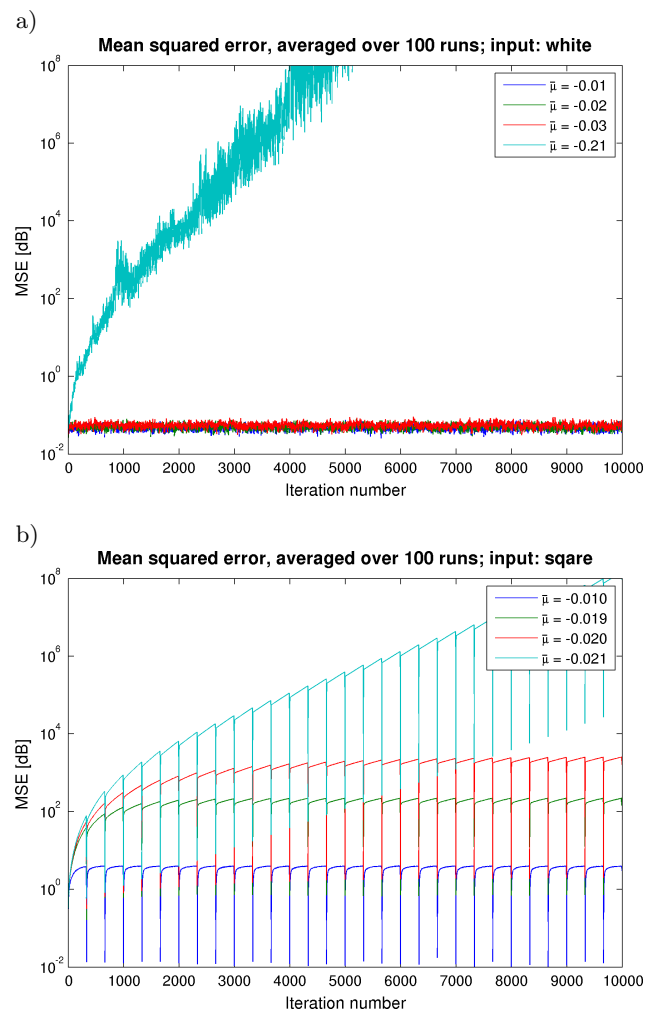


Fig. 4. Identification experiment showing MSE plots for an IIR filter,  $\gamma = 0.98$ , for different normalized step sizes: a) white noise excitation, b) square wave excitation.

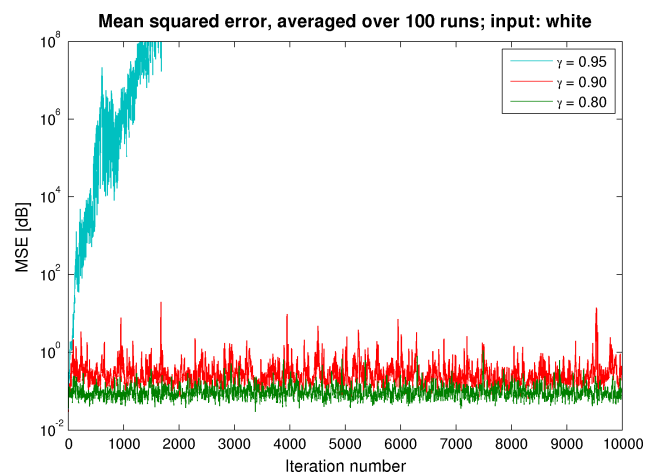


Fig. 5. Identification experiment showing MSE plot for an IIR filter, white noise excitation,  $\mu = -0.5$ , for different leakage factors.

and in both cases, the leaky NLMS algorithm is stable. The latter two leakage factors are, however, unusually low and not suitable for practical applications, which

clearly demonstrates the correctness of the developed theory, and is important from a cognitive point of view.

Many additional simulations were also performed to make sure the adaptive filter remains stable in different scenarios if the step size obeys the derived stability condition. These included different identified transfer functions, different filter lengths (including long filters with more than 100 taps), different excitation signals, different experiment duration and different numbers of averages. In all cases, a small negative step size allowed for stable operation of the leaky NLMS algorithm.

### 5.3. Adaptive line enhancement

Another experiments were performed for a different application of the leaky NLMS algorithm – the adaptive line enhancer (ALE) (HAYKIN, 2002). The ALE is a very popular application, frequently used in audio and speech processing (LATOS, PAWELCZYK, 2010). The input signal for the ALE was a speech recording, sampled at 8 kHz, disturbed by four sinusoids of different constant frequencies. This application was chosen due to the fact that a speech signal is nonstationary, and therefore neither the independence theory, nor the small-step-size theory applies. Since the derivations presented in this paper do not use either the independence theory assumptions, or the small-step-size theory assumptions, the results can be applied to such nonstationary signals as well.

For the experiments presented below, the ALE length and the decorrelation delay were both equal to 10. The leakage factor was again  $\gamma = 0.98$ , and the stability and instability conditions remain the same as described above. The experimental results are presented in Fig. 6. The leaky NLMS algorithm remains stable for  $\bar{\mu} \geq -0.05$ , but a step size equal to  $-0.11$  makes it unstable. This is again in agreement with the theoretical stability and instability sufficient conditions developed above.

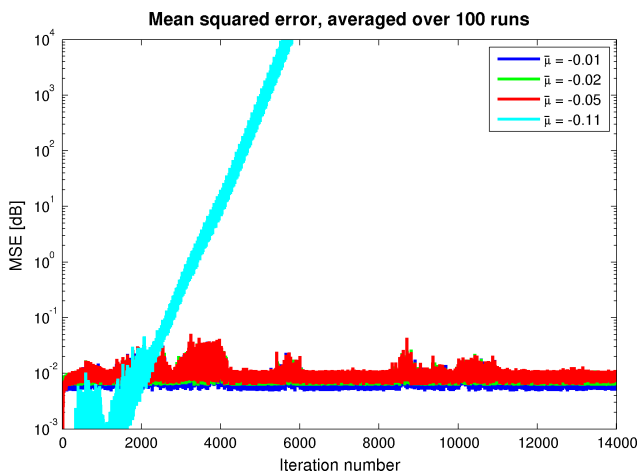


Fig. 6. Simulations of the adaptive line enhancer with different normalized step sizes.

### 5.4. Active noise control

Active noise control (ANC) applications usually use the filtered-x LMS (FxLMS) structure, which differs from classical adaptive filtering applications by the presence of a secondary path and its estimate (KUO, MORGAN, 1996). This difference seriously complicates the stability and convergence analysis. However, it might be expected that the conclusions from Subsec. 5.1 apply to ANC applications as well: the leakage also stabilizes the system with the FxLMS adaptation (ELLIOTT *et al.*, 1987). To confirm this, the following simulations were performed.

The ANC system considered was a classical feed-forward system (refer to (KUO, MORGAN, 1999) or (KUO, MORGAN, 1999) for details), using a sampling frequency of 1 kHz. The primary and secondary path transfer functions were modeled as FIR filters with 256 coefficients – the models were obtained during the identification experiments, using real data acquired from the active casing (WRONA, PAWELCZYK, 2013). A perfect secondary path model was initially assumed. The reference signal consisted of narrowband noise, with energy concentrated between 150 and 250 Hz, and a sinusoid signal, with frequency 200 Hz. The variance of the noise was 0.01, and the amplitude of the sinusoid was 1. The ANC filter length was 64 and it was adapted using the leaky NLMS algorithm, with a leakage factor  $\gamma = 0.98$ , as previously considered.

The simulation results are presented in Fig. 7. The NLMS algorithm (without leakage) converges in about one second to a value of approximately  $10^{-3}$ . The leaky NLMS algorithm with  $\bar{\mu} = -0.02$  (which is within the stability sufficient limits) and with  $\bar{\mu} = -0.03$  (which is outside the stability sufficient limits) does not converge, but the MSE does not grow unboundedly. The leaky NLMS algorithm with  $\bar{\mu} = -0.04$  and  $-0.05$  is unstable.

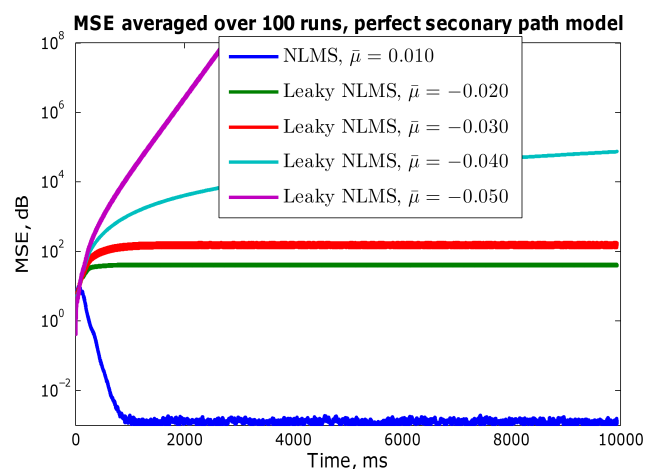


Fig. 7. Simulations of active noise control system, with perfect secondary path model.

As mentioned above, the presence of the secondary path complicates the behavior and analysis of the FxLMS algorithm. Moreover, imperfect secondary path modeling adds another degree of complication, as can be concluded from Fig. 8a, showing simulation results for the same ANC system, but with an imperfect secondary path model, where only the first 70 coefficients (out of 256) of the impulse response were used. In this case, the leaky NLMS algorithm is stable even with  $\bar{\mu} = -0.04$ .

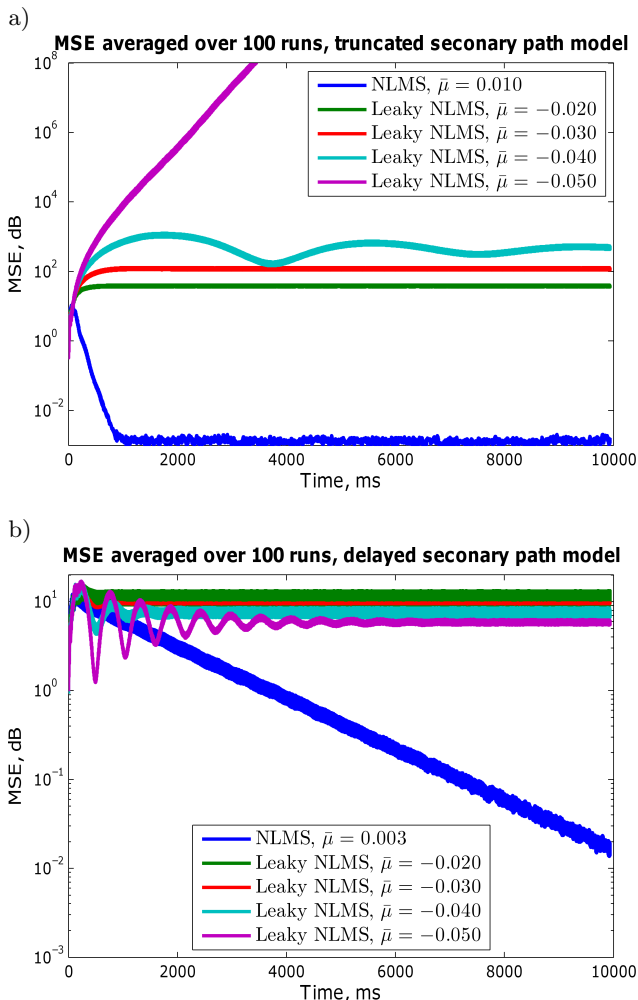


Fig. 8. Simulations of active noise control system: a) with truncated secondary path model, b) with error in the delay estimation.

Finally, in the experiments presented in Fig. 8b, the secondary path impulse response estimate was assumed to be identical to the actual secondary path, except that it is shifted by an additional discrete time delay ( $\hat{S}(z^{-1}) = z^{-1}S(z^{-1})$ ). The error in the time delay estimate constitutes a very difficult condition for operation of the LMS algorithm because it results in an error in the phase response, which increases with the frequency. Therefore, in this case, the convergence time of the NLMS algorithm (without delay) is more than ten times longer than in the previous cases. How-

ever, the influence on the leaky NLMS algorithm with the negative step size is the opposite: the algorithm is stable for all selected step sizes, and the final MSE is even lower than in the previous cases (less amplification was achieved).

The authors do not foresee any application where a negative step size selected on purpose could be advantageous over a positive step size. However, the authors are aware of an application where knowledge that a negative step size does not lead to immediate divergence could be advantageous. This application is an ANC system without the secondary path model, where a decision about the step size sign change is made based on current error behavior (convergence or divergence). If the Leaky LMS algorithm is used, it must be accounted for that for some value of the step size there may be no divergence even if the step size has a wrong sign.

## 6. Conclusion

In this paper, we have shown that the commonly used stability necessary condition expressed as  $\mu > 0$  is improper for the leaky LMS algorithm. With this algorithm, the step size may take a small negative value (depending on the leakage factor) and the algorithm may still be stable in the sense that the MSE will not grow unboundedly. However, in all the simulations the final MSE was worse when a negative step size was used, compared to simulations with positive step size values. Stability and instability sufficient conditions have been derived based on control system theory. These theoretical results were achieved with a very small set of assumptions; it was only assumed that the signals are real valued and bounded. Although these results do not give direct guidelines on how to choose the step size for practical applications, they contribute to an understanding of the behaviour of the leaky LMS algorithm. Knowledge about the allowed small negative step-size values can change parameterisation of the leaky-LMS-based algorithms used for active control of sound or vibration (as well as for other applications), with and without secondary path modeling which is of great practical importance.

## Appendix. Proof of the main theorem

In the following proof, for clarity of a presentation, the LMS stability matrix (5) will be expressed as:

$$\mathbf{A} = \gamma\mathbf{I} - \mu\mathbf{u}\mathbf{u}^T, \quad (26)$$

(the time index  $n$  has been omitted). It is assumed that the adaptive filter length, and therefore also the input vector length, and both dimensions of the LMS stability matrix  $\mathbf{A}$  are equal to  $L$ .

First, consider that each of the columns (or rows) of the matrix  $\mathbf{u}\mathbf{u}^T$  is linearly dependent on all other



columns (rows); therefore the rank of this matrix is equal to 1, and therefore only one of its eigenvalues is non-zero. The direct and easy-to-prove result is that the LLMS stability matrix (26) has  $L-1$  eigenvalues equal to  $\gamma$ .

Now consider right-multiplication of the LMS stability matrix (26) by the vector  $\mathbf{u}$ :

$$\begin{aligned} \mathbf{A}\mathbf{u} &= (\gamma\mathbf{I} - \mu\mathbf{u}\mathbf{u}^T)\mathbf{u} = \gamma\mathbf{u} - \mu\mathbf{u}\mathbf{u}^T\mathbf{u} \\ &= \mathbf{u}(\gamma - \mu\mathbf{u}^T\mathbf{u}). \end{aligned} \quad (27)$$

As  $\mathbf{u}^T\mathbf{u}$  on the right-hand side of the above equation is a scalar, being an inner product of the vector  $\mathbf{u}$  by itself, the above equation can be expressed as:

$$\mathbf{A}\mathbf{u} = \left( \gamma - \mu \sum_{i=0}^{L-1} u_i^2 \right) \mathbf{u}, \quad (28)$$

where  $u_i$  denotes  $u(n-i)$ . Equation (28) may also be viewed as a definition of the eigenvalue and the associated eigenvector. This concludes the proof.

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