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Numerical approach to reliability evaluation of two-state consecutive "*k* out of *n*: F" systems

Keywords

two-state system, consecutive "k out of n : F" system, reliability, algorithm

Abstract

An approach to reliability analysis of two-state systems is introduced and basic reliability characteristics for such systems are defined. Further, a two-state consecutive "k out of n: F" system composed of two-state components is defined and the recurrent formulae for its reliability function are proposed. The algorithm for numerical approach to reliability evaluation is given. Moreover, the application of the proposed reliability characteristics and formulae to reliability evaluation of the system of pump stations composed of two-state components is illustrated.

1. Introduction

The assumption that the systems are composed of two-state components gives the possibility for basic analysis and diagnosis of their reliability. This assumption allows us to distinguish two states of system reliability. The system works when its reliability state is equal to 1 and is failed when its reliability state is equal to 0. In the stationary case the system reliability is the independent of time probability that the system is in the reliability state 1. The main results determining the stationary reliability and the algorithms for numerical approach to this reliability evaluation for consecutive "*k* out of *n*: F" systems are given for instance in [1], [5]-[6]. An exemplary technical consecutive "*k* out of *n*: F" system can be found in [3]. There is considered the ordered sequence of n relay stations $E_1, E_2, \ldots, E_n$, which have to reroute a signal from a source station E_0 to a target station E_{n+1}. A range of each station is equal to *k*. It means, when the station E_i, $i = 0,1,...,n,$ is operating, it sends a signal directly to a station $E_{i+1},..., E_{\min(i+k,n)}$. The failed station does not send any signal. The probability of efficiency of the stations E_0 and E_{n+1} is equal to 1. The signal from E_0 to E_{n+1} cannot be sent, if at least *k* consecutive stations out of $E_1, E_2, \ldots, E_n$, are damaged.

The paper is devoted to extension of these stationary results to the non-stationary case and applying them in transmitting then for two-state consecutive "k out of n: F" systems with dependent of time reliability functions of system components ([3]). Then, the reliability function, the lifetime mean value and the lifetime standard deviation are basic characteristics of the system.

2. Reliability of two-state consecutive "k out of n: F" systems

In the non-stationary case of two-state reliability analysis of consecutive "*k* out of *n*: F" systems we assume that ([3]):

- n is the number of system components,
- E_i, $i = 1,2,...,n,$ are components of a system,
- T_i are independent random variables representing the lifetimes of components E_i, $i = 1,2,...,n,$
- $R_i(t) = P(T_i > t), t \in < 0, \infty),$ is a reliability function of component E_i, $i = 1,2,...,n,$
- $F_i(t) = 1 - R_i(t) = P(T_i \le t), t \in < 0, \infty),$ is the

distribution function (unreliability function) of component E_i, $i = 1,2,...,n.$

Definition 1. A two-state system is called a two-state consecutive "k out of n: F" system if it is failed if and only if at least its k neighbouring components out of n its components arranged in a sequence of E_1, E_2, ..., E_n, are failed.

The following auxiliary theorem is proved in [3], [6].

Lemma 1. The stationary reliability of the two-state consecutive "k out of n: F" system composed of components with independent failures is given by the following recurrent formula

$$\boldsymbol{R}_{k,n} = \begin{cases} 1 & \text{for } n < k, \\ 1 - \prod_{j=1}^{n} q_j & \text{for } n = k, \\ p_n \boldsymbol{R}_{k,n-1} & \\ + \sum_{i=1}^{k-1} p_{n-i} \boldsymbol{R}_{k,n-i-1} & \\ \cdot \prod_{j=n-i+1}^{n} q_j & \text{for } n > k, \end{cases} \quad (1)$$

where

– p_i is a stationary reliability coefficient of component E_i, $i = 1,2,...,n,$
– q_i is a stationary unreliability coefficient of component E_i, $i = 1,2,...,n,$
– $R_{k,n}$ is the stationary reliability of consecutive "k out of n: F" system.

After assumption that:

– $T_{k,n}$ is a random variable representing the lifetime of a consecutive "k out of n: F" system,
– $R_{k,n}(t) = P(T_{k,n} > t), t \in < 0, \infty),$ is the reliability function of consecutive "k out of n: F" system,
– $F_{k,n}(t) = 1 - R_{k,n}(t) = P(T_{k,n} \le t), t \in < 0, \infty),$ is the distribution function of consecutive "k out of n: F" system,

we can formulate the following result.

Lemma 2. The reliability function of the two-state consecutive "k out of n: F" system composed of components with independent failures is given by the following recurrent formula

$$\boldsymbol{R}_{k,n}(t) = \begin{cases} 1 & \text{for } n < k, \\ 1 - \prod_{j=1}^{n} F_j(t) & \text{for } n = k, \\ R_n(t) \boldsymbol{R}_{k,n-1}(t) & \\ + \sum_{i=1}^{k-1} R_{n-i}(t) \boldsymbol{R}_{k,n-i-1}(t) & \\ \cdot \prod_{j=n-i+1}^{n} F_j(t) & \text{for } n > k, \end{cases} \quad (2)$$

for $t \in < 0, \infty).$

Motivation. When we assume in formula (1) that

$p_i(t) = R_i(t),\ q_i(t) = F_i(t)$ for $t \in < 0, \infty),$
$i = 1,2,...,n,$

we get formula (2).

From the above theorem, as a particular case for the system composed of components with identical reliability functions, we immediately get the following corollary.

Corollary 1. If components of the two-state consecutive "k out of n: F" system are independent and have identical reliability functions, i.e.

$R_i(t) = R(t),\ F_i(t) = F(t)$ for $t \in < 0, \infty),$
$i = 1,2,...,n,$

then the reliability function of this system is given by

$$\boldsymbol{R}_{k,n}(t) = \begin{cases} 1 & \text{for } n < k, \\ 1 - [F(t)]^n & \text{for } n = k, \\ R(t) \boldsymbol{R}_{k,n-1}(t) & \\ + R(t) \sum_{i=1}^{k-1} F^i(t) & \\ \cdot \boldsymbol{R}_{k,n-i-1}(t) & \text{for } n > k, \end{cases} \quad (3)$$

for $t \in < 0, \infty).$

In further considerations we will used the following reliability characteristics:

- the mean value of the system lifetime,

$$E[T_{k,n}] = \int_0^\infty \boldsymbol{R}_{k,n}(t)dt, \quad (4)$$

- the second order ordinary moment of the system lifetime,

$$E[T_{k,n}^2] = 2\int_0^\infty t\,\boldsymbol{R}_{k,n}(t)dt, \quad (5)$$

- the standard deviation of the system lifetime,

$$\sigma = \sqrt{D[T_{k,n}]}, \quad (6)$$

where

$$D[T_{k,n})] = E[T_{k,n}^2] - (E[T_{k,n}])^2. \quad (7)$$

3. Algorithm for reliability evaluation of a two-state consecutive „*k* out of *n*: F" system

For numerical approach to evaluation of the reliability characteristics, given by (3)-(6), we use the trapezium rule of numerical integration.
In particular situation, for $t_0 = 0,$ step h, we have

$$E[T_{k,n}] = \int_0^\infty \boldsymbol{R}_{k,n}(t)dt$$

$$= \frac{h}{2}\sum_{i=0}^{n-1}\left[R_{k,n}(t_0 + ih) + R_{k,n}(t_0 + (i+1)\cdot h)\right], \quad (8)$$

$$E[T_{k,n}^2] = 2\int_0^\infty t\,\boldsymbol{R}_{k,n}(\mathrm{t})dt$$

$$= h\sum_{i=0}^{n-1}\{(t_0 + ih)\cdot R_{k,n}(t_0 + ih)$$

$$+ (t_0 + (i+1)\cdot h)\cdot R_{k,n}(t_0 + (i+1)\cdot h)\}. \quad (9)$$

Necessary in (7)-(8) values of function $\boldsymbol{R}_{k,n}$(t) are calculated from (2) using the following algorithm.

Algorithm 1.

1. Given: t, k, n, F(t), R(t);
2. If k > n then $\boldsymbol{R}_{k,n}(t) = 1$
3. else if k = n $\boldsymbol{R}_{k,n}(t) = 1 - [F(t)]^n$
4. else
5. for i = 0 to t do
6. {
7. for j = 1 to k – 1 do
8. temp := temp + $[F(i)]^j \cdot R_{k,n-j-1}(i)$;
9. $\boldsymbol{R}_{k,n}(i) = R(i)\cdot R_{k,n-1}(i) +$ temp;
10. }

where

- k is a length of the sequence of consecutive components,
- n is a number of all components in sequence,
- t is an end of the time interval,
- *F(t)* is a distribution function of components,
- *R(t)* is a reliability function of components.

Example implementation of *Algorithm 1* and formulas (3), (8)-(9) in the D programming language is given in Appendix.

4. Application

From *Corollary 1*, in a particular case, substituting $k = 3$ in (3), we get:

- for $n = 1$

$$\boldsymbol{R}_{3,1}(t) = 1 \text{ for } t \in <0,\infty), \quad (10)$$

- for $n = 2$

$$\boldsymbol{R}_{3,2}(t) = 1, \text{ for } t \in <0,\infty), \quad (11)$$

- for $n = 3$

$$\boldsymbol{R}_{3,3}(t) = 1 - F^3(t) \text{ for } t \in <0,\infty), \quad (12)$$

- for $n \geq 4$

$$\boldsymbol{R}_{3,n}(t) = R(t)\,\boldsymbol{R}_{3,n-1}(t) + R(t)F(t)\,\boldsymbol{R}_{3,n-2}(t)$$

$$+ R(t)[F(t)]^2\,\boldsymbol{R}_{3,n-3}(t) \text{ for } t \in <0,\infty), \quad (13)$$

Example 1. Let us consider the pump stations system with $n = 20$ pump stations $E_1, E_2, ..., E_{20}$. We assume that this system fails when at least 3 consecutive pump stations are down. Thus, the considered pump stations system is a two-state consecutive "3 out of 20: F" system, and according to (9)-(12), its the reliability function is given by

$$\boldsymbol{R}_{3,20}(t) = R(t)\ \boldsymbol{R}_{3,19}(t) + R(t)F(t)\ \boldsymbol{R}_{3,18}(t) + R(t)[F(t)]^2\ \boldsymbol{R}_{3,17}(t) \quad (14)$$

for $t \in < 0, \infty)$.

In the particular case when the lifetimes T_i, of the pump stations E_i, $i = 1,2,\ldots,20$ have exponential distributions of the form

$$F(t) = 1 - e^{-0.01t} \text{ for } t \geq 0,$$

i.e. if the reliability functions of the pump stations E_i, $i = 1,2,\ldots,20$ are given by

$$R(t) = e^{-0.01t} \text{ for } t \geq 0,$$

considering (9)-(12), (13) we get the following recurrent formula for the reliability $\boldsymbol{R}_{3,20}(t)$ of pump stations system

$$\boldsymbol{R}_{3,1}(t) = 1 \text{ for } t \in < 0, \infty), \quad (15)$$

$$\boldsymbol{R}_{3,2}(t) = 1 \text{ for } t \in < 0, \infty), \quad (16)$$

$$\boldsymbol{R}_{3,3}(t) = 1 - [1 - e^{-0.01t}]^3 \text{ for } t \in < 0, \infty), \quad (17)$$

$$\boldsymbol{R}_{3,n}(t) = e^{-0.01t}\ \boldsymbol{R}_{3,n-1}(t) + e^{-0.01t}[1 - e^{-0.01t}]\ \boldsymbol{R}_{3,n-2}(t) + e^{-0.01t}[1 - e^{-0.01t}]^2\ \boldsymbol{R}_{3,n-3}(t) \text{ for } t \in < 0, \infty), \quad (18)$$
$n = 4,5,...,20.$

The values of reliability function of the system of pump stations given by (14), calculated by the computer programme based on the formulae (10)-(18) and *Algorithm 1*, are presented in *Table 1* and illustrated in *Figure 1*.

Table 1. The values of the two-state reliability function of the pump stations system for $\lambda = 0.01$

t	$\boldsymbol{R}_{3,20}(t)$	$2t\ \boldsymbol{R}_{3,20}(t)$
0.0	1.0000	0.0000
5.0	0.9980	9.9800
10.0	0.9859	19.7189
15.0	0.9583	28.7499
20.0	0.9137	36.5474
25.0	0.8535	42.6743
30.0	0.7811	46.8657
35.0	0.7008	49.0561
40.0	0.6170	49.3614
45.0	0.5337	48.0347
50.0	0.4541	45.4117
55.0	0.3805	41.8584
60.0	0.3144	37.7282
65.0	0.2564	33.3331
70.0	0.2066	28.9274
75.0	0.1647	24.7024
80.0	0.1299	20.7893
85.0	0.1016	17.2662
90.0	0.0787	14.1688
95.0	0.0605	11.5004
100.0	0.0462	9.2416
105.0	0.0350	7.3588
110.0	0.0264	5.8107
115.0	0.0198	4.5531
120.0	0.0148	3.5426
125.0	0.0109	2.7385
130.0	0.0081	2.1044
135.0	0.0060	1.6082
140.0	0.0044	1.2229
145.0	0.0032	0.9255
150.0	0.0023	0.6974
155.0	0.0017	0.5235
160.0	0.0012	0.3916
165.0	0.0009	0.2918
170.0	0.0006	0.2168
175.0	0.0004	0.1607
180.0	0.0003	0.1188
185.0	0.0002	0.0876
190.0	0.0002	0.0644
195.0	0.0001	0.0473
200.0	0.0000	0.0347
205.0	0.0000	0.0253
210.0	0.0000	0.0185
215.0	0.0000	0.0135

220.0	0.0000	0.0098
225.0	0.0000	0.0072
230.0	0.0000	0.0052
235.0	0.0000	0.0038
240.0	0.0000	0.0027
245.0	0.0000	0.0020
250.0	0.0000	0.0014
255.0	0.0000	0.0010
260.0	0.0000	0.0007
265.0	0.0000	0.0005
270.0	0.0000	0.0004
275.0	0.0000	0.0003
280.0	0.0000	0.0002
285.0	0.0000	0.0001

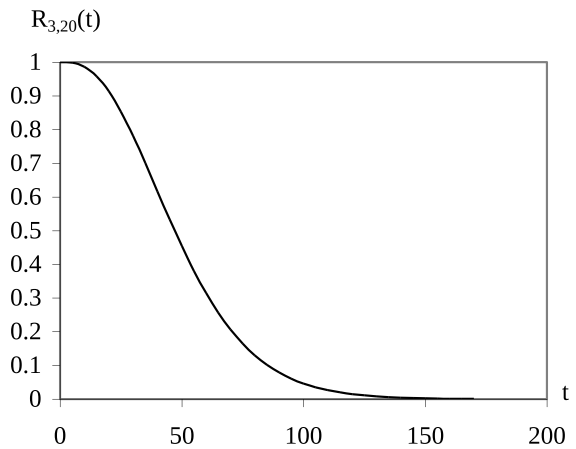

Figure 1. The graph of the pump stations system reliability function

Using the values given in the *Table 1*, the formulae (4)-(9) and numerical integration we find:

- the mean value of the pump stations system lifetime

$$E[T_{3,20}] = \int_0^\infty \boldsymbol{R}_{3,20}(t)dt \cong 50.8639,$$

- the second order ordinary moment of the pump stations system lifetime

$$E[T^2_{3,20}(1)] = 2\int_0^\infty t\,\boldsymbol{R}_{3,20}(t)dt \cong 3246.69,$$

- the standard deviation of the pump stations system lifetime

$$\sigma = \sqrt{D[T_{3,20}]} = \sqrt{659.558} \cong 25.6819.$$

5. Conclusion

Two recurrent formulae for two-state system reliability functions, a general one for non-homogeneous and its simplified form for homogeneous two-state consecutive "k out of n: F" systems have been proposed. The algorithm for reliability evaluation of two-state consecutive "k out of n: F" system has been shown as well. The formulae and algorithm for two-state reliability function of a homogeneous two-state consecutive "k out of n: F" system have been applied to reliability evaluation of the pump stations system. The considered pump stations system was a two-state consecutive "3 out of 20: F" system composed of components with exponential reliability functions. On the basis of the recurrent formula and the algorithm for two-state pump stations system reliability function the approximate values have been calculated and presented in table and illustrated graphically. On the basis of these values the mean value and standard deviation of the pump stations system lifetime have been estimated.

The input structural and reliability data of the considered pump stations system have been assumed arbitrarily and therefore the obtained its reliability characteristics evaluations should be only treated as an illustration of the possibilities of the proposed methods and solutions.

The proposed methods and solutions and the software are general and they may be applied to any two-state consecutive "*k* out of n: F" systems.

Appendix

We present the D programming language code for formulas (3), (8)-(9) and *Algorithm 1*.

```
import std.stdio;
import std.stream;
import std.math;
import std.string;

const real LAMBDA1 = 0.01;

real Ft(real t) {
   return (1-exp(-(LAMBDA1)*t));
}


real Rt(real t) {
   return exp(-(LAMBDA1)*t);
}

real SigmaFi(real ii, real k, real t, real n) {
   real result = 0;
   for(real i = ii; i < k; i++) {
      result += pow(Ft(t),i)*Rkn(t,k,n-i-1);
   }
```

```
   return result;
}


real Rkn(real t, real k, real n) {
   if (n < k)
      return 1;
   if (n == k)
      return 1 – pow(Ft(t),n);
   return Rt(t)*(Rkn(t, k, n-1) + SigmaFi(1, k, t, n));
}


real trapeziumT(real k, real n, uint p, real t){
   real integ = 0;
   real step = 0;

   step=t/p;

   for(real i = 0; i < p; i = i + step){
      integ += (((Rkn(i, k, n) +
        Rkn(i + step, k, n))*step)/2);
   }
   return integ;
}


real trapezium2T(real k, real n, uint p, real t){
   real integ = 0;
   real step = 0;

   step=t/p;

    for(real i=0; i < p; i = i + step){
     integ +=  (((i*Rkn(i, k, n) +
      (i + step)*Rkn(i + step, k, n))*step));
    }
    return integ;
}


int main(char[][] args) {
   real integral = 0;
   real integral1 = 0;
   real dif = 0;
   real sq = 0;

   if (args.length < 3) {
      writefln("Usage:\n   "~ args[0] ~" t k n\n");
      return 0;
   }


   for(real i = 0; i < atoi(args[1]); i = i + 5){
     writefln("%s\t%4s\t%4s\t%s", i, Rkn(i,
              atoi(args[2]), atoi(args[3])), 2*i*Rkn(i,
              atoi(args[2]),atoi(args[3])), 1 - Rkn(i,
              atoi(args[2]), atoi(args[3])) );
   }
   integral=trapeziumT(atoi(args[2]), atoi(args[3]),
    atoi(args[4]) );

   integral1=trapezium2T(atoi(args[2]),  atoi(args[3]),
   atoi(args[4]) );

   diff=(integral1)-pow(integral,2);
   sq=sqrt(diff);

   writefln("The mean value of the system lifetime");
   writefln("%s", integral );
   writefln("The second order ordinary moment of the
       system lifetime");
   writefln("%s", integral1 );
   writefln("%s", diff);
   writefln("The standard deviation of the system
       lifetime");
  writefln("%s",sq);

   return 0;
}
```

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