Economical assigning weapons to targets

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Some upper and lower bound are proposed to use in branch and bound method for solving Weapon-Target Assignment (WTA) problems. Analyze of WTA inspires to formulate another problem to economize the number of weapons under the condition that the establish threshold value of destroying targets is achieved. This requirement generates an additional constrain generating the set of feasible solutions of WTA.

Keywords: WTA problem, assignment, economize.

1. Introduction

Weapon-Target Assignment problem belongs to main tools that support making decisions by commanders. From the mathematical point of view WTA is the special case of nonlinear integer optimization problem. Exact and heuristic algorithms for solving WTA are presented in [1]. Combinatorial optimization techniques applied to solve WTA we can find in [9]. Static and dynamic models of WTA are considered in [7]. Static models do not take into account the opponent's response. The examples of attrition process during the battle are described in [2], [4].

2. WTA problem

We can formulate the WTA problem in following form

$$\max \sum_{j=1}^{n} V_j (1 - \prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}}) \quad (1)$$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i \quad i = \overline{1,m} \quad (2)$$

$$x_{ij} \ge 0 \quad \text{and integer } i = \overline{1,m}, \quad j = \overline{1,n} \quad (3)$$

where

x_{ij} – the number of weapons of type i to be assigned to destroy the target j,

p_{ij} – the probability of destroying target j by a single weapon (shoot) of type i

V_j – the value of the target j

a_i – the number of weapons of type i available to be assigned to destroy targets.

The total expected value of destroyed targets can be expressed in form

$$\sum_{j=1}^{n} V_j (1 - \prod_{i=1}^{m} (q_{ij})^{x_{ij}}) = \\ = \sum_{j=1}^{n} V_j - \sum_{j=1}^{n} V_j (\prod_{i=1}^{m} (q_{ij})^{x_{ij}}) \quad (4)$$

where

q_{ij} – the probability of survival of target j when a single weapon of type i is assigned to it.

Because the first part of (4) is constant the problem (1)–(3) can be transformed into following problem

$$\min \sum_{j=1}^{n} V_j (\prod_{i=1}^{m} (q_{ij})^{x_{ij}}) \quad (5)$$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i \quad i = \overline{1,m} \quad (6)$$

$$x_{ij} \ge 0 \quad \text{and integer } i = \overline{1,m}, \quad j = \overline{1,n} \quad (7)$$

This resulting problem we can meet for example in [1] and [7].

Let us denote by S the set of feasible solution of problem (5)-(7)

$S = \{x = (x_{ij})_{m \times n} : x \ \ satisfies \quad (6),(7)\}$.

We observe that the set

$$T = \left\{ \begin{array}{l} x = (x_{ij})_{m \times n} : \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \le \sum_{i=1}^{m} a_i = A \\ x_{ij} \ge 0, \quad and \ \text{int}.\ i = \overline{1,m},\ j = \overline{1,n} \end{array} \right\} \quad (8)$$

contains the set S ($S \subset T$).

So the problem

$$\min_{x \in T} \sum_{j=1}^{n} V_j \left(\prod_{i=1}^{m} (q_{ij})^{x_{ij}}\right) \qquad (9)$$

can be seen as a relaxation of the problem (5)–(7).
An optimal solution of this problem provides an lower bound which can be used in branch and bound method for solving (5)–(7) problem.

3. Case of uniform weapons

The relaxation (9) is correctly done from the mathematical point of view but the formula (8) has the practical meaning when the weapons are uniform.
In this case

$p_{ij} = p_j \quad i = \overline{1,m}$ and $q_{ij} = q_j \quad i = \overline{1,m}$.

The index i can mean the i-th place for the deployment of weapons, therefore

x_{ij} – the number of weapons from the place i to be assigned to destroy the target j .

Assuming that the weapons are **uniform** we can construct the following problem

$$\min \sum_{j=1}^{n} V_j (q_j)^{x_j} \qquad (10)$$

subject to

$$\sum_{j=1}^{n} x_j \le A \qquad (11)$$

$$x_j \ge 0 \text{ and integer } \quad i = \overline{1,m}, \quad j = \overline{1,n} \qquad (12)$$

where

$$x_j = \sum_{i=1}^{m} x_{ij} \ . \qquad (13)$$

The problem (10)–(12) belongs to the class of nonlinear knapsack problems and can be solved by methods described for example in [3].
Two special cases of the problem (10)–(12) are important.
Let us introduce the following notations

$q_j^1 = \min_{i,j} q_{ij}, \qquad q_j^2 = \max_{i,j} q_{ij}$.

Taking into account (13) we can formulate two problems

$$\min \sum_{j=1}^{n} V_j (q_j^1)^{x_j} \qquad (14)$$

subject to

$$\sum_{j=1}^{n} x_j \le A \qquad (15)$$

$$x_j \ge 0 \text{ and integer } \quad i = \overline{1,m}, \quad j = \overline{1,n} \qquad (16)$$

and

$$\min \sum_{j=1}^{n} V_j (q_j^2)^{x_j} \qquad (17)$$

subject to

$$\sum_{j=1}^{n} x_j \le A \qquad (18)$$

$$x_j \ge 0 \text{ and integer } \quad i = \overline{1,m}, \quad j = \overline{1,n} \ . \qquad (19)$$

An optimal solution x_{ij}^1 of the first problem (14)–(16) provides a lower bound for the WTA problem.
Similarly, an optimal solution x_{ij}^2 of the second problem (17)–(19) provides an upper bound for WTA problem.
These two auxiliary problems allow to apply branch and bound method for solving WTA problem (5)–(7).
We should add that useful upper bound for WTA problem one can obtain taking an feasible solution $x^{'} \in S$.
The value of such upper bound equals

$$\sum_{j=1}^{n} V_j \left(\prod_{i=1}^{m} (q_{ij})^{x_{ij}^{'}}\right) . \qquad (20)$$

4. Economical assigning weapons

Modern weapons are very expensive. This fact obliges decision makers to take into account the cost of assigning weapons.
To realize such requirements we propose to formulate and solve the following problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (21)$$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i \quad i = \overline{1,m} \qquad (22)$$

$$\sum_{j=1}^{n} V_j \left(1 - \prod_{i=1}^{m} (1 - p_{ij})^{x_{ij}}\right) \ge V^T \qquad (23)$$

$$x_{ij} \ge 0 \text{ and integer, } i = \overline{1,m}, \ j = \overline{1,n} \qquad (24)$$

where

V^T – the threshold value of destroying targets.
Now, we can write the equivalent problem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (25)$$

subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i \quad i = \overline{1,m} \qquad (26)$$

$$\sum_{j=1}^{n} V_j \prod_{i=1}^{m} (q_{ij})^{x_{ij}} \leq \sum_{j=1}^{n} V_j - V^T \qquad (27)$$

$$x_{ij} \geq 0 \quad \text{and integer}, i = \overline{1,m}, \quad j = \overline{1,n} \qquad (28)$$

Let us denote by

$S^E = \{x = (x_{ij})_{m \times n} : x \text{ } satysfies \text{ } (26) - (28)\}$

the set of feasible solutions of the problem (25)–(28).

We can obtain two sets related to S^E.

Setting q_j^1 into (27) we have

$$\sum_{j=1}^{n} V_j \prod_{i=1}^{m} (q_j^1)^{x_{ij}} \leq \sum_{j=1}^{n} V_j - V^T$$ which is equivalent to

$$\sum_{j=1}^{n} V_j (q_j^1)^{\sum_{i=1}^{m} x_{ij}} \leq \sum_{j=1}^{n} V_j - V^T. \qquad (29)$$

Setting q_j^2 into (27) we have

$$\sum_{j=1}^{n} V_j \prod_{i=1}^{m} (q_j^2)^{x_{ij}} \leq \sum_{j=1}^{n} V_j - V^T$$ which is equivalent to

$$\sum_{j=1}^{n} V_j (q_j^2)^{\sum_{i=1}^{m} x_{ij}} \leq \sum_{j=1}^{n} V_j - V^T. \qquad (30)$$

One can observe the following relations.

The set

$T^1 = \{x = (x_{ij})_{m \times n} : x \text{ } satisfies \text{ } (26), (29), (28)\}$

contains the set S^E ($S^E \subset T^1$).

The set S^E contains the set

$T^2 = \{x = (x_{ij})_{m \times n} : x \text{ } satisfies \text{ } (26), (30), (28)\}$

($T^2 \subset S^E$).

These observations allow us to use the problem

$$\min_{x \in T^1} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (31)$$

for computing a lower bound in branch and bound method to solve problem (25)–(28).

Respectively, the problem

$$\min_{x \in T^2} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (32)$$

allows to compute an upper bound in procedure of branch and bound.

If we can determine the threshold value V_j^T for j-th target,

where

$$V^T = \sum_{j=1}^{n} V_j^T, \quad V_j^T \leq V_j, \quad j = \overline{1,n}$$

then economical assigning problem expresses in the form

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (33)$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leq a_i \quad i = \overline{1,m} \qquad (34)$$

$$\prod_{i=1}^{m} (q_{ij})^{x_{ij}} \leq \frac{V_j - V_j^T}{V_j}, \; j = \overline{1,n} \qquad (35)$$

$$x_{ij} \geq 0 \quad \text{and integer}, i = \overline{1,m}, \quad j = \overline{1,n} \qquad (36)$$

The constrain (31) can be transformed into form

$$\sum_{i=1}^{m} \log(q_{ij})^{x_{ij}} \leq \log \frac{V_j - V_j^T}{V_j}, \; j = \overline{1,n} \qquad (37)$$

and finally

$$\sum_{i=1}^{m} x_{ij} \log q_{ij} \leq \log \frac{V_j - V_j^T}{V_j}, \; j = \overline{1,n} \qquad (38)$$

The problem (33), (34), (36), (38) belongs to the class of linear integer optimization problems and can be solved using branch and bound method.

When we assume $c_{ij} = 1$, then the number of weapons will be minimized.

5. Conclusions

Presented optimization problems belong to NP-hard class. To solve them we are forced to apply methods belonging to the following class: cutting planes, heuristics or branch and bound. Some proposal of combination of these methods contains [8]. We propose to use branch and bound because the structure of problems allows to construct an upper and a lower bound.

6. Bibliography

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Ekonomiczny przydział środków do niszczenia celów

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W pracy zaproponowano górne i dolne oszacowanie optymalnej wartości funkcji celu użyteczne w zastosowaniu do metody podziału i oszacowań rozwiązywania zadania przydziału środków ataku na cele nieprzyjaciela.
Analiza tego problemu (WTA) zainspirowała do sformułowania zadania minimalizacji liczby środków niszczenia przy spełnieniu warunku osiągnięcia lub przekroczenia założonego progu wartości zniszczonych celów. To wymaganie wymusza konieczność wprowadzenia dodatkowego ograniczenia do zestawu ograniczeń definiujących dopuszczalne rozwiązania problemu WTA.

Słowa kluczowe: WTA problem, przydział, przydział ekonomiczny.