Realization of uniform approximation by applying mean-square approximation

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In the paper the polynomial mean-square approximation method was applied, where the applied criterion was the value of the maximum error of the obtained approximation. The value of this error depends on the number of approximation points within the range. By changing the number of points within the range, it can be noticed that the value of the maximum error has the minimum value for a particular value of L number of considered points. For a polynomial of N degree, the optimum number of equidistant points of approximation L and the maximum error of approximation are determined. The proposed method was compared with a uniform approximation method, namely the Chebyshev polynomial. The examples included in the paper show that the proposed method yields smaller values of the maximum error than Chebyshev polynomial.

KEYWORDS: mean-square approximation, uniform approximation, Chebyshev polynomial

1. Mean-square approximation uniform approximation

It is assumed that the series of points $x_0, x_1, ..., x_L$ and the values of function *f(x)* in these points are given:

$$f_i = f(x_i) \; ; \; i = 0,1,...,L \tag{1}$$

The following polynomial approximation is applied:

$$fa(x) = \sum_{j=0}^{N} a_j x^j \tag{2}$$

By replacing function f(x) with approximating function *fa(x)* the following error is obtained:

$$\varepsilon_i = fa(x_i) - f_i \tag{3}$$

Mean-square approximation means minimization of the following expression:

$$\sum_{i=0}^{L} \varepsilon_i^{\,2} = \min \tag{4}$$

Using the necessary condition of existence of a multiple-variable function extremum, from equations (1), (2), (3), (4), the following system of equations is obtained:

$$\sum_{j=0}^{N} c_{jk} a_j = b_k \ ; \quad k = 0,1,...,N \tag{5}$$

where: $c_{jk} = \sum_{i=0}^{L} x_i^{\,j+k}$; $b_k = \sum_{i=0}^{L} x_i^{\,k} f_i$.

The solution of the system of equations (5) in the form of a matrix is described by the formula:

$$A = C^{-1} B \tag{6}$$

where $A = \lfloor a_j \rfloor$ - vector of coefficients of the polynomial fa(x) (equation (2)), $C = \lfloor c_{jk} \rfloor$; $B = \lfloor b_k \rfloor$.

For the case when the number of points x_i is larger than the degree of polynomial N ($L > N$), it is actually approximation. However, for $L = N$ there is the case of interpolation. The equations provided above are applicable for both the approximation and the interpolation.

Apart from a mean-square approximation (equation (4)) there is also a uniform approximation, which is defined by the condition:

$$\max_{0 \le i \le L} |\varepsilon_i| = \min \tag{7}$$

where ε_i is described by (3).

In accordance with equation (7) the point is to minimize the maximum value of error within the interval under study. Among various methods applied in the uniform approximation, there are: Remez algorithm [2, 3], Pade approximations, Maclaurin series and Chebyshev polynomials [1, 3].

The scope of this paper is limited to Chebyshev polynomials, where their application consisted in solving the problem of interpolation (L = N), where the knots fulfill the condition:

$$x_i = \cos\left[\frac{(2i+1)\pi}{2(N+1)}\right] \quad \text{where: } i = 0,1,...,N \tag{8}$$

i.e. they are roots of the Chebyshev polynomial.
The solution is still determined by equation (6).

In the paper a method is proposed which draws on the fact that the maximum error of a solution obtained using the mean-square approximation method is highly dependent on the number of included points L + *1*. Assuming the degree of polynomial N, figure L is changed and the value of the maximum error is determined. For a determined optimum value L_o, which ensures the smallest value of the maximum error, the values of polynomial (2) coefficients a_j are determined. The method is based on equations (5) and (6), which refer to the mean-square approximation, but which refers to minimization of the maximum error. The results of the proposed method will be compared with the results obtained for the Chebyshev polynomials.

2. Calculation examples

Due to the application of the Chebyshev polynomials, the following variability interval is assumed

$$x \in \langle -1,1 \rangle \tag{9}$$

Function *f(x)* is given by equation:

$$f(x) = arctg(4x) \tag{10}$$

The proposed method consists in analyzing the value of error:

$$|eQ_i| = |fQ(x_i) - f_i| \tag{11}$$

where $fQ(x_i)$ - result of the mean-square approximation including *LA* points of approximation.

Symbol EQ refers to the maximum error for a given value *LA*:

$$EQ = \max_{0 \le i \le LA} |eQ_i| \tag{12}$$

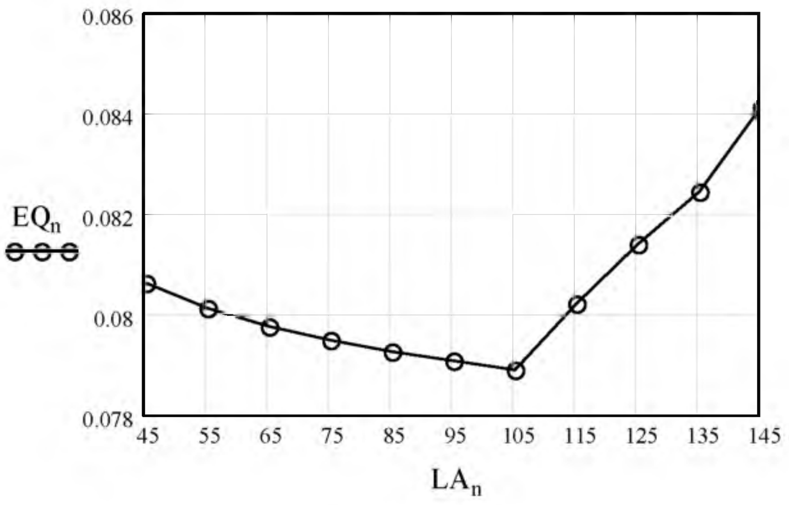


Fig. 1. Dependency of the value of maximum error EQ on the number of included points of approximation LA for a polynomial of degree five (N = 5)

Using Fig. 1 such a value LA is found which corresponds to the minimum value of the maximum error EQ. The optimum value of the number of approximation points equals LO = 105.

In Fig. 2 absolute values of the error obtained with the proposed method $|eQ_i|$ and the error of the Chebyshev polynomials method were compared:

$$|eC_i| = |fC(x_i) - f_i| \tag{13}$$

where: $fC(x_i)$- approximating function obtained for the Chebyshev polynomials.

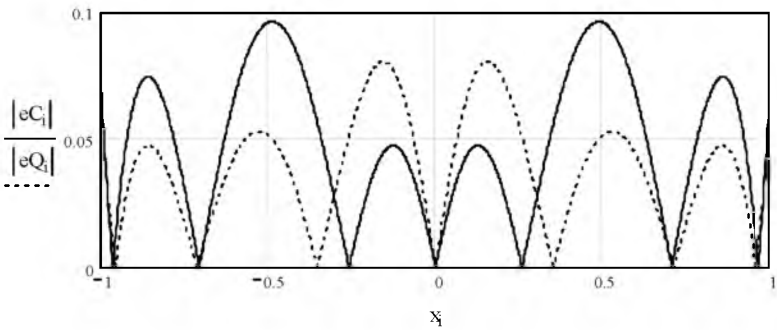


Fig. 2. Absolute values for the polynomial of degree five (N = 5): solid line $|eC_i|$ (equation (13)) and dashed line $|eQ_i|$ (equation (11), LO = 105)

On the basis of Fig. 2 the maximum values of errors are determined, which equal EC = 0.0963 for the Chebyshev method EQ = 0.0789 for the proposed method. The determined value EQ = 0.0789 corresponds to the minimum value in Fig. 1.

The described process is repeated for the following values N = 3, 4, 14 (degree of the polynomial). The results of these calculations are presented in Fig. 3, where EA refers to the maximum value of the error for the classic mean-square approximation conducted for L = 5000 points.

Fig. 3 shows that in the case of the Chebyshev polynomials, the error for the polynomial of degree three (N = 3) is smaller than for degree four N = 4. This situation reoccurs for the following pairs: N = 5/N = 6; N = 7/N = 8... It results from the fact that function $f(x) = arctg(4x)$ is an odd function in the interval under study $x \in \langle -1,1 \rangle$. However, for error EQ (EA) it can be observed that the value of error determined for N = 3 is identical to the one determined for N = 4. Therefore, it is advisable to include only odd values of a degree of polynomial N, which is the case in Fig. 4.

Fig. 4 proves that, when considering all the three methods, the proposed method yields the smallest value of the maximum error: $EQ < EC < EA$. It should be noticed that the classic approximation (EA) is only slightly worse than the Chebyshev polynomials method (EC).

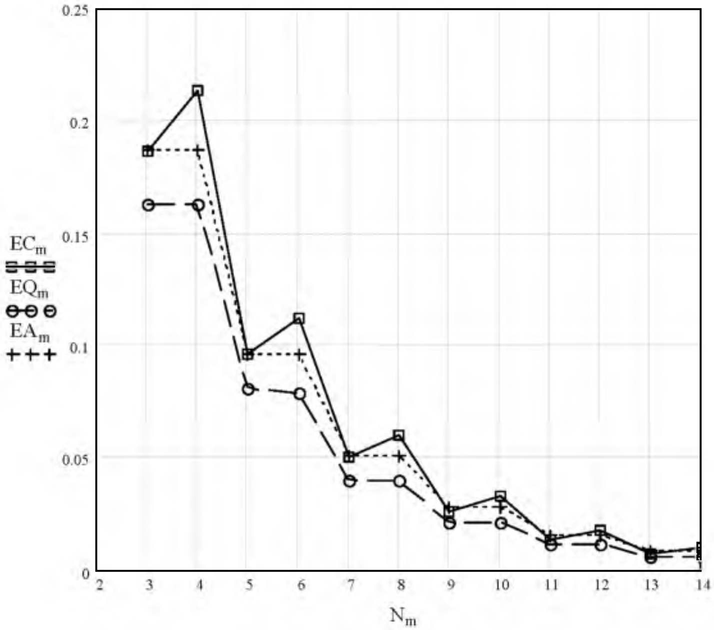


Fig. 3. Maximum absolute values of error in a function of polynomial degree N. Solid line with rectangles EC represents the Chebyshev polynomial, dashed line with circles EQ shows the error of an optimum polynomial and dashed line with pluses EA shows the error of the classic approximation

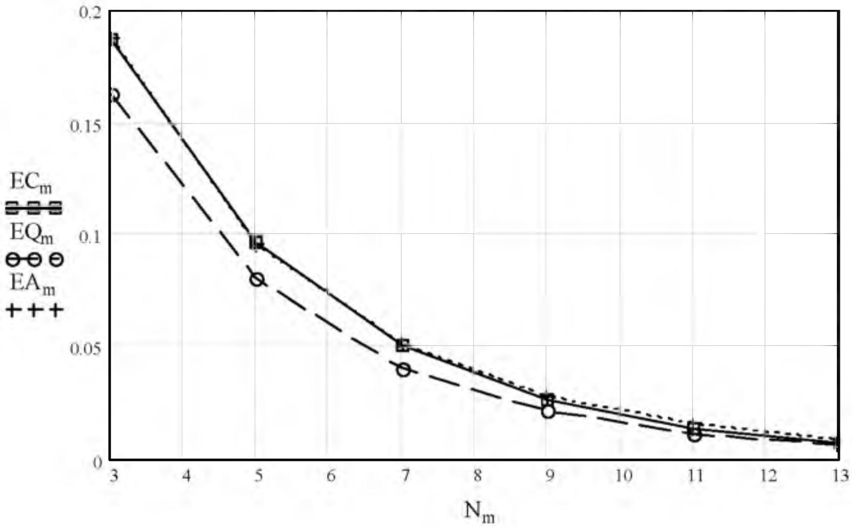


Fig. 4. Maximum absolute values of the error in a function of a polynomial degree for odd N The labels used in this figure are identical to the ones in Fig. 3

Figure 5 presents the dependence of the number of optimum points of approximation LO on the degree of polynomial N. This dependence can be described by a linear function of a form:

$$LT_m = -1.495 + 21.229 \cdot N_m \tag{14}$$

As another example the following function was examined:

$$f(x) = \frac{1}{1 + 25 \cdot x^2} \tag{15}$$

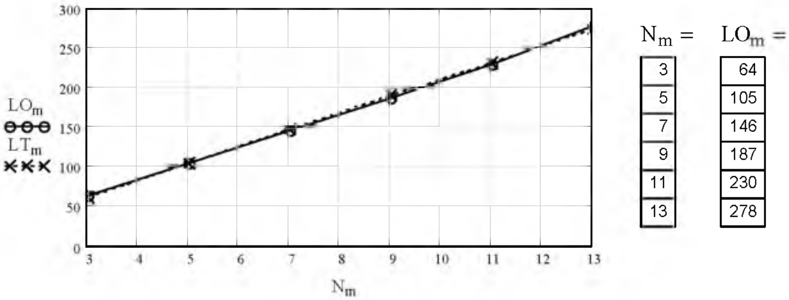


Fig. 5. Dependence of the number of optimum points of approximation LO on the degree of polynomial N. Solid line with circles represents empirical values and dotted line with x represents the results of the linear approximation (theoretical dependence, equation (14))

Fig. 6 includes the maximum absolute values of the error in the function of a degree of polynomial N. The situation is similar to the one in Fig. 3 but the difference is that the smaller values of error are obtained for even values of the degree of polynomial N. The reason for the difference lies in the fact that the function described by equation (15) is even.

Figure 7 shows that the results (EQ) of the proposed method are burdened with the smallest values of the maximum error. Interestingly, the classic mean-square approximation method yields a smaller maximum error than the Chebyshev polynomials method (EA < EC).Perhaps the reason lies in the fact that in the process, L = 5000 points of approximation were taken into account, which approximately corresponds to the approximation conducted for a continuous function. This fact may be important, as in the case of interpolation conducted for function (15) a strong Runge-Kutta effect can be observed. As for the Chebyshev polynomials method, it is an example of interpolation with unevenly spaced knots.

Fig. 8 presents the dependence of the number of optimum points of approximation LO on the degree of polynomial, where N is an even number.

Yet another example consists in determining errors of approximation for a function described by the following dependence

$$f(x) = \ln(x + 1.01) \tag{16}$$

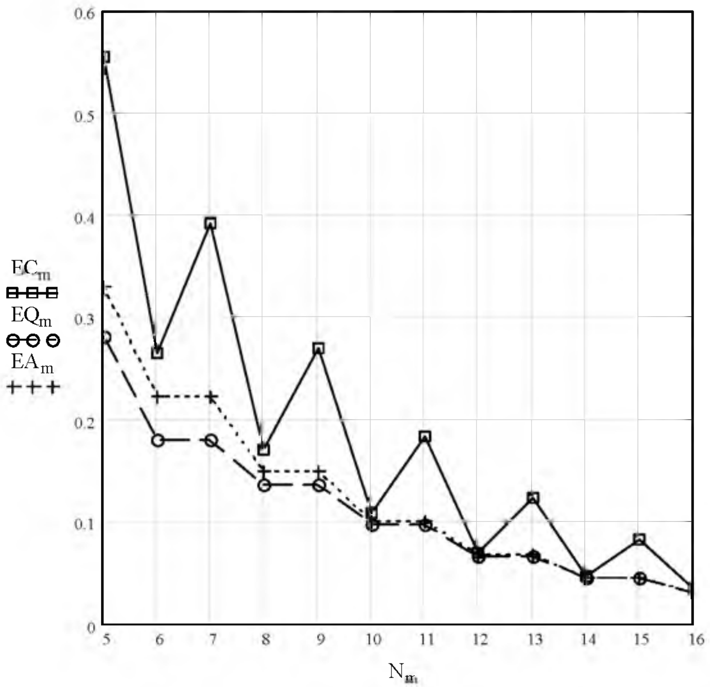


Fig. 6. Maximum absolute values of error in the function of a degree of polynomial N. The labels used in this figure are identical to the ones in Fig. 3

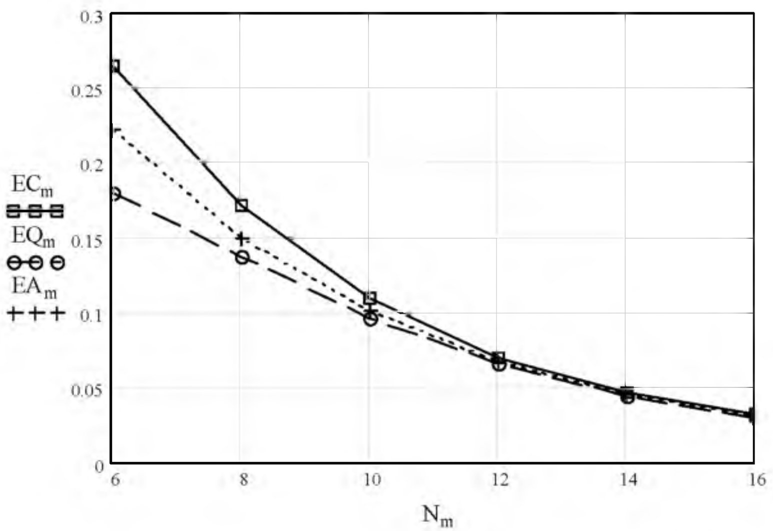
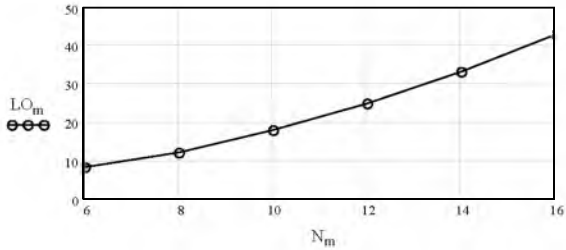


Fig. 7. Maximum absolute values of error in the function of a degree of polynomial N, where N is an even value. The labels used in this figure are identical to the ones in Fig. 3



N_m =	LO_m =
6	8
8	12
10	18
12	25
14	33
16	43

Fig. 8. Dependence of the number of optimum points of approximation LO on a degree of polynomial N

Figure 9 presents the curve of the function described by equation (16) for a given interval, $x \in \langle -1,1 \rangle$.

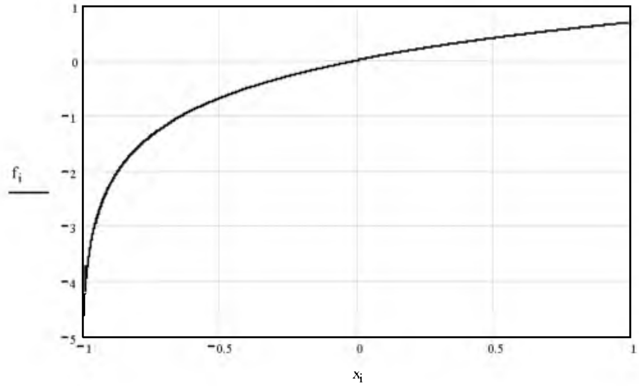


Fig. 9. Curve of the function described by equation (16) for a given interval, $x \in \langle -1,1 \rangle$

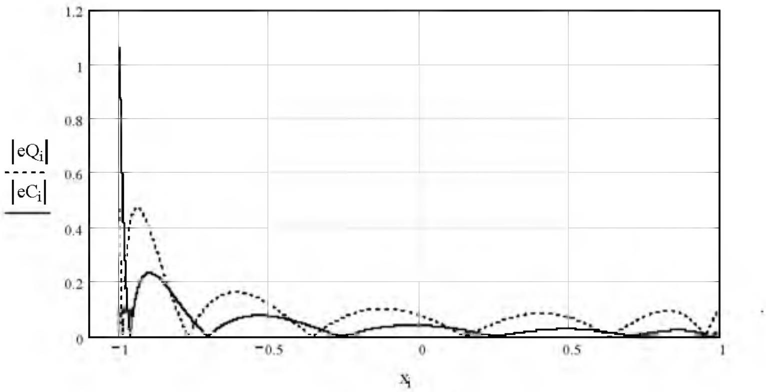


Fig. 10. Absolute values of the error for the polynomial of degree 5 (N=5): solid line $|eC_i|$ and dashed line $|eQ_i|$ (LO=31)

Figure 10 includes examples of absolute values of the error for a polynomial of degree 5 (N = 5): solid line $|eC_i|$ and dashed line $|eQ_i|$ (LO = 31). From Fig. 10 maximum values of errors can be identified, which equal EC = 1.063 for the Chebyshev method and EQ = 0.474 for the proposed method.

Fig. 11 presents the values of the maximum error for particular methods. It clearly shows the advantage of the proposed method over the other methods. Apart from that, the error of the Chebyshev method is similar to the error of the classic mean-square approximation method.

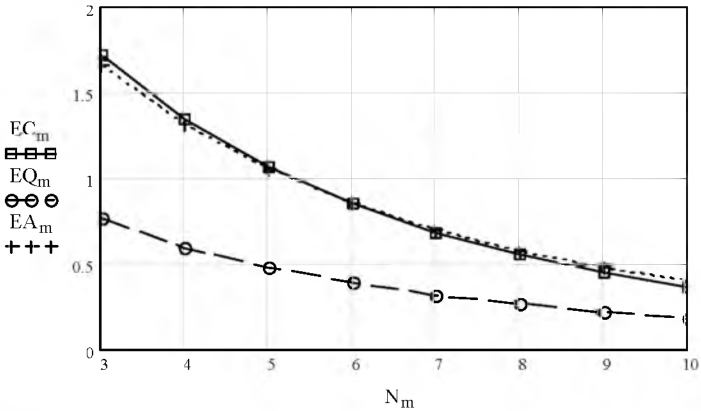
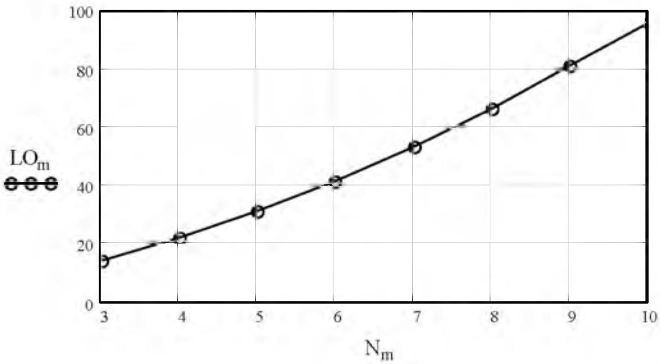


Fig. 11. Values of the maximum error for particular methods. Labels as in Fig. 3



N_m =	LO_m =
3	14
4	22
5	31
6	41
7	53
8	66
9	81
10	96

Fig. 12. Dependence of the number of optimum points of approximation LO on a degree of polynomial N

By introducing a new variable identified as:

$$L_m = \frac{LO_m}{N_m} \tag{17}$$

for which the linear approximation was conducted, the following was obtained:

$$LT_m = 2.643 + 0.701 \cdot N_m \tag{18}$$

Based on Fig.13, which illustrates the dependence of variable L on a degree of polynomial N, it can be observed that equation (18) fits empirical data.

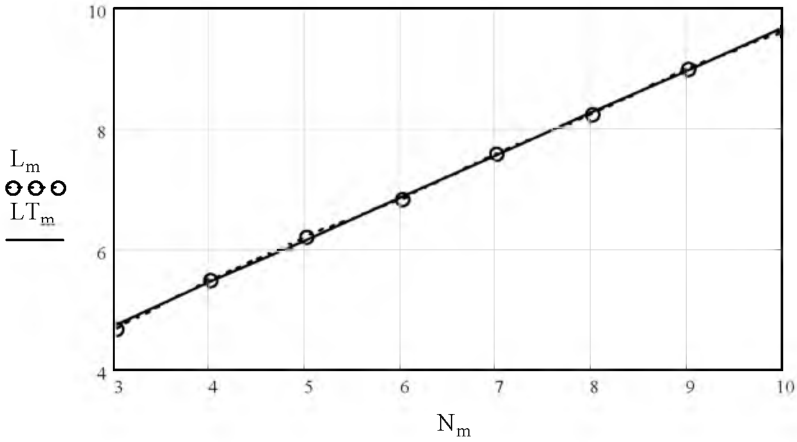


Fig. 13. Dependence of variable L (equation (17)) on the degree of polynomial N. Solid line with circles represents empirical values and dotted line with x represents the results of the linear approximation (theoretical dependence, equation(18))

As the final example, a following function is examined:

$$f(x) = \cos\left(\frac{\pi}{2} \cdot x\right) \tag{19}$$

Taking into account the evenness of cosine function, only even values of a degree of polynomial N were considered. Fig. 14 presents maximum values of error for particular methods. Due to small values of the error, a logarithmic scale was applied. Fig. 14 indicates close similarity of error values for the methods under study. With a view of explaining this problem, quotients of errors are introduced in the forms:

$$ec_m = \frac{EC_m}{EQ_m}; \qquad ea_m = \frac{EA_m}{EQ_m} \tag{20}$$

Figure 15 shows that the ratio of the maximum error for the Chebyshev method to the maximum error for the proposed method falls within interval $ec_m \in \langle 1.39, 1.89 \rangle$. Similarly, for the classic approximation method, the ratio falls within $ea_m \in \langle 1.74, 2.13 \rangle$. It clearly proves a considerable advantage of the proposed method over the other methods, which was not clearly visible in Fig.14.

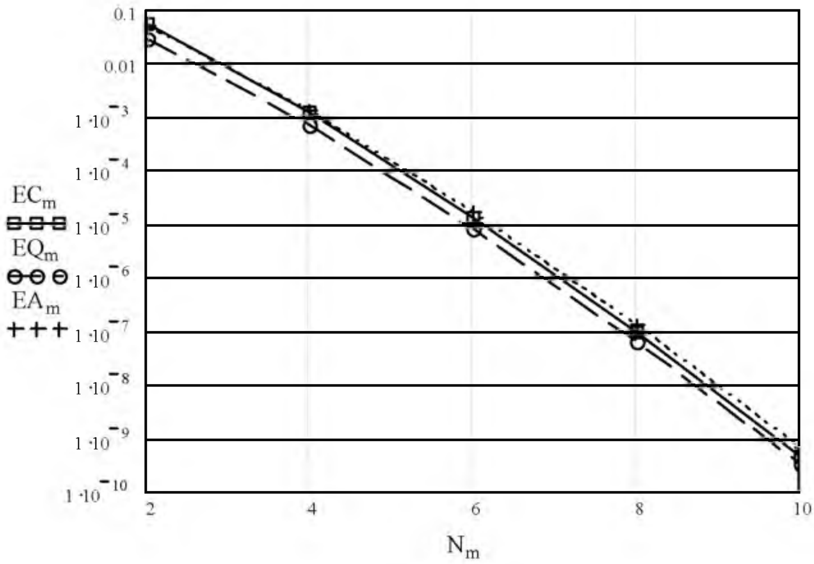


Fig. 14. Values of the maximum error for particular methods. Labels as in Fig. 3

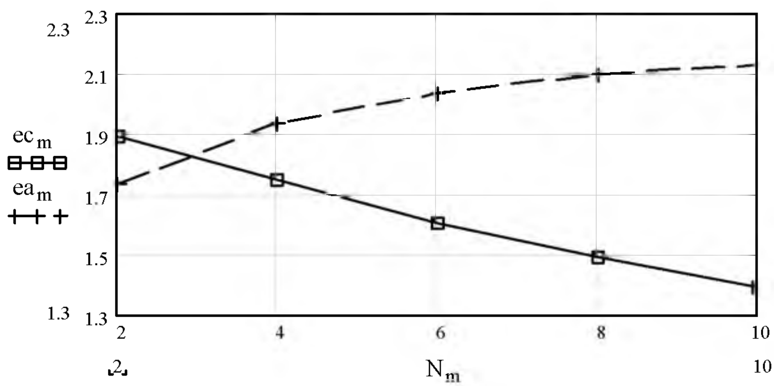


Fig. 15. Values of quotients of maximum errors (equation (20))

3. Summary

On the basis of the aforementioned examples, it can be concluded that the Chebyshev polynomials method has no considerable advantage over the mean-square approximation method: smaller values of the maximum error EC than EA were observed for the function described by equation (19), similar values EC and EA were obtained for function (10) and (16), however for function (15) EA < EC is the case.

In all the examined cases, the proposed method yields smaller values of the maximum error, yet the most substantial difference is observed for function (16), where quotients derived from equation (20) fulfill inequality $ec_m > 2$; $ea_m > 2$.

While analyzing particular examples, the dependence of the optimum number of approximation points LO on a degree of polynomial N was provided. However no theoretical dependence, true for all the examples, could be found. Hence the only way is to determine optimum LO empirically. The algorithm is as follows:

1. As the number of approximation points assume LA=L and LA=L+1, if EQ(L) > EQ(L+1), then (according to Fig.1) assume LA=L+2;
2. if EQ(LA) < EQ(LA+1),then take LA as a result;
3. the algorithm can be made faster by enlarging the step size, e.g. from one to five.

References

[1] Fortuna Z., Macukow B., Wąsowski J, Numerical methods, WNT, Warsaw, 1993 (in Polish).

[2] Jankowscy J. i M., Overview of numerical methods and algorithms Vol.1, WNT, Warsaw, 1981 (in Polish).

[3] Ralston A. R., Introduction to numerical analysis, PWN, Warsaw, 1983 (in Polish).