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NOTE ON THE STABILITY OF THE SYSTEM OF FUNCTIONAL EQUATIONS

Abstract. We deal with the system of functional equations connected with additive and quadratic mappings. We correct some mistakes made in the paper [W. Fechner, *On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings*, J. Math. Anal. Appl. 322 (2006), 774–786] and provide accurate statements of those results. Moreover, we get the improvement of the Hyers-Ulam stability result of the considered system of functional equations.

1. Introduction

Let $(G, +)$ be an Abelian group and for the rest of this paper we assume that $(X, ||\cdot||)$ is a Banach space. We recall some basic definitions. A map $A\colon G \to X$ is said to be additive iff it satisfies the Cauchy functional equation

$$A(x + y) = A(x) + A(y), \quad x, y \in G.$$

A map $Q\colon G \to X$ is said to be quadratic iff it satisfies the following functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad x, y \in G.$$

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We consider some stability problem connected with the system of two equations

$$\begin{cases} \varphi(x+y) - \varphi(x) - \varphi(y) = 2B(x,y), \\ B(x,-y) = -B(x,y) \end{cases} \tag{1}$$

for all $x, y \in G$, where $B\colon G \to X$ is a biadditive and symmetric mapping and $\varphi\colon G \to X$. The above system of functional equations has been investigated by W. Fechner [4]. Equations (1) are closely associated with characterization of the quadratic mappings (see, e.g., [1]) – it can be easily checked that a function $\varphi = A + Q$ satisfies (1), where A is an additive mapping and Q is a quadratic one. Conversely, under certain assumptions imposed upon G and X considered system of two equations has its general solution of the form $\varphi = A + Q$. Moreover, (1) implies the well-known equation of Drygas (see [2,3]).

Applying some results from [4,5] we give an alternative proof of the Hyers-Ulam stability of (1). Moreover, we obtain some improvements of the approximating constants.

Throughout this paper, by $\mathbb{R}_+$ we denote the set of nonnegative real numbers.

2. Stability of (1)

We start this section with the following auxiliary result which we will use in the sequel.

Lemma 1 (cf. [5]). *Let $(G,+)$ be a group. Assume that $f\colon G \to X$ satisfies the condition*

$$\left\| f(x) - \frac{a+1}{2a^2} f(ax) + \frac{a-1}{2a^2} f(-ax) \right\| \le \delta, \quad x \in G, \tag{2}$$

where a is an integer different from $-1, 0, 1$ and δ is a nonnegative constant. Then there exists a uniquely determined function $\varphi\colon G \to X$ such that

$$\varphi(x) = \frac{a+1}{2a^2}\varphi(ax) - \frac{a-1}{2a^2}\varphi(-ax), \quad x \in G$$

and

$$\|f(x) - \varphi(x)\| \le \frac{|a|}{|a|-1}\delta, \quad x \in G.$$

The following result has been proved by W. Fechner (see [4], Theorem 6).

Theorem 2. *Assume that* $(G,+)$ *is an Abelian group and mappings* $\varepsilon,\eta\colon G\times G\to\mathbb{R}_+$ *satisfy the following conditions*

$$\begin{aligned}
&\lim_{k\to\infty}2^{-k}\varepsilon(2^kx,2^ky)=\lim_{k\to\infty}2^{-k}\eta(2^kx,2^ky)=0, && x,y\in G,\\
&\sum_{k=0}^{\infty}2^{-k}\varepsilon(2^kx,2^kx)<\infty, && x\in G,\\
&\sum_{k=0}^{\infty}2^{-k}\eta(2^kx,2^kx)<\infty, && x\in G,\\
&\sum_{k=0}^{\infty}2^{-k}\varepsilon(2^kx,-2^kx)<\infty, && x\in G.
\end{aligned}$$

If $f\colon G\to X$ *and* $\phi\colon G\to X$ *solve the inequalities*

$$\begin{aligned}
\|f(x+y)-f(x)-f(y)-2\phi(x,y)\|&\le\varepsilon(x,y), && x,y\in G,\\
\|\phi(x,y)+\phi(x,-y)\|&\le\eta(x,y), && x,y\in G,
\end{aligned}$$

then there exist unique functions $\varphi\colon G\to X$ *and* $B\colon G\times G\to X$ *such that (1) holds true and*

$$\|f(x)-\varphi(x)\|\le\frac14\Delta(x)+\frac18\Gamma(x),\quad x\in G,$$

where $\Delta\colon G\to\mathbb{R}_+$ *and* $\Gamma\colon G\to\mathbb{R}_+$ *are given by formulae*

$$\Delta(x)=\sum_{k=0}^{\infty}2^{-k}\left[\delta(2^kx)+\delta(-2^kx)\right],\quad \Gamma(x)=\sum_{k=0}^{\infty}4^{-k}\left[\delta(2^kx)-\delta(-2^kx)\right]$$

for all $x\in G$*, respectively. Moreover,*

$$\delta(x)=\varepsilon(x,x)+\varepsilon(x,-x)+\varepsilon(0,0)+2\eta(x,x)+\frac12\eta(0,0),\quad x\in G$$

and

$$\begin{aligned}
\|\phi(x,y)-B(x,y)\|\le\frac12\varepsilon(x,y)&+\frac18\big[\Delta(x+y)+\Delta(x)+\Delta(y)\big]+\\
&+\frac1{16}\big[\Gamma(x+y)+\Gamma(x)+\Gamma(y)\big],\qquad x,y\in G.
\end{aligned}$$

The approximating mapping δ occuring above has been calculated incorrectly and should be replaced by the following one:

$$\delta(x) = \varepsilon(x,x) + \varepsilon(x,-x) + \varepsilon(0,0) + 2\eta(x,x) + \eta(0,0), \quad x \in G.$$

The failure comes from an inccorect calculation made in the proof of this theorem (see [4], page 781, line 7 from above – there should be $||\phi(0,0)|| \leq \eta(0,0)$ instead of $||\phi(0,0)|| \leq \frac{1}{2}\eta(0,0)$).

As a corollary to Theorem 2 one can obtain the following result (see [4], Corollary 3).

Corollary 3. *Assume that $(G,+)$ is an Abelian group and $\varepsilon, \eta > 0$. If $f\colon G \to X$ and $\phi\colon G \times G \to X$ satisfy*

$$\|f(x+y) - f(x) - f(y) - 2\phi(x,y)\| \leq \varepsilon, \qquad x, y \in G, \tag{3}$$

$$\|\phi(x,y) + \phi(x,-y)\| \leq \eta, \qquad x, y \in G, \tag{4}$$

then there exist unique functions $\varphi\colon G \to X$ and $B\colon G \times G \to X$ such that (1) holds and

$$\|f(x) - \varphi(x)\| \leq 3\varepsilon + \frac{3}{4}\eta, \quad x \in G. \tag{5}$$

Moreover,

$$\|\phi(x,y) - B(x,y)\| \leq \frac{37}{4}\varepsilon + \frac{9}{4}\eta, \quad x, y \in G. \tag{6}$$

Unfortunately, the above two constants occuring in (5) and (6) have also been calculated incorrectly and they should be equal to $3\varepsilon + 3\eta$ and $5\varepsilon + \frac{9}{2}\eta$, respectively (taking into account the correct form of the mapping δ because this erroneous fact derives also from Theorem 2). The following theorem improves the appropriate approximating constants obtained in Corollary 3 but we prove it in a different way applying Lemma 1 and Theorem 2.

Theorem 4. *Assume that $(G,+)$ is an Abelian group and $\varepsilon, \eta > 0$. If $f\colon G \to X$ and $\phi\colon G \times G \to X$ satisfy (3) and (4), then there exist unique functions $\varphi\colon G \to X$ and $B\colon G \times G \to X$ such that (1) holds and*

$$\|f(x) - \varphi(x)\| \leq 2\varepsilon + 2\eta, \quad x \in G. \tag{7}$$

Moreover,

$$\|\phi(x,y) - B(x,y)\| \leq \frac{7}{2}\varepsilon + 3\eta, \quad x, y \in G. \tag{8}$$

Proof. Apply (4) with $x = y = 0$ to get that $\|\phi(0,0)\| \le \frac{1}{2}\eta$. Now, put $x = y = 0$ in (3) to obtain $\|f(0)\| \le \varepsilon + \eta$. Substitute (x, x) and then $(x, -x)$ in the place of (x, y) in (3), in order to get

$$\|f(2x) - 2f(x) - 2\phi(x,x)\| \le \varepsilon, \qquad x \in G, \tag{9}$$

$$\|f(0) - f(x) - f(-x) - 2\phi(x,-x)\| \le \varepsilon, \qquad x \in G. \tag{10}$$

Moreover, from (4) we have

$$\|\phi(x,x) + \phi(x,-x)\| \le \eta, \quad x \in G. \tag{11}$$

Combining (9), (10) and (11) with $\|f(0)\| \le \varepsilon + \eta$ we get

$$\|f(2x) - 3f(x) - f(-x)\| \le 3(\varepsilon + \eta), \quad x \in G. \tag{12}$$

Replacing x by $-x$ in the above inequality gives

$$\|f(-2x) - 3f(-x) - f(x)\| \le 3(\varepsilon + \eta), \quad x \in G. \tag{13}$$

Substitute in the sequel $(x, 2x)$ and $(x, -2x)$ in the place of (x, y) in (3), in order to obtain

$$\|f(3x) - f(x) - f(2x) - 2\phi(x,2x)\| \le \varepsilon, \qquad x \in G, \tag{14}$$

$$\|f(-x) - f(x) - f(-2x) - 2\phi(x,-2x)\| \le \varepsilon, \qquad x \in G. \tag{15}$$

Moreover, from (4) we also have

$$\|\phi(x,-2x) + \phi(x,2x)\| \le \eta, \quad x \in G. \tag{16}$$

Combining (14), (15) and (16) yields

$$\|f(3x) - 2f(x) - f(2x) - f(-2x) + f(-x)\| \le 2(\varepsilon + \eta), \quad x \in G. \tag{17}$$

Replacing x by $-x$ in the above inequality gives

$$\|f(-3x) - 2f(-x) - f(2x) - f(-2x) + f(x)\| \le 2(\varepsilon + \eta), \quad x \in G. \tag{18}$$

The inequality (18) together with (17) multiplied by 2 lead to

$$\|2f(3x) - f(-3x) - 5f(x) - f(2x) + 4f(-x) - f(-2x)\| \le 6(\varepsilon + \eta), \quad x \in G. \tag{19}$$

Therefore from (12), (13) and (19) we get

$$\|2f(3x) - f(-3x) - 9f(x)\| \le 12(\varepsilon + \eta), \quad x \in G,$$

i.e.

$$\|f(x) - \frac{2}{9}f(3x) + \frac{1}{9}f(-3x)\| \leq \frac{4}{3}(\varepsilon + \eta), \quad x \in G,$$

so, we have got (2) with $\delta = \frac{4}{3}(\varepsilon + \eta)$ and $a = 3$.

From Lemma 1 there exists a uniquely determined function $\varphi\colon G \to X$, satisfying (7) and

$$\varphi(x) = \frac{2}{9}\varphi(3x) - \frac{1}{9}\varphi(-3x), \quad x \in G.$$

We have to prove that (1) holds. It is clear that the assumpions of Theorem 2 are satisfied. Thus, by repeating the arguments presented in the proof of this theorem (see [4]) to the inequality (12), we can get that the functions φ and B satisfy (1).

Moreover, on account of (3) and (7) for fixed $x, y \in G$ we obtain

$$\begin{aligned}\|2B(x,y) - 2\phi(x,y)\| &= \|\varphi(x+y) - \varphi(x) - \varphi(y) - 2\phi(x,y)\| \leq \\ &\leq \|f(x+y) - f(x) - f(y) - 2\phi(x,y)\| + \|f(x+y) - \varphi(x+y)\| + \\ &\quad + \|f(x) - \varphi(x)\| + \|f(y) - \varphi(y)\| \leq 7\varepsilon + 6\eta,\end{aligned}$$

which means that (8) holds. The proof is completed. □

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