Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XIV, Częstochowa 2010

A CHARACTERIZATION OF A HOMOGRAPHIC TYPE FUNCTION

Katarzyna Domańska

Institute of Matematics and Computer Science
Jan Długosz University in Częstochowa
Armii Krajowej 13/15, 42 − 201 *Częstochowa, Poland*
e-mail: k.domanska@ajd.czest.pl

Abstract. We deal with a functional equation of the form

$$f(x+y) = F(f(x), f(y))$$

(the so called addition formula) assuming that the given binary operation F is associative but its domain of definition is not necessarily connected. In the present paper we shall restrict our consideration to the case when

$$F(u,v) = \frac{u+v+2uv}{1-uv}.$$

These considerations may be viewed as counterparts of Losonczi's [7] and Domańska's [3] results on local solutions of the functional equation

$$f(F(x,y)) = f(x) + f(y)$$

with the same behaviour of the given associative operation F. In this paper we admit fairly general structure in the domain of the unknown function.

1. Introduction

If $(G, \star)$ is a group or a semigroup and F stands for an arbitrary binary operation in some set H, then a solution of the functional equation

$$f(x \star y) = F(f(x), f(y))$$

is called a homomorphism of structures $(G, \star)$ and (H, F). We consider here a rational function $F : \{(x, y) \in \mathbb{R} : \quad xy \neq 1\} \longrightarrow \mathbb{R}$ of the form

$$F(u,v) = \frac{u+v+2uv}{1-uv}.$$

This is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$, that satisfies the equation

$$F(F(x,y),z) = F(x, F(y,z))$$

for all $(x,y,z) \in \mathbb{R}^3$ such that products $xy, yz, F(x,y)z, xF(y,z)$ are not equal to 1. Rational functions with such or similar properties are termed associative operations. The class of the associative operations was described by Chéritat [2], and his work was followed by the author.

A homografic function $\varphi : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ given by the formula

$$\varphi(x) = \frac{x}{1-x}, \quad x \neq 1$$

satisfies the functional equation

$$f(x+y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)}$$

for every pair $(x,y) \in \mathbb{R}^2 \setminus D$, where

$$D = \{(x, 1-x) : \quad x \in \mathbb{R}\} \cup \{(x,1) : \quad x \in \mathbb{R}\} \cup \{(1,x) : \quad x \in \mathbb{R}\}.$$

We shall determine all functions $f : G \longrightarrow \mathbb{R}$, where $(G, \star)$ is a group, that satisfy the functional equation

$$f(x \star y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)}. \tag{1}$$

A neutral element of a group $(G, \star)$ will be written as 0.

By a solution of the functional equation (1) we understand any function $f : G \longrightarrow \mathbb{R}$ that satisfies the equality (1) for every pair $(x,y) \in G^2$ such that $f(x)f(y) \neq 1$. Thus we deal with the following conditional functional equation:

$$f(x)f(y) \neq 1 \quad \text{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)} \tag{E}$$

for all $x, y \in G$. Some results on addition formulas can be found for example in Aczél's monography [1] and in the work of Domańska and Ger [4].

The following lemma will be useful in the sequel (see Ger [6]).

Lemma (on a characterization on subgroups). *Let $(G,+)$ be a group. Then $(H,+)$ is a subgroup of group $(G,+)$ if and only if $G \supset H \neq \emptyset$ and*

$$H + H' \subset H',$$

where $H' := G \setminus H$.

2. Main result

We proceed with a description of solutions of (E).

Theorem. *Let $(G, \star)$ be a group. A function $f : G \longrightarrow \mathbb{R}$ yields a nonconstant solution to the functional equation*

$$f(x)f(y) \neq 1 \quad \textit{implies} \quad f(x \star y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)} \tag{E}$$

for all $x, y \in G$ if and only if either

$$f(x) := \begin{cases} 1 & \textit{for } x \in H, \\ -1 & \textit{for } x \in G \setminus H \end{cases}$$

or

$$f(x) := \begin{cases} \dfrac{A(x)}{1 - A(x)} & \textit{for } x \in \Gamma \\ -1 & \textit{for } x \in G \setminus \Gamma \end{cases}$$

or

$$f(x) := \begin{cases} 1 & \textit{for } x \in \Gamma \setminus Z \\ 0 & \textit{for } x \in Z \\ -1 & \textit{for } x \in G \setminus \Gamma, \end{cases}$$

where $(H, \star), (\Gamma, \star)$ are subgroups of the group $(G, \star)$, $(Z, \star)$ is a subgroup of the group $(\Gamma, \star)$, and $A : \Gamma \longrightarrow \mathbb{R}$ is a homomorphism such that $1 \notin A(\Gamma)$.

Proof. Assume that f is a nonconstant solution of equation (E). First we show that $f(0) \in \{-1, 0, 1\}$. Indeed, setting $x = y = 0$ in (E), we obtain

$$f^2(0) = 1 \quad \text{or} \quad f(0) = \frac{2f(0) + 2f^2(0)}{1 - f^2(0)}.$$

Put $c := f(0)$. By equality

$$c = 2c\,\frac{1 + c}{1 - c^2}$$

we have $c = 0$ or $2(1 + c) = 1 - c^2$, whence $c \in \{0, -1\}$ which jointly with the equality $c^2 = 1$ implies $f(0) \in \{-1, 0, 1\}$, which was to be shown.

If $f(0) = -1$, then setting $y = 0$ in (E) we obtain

$$f(x) = -1 \quad \text{or} \quad f(x) = \frac{f(x) - 1 - 2f(x)}{1 + f(x)} = -1$$

for all $x \in G$, whence $f = -1$, a contradiction because we were assuming f to be nonconstant.

Now assume that $f(0) = 1$. We show that $f(G) \subset \{-1, 1\}$. Indeed, putting $y = 0$ in (E), we obtain

$$f(x) = 1 \quad \text{or} \quad f(x) = \frac{3f(x) + 1}{1 - f(x)}$$

for all $x \in G$ and by the equality

$$c = \frac{3c + 1}{1 - c}$$

we have $c = -1$, whence

$$f(x) = 1 \quad \text{or} \quad f(x) = -1$$

for all $x \in G$. By setting

$$H := \{x \in G : \quad f(x) = 1\},$$

we have

$$H' = \{x \in G : \quad f(x) = -1\}$$

and we show that $H \star H' \subset H'$, which implies that H is a subgroup of the group G (see Lemma). Fix arbitrarily elements $x \in H$ and $y \in H'$. Since $f(x)f(y) = -1$, we get by (E) $f(x \star y) = -1$, i.e. $x \star y \in H'$, which was to be shown. So, in this case we have

$$f(x) := \begin{cases} 1 & \text{for } x \in H, \\ -1 & \text{for } x \in G \setminus H. \end{cases}$$

Let now $f(0) = 0$. Put

$$\Gamma := \{x \in G : \quad f(x) \neq -1\}.$$

We are going to show that the complement Γ' of the set Γ enjoys the property $\Gamma \star \Gamma' \subset \Gamma'$, which implies (see Lemma) that Γ is a subgroup of the group G. Fix arbitrarily $x \in \Gamma$ and a $y \in \Gamma'$. Since $f(x)f(y) = -f(x) \neq 1$, we obtain by (E)

$$f(x \star y) = \frac{f(x) - 1 + 2f(x)(-1)}{1 - f(x)(-1)} = \frac{-1 - f(x)}{1 + f(x)} = -1,$$

i.e. $x \star y \in \Gamma'$, which was to be shown. Since $-1 \notin f(\Gamma)$ and $f|_\Gamma$ satisfies (E), a straightforward verification shows that

$$f(x)f(y) \neq 1 \quad \text{implies} \quad \frac{f(x \star y)}{1 + f(x \star y)} = \frac{f(x)}{1 + f(x)} + \frac{f(y)}{1 + f(y)}$$

for all $x, y \in \Gamma$, which jointly with

$$1 - \frac{f(x)}{1+f(x)} - \frac{f(y)}{1+f(y)} = 1 - \frac{f(x)+2f(x)f(y)+f(y)}{(1+f(x))(1+f(y))}$$
$$= \frac{1-f(x)f(y)}{(1+f(x))(1+f(y))},$$

i.e.

$$f(x)f(y) = 1 \Longleftrightarrow \frac{f(x)}{1+f(x)} + \frac{f(y)}{1+f(y)} = 1,$$

states that the function $A : \Gamma \longrightarrow \mathbb{R}$ of the form

$$A(x) := \frac{f(x)}{1+f(x)}, \quad x \in \Gamma$$

yields a solution of equation

$$A(x) + A(y) \neq 1 \quad \text{implies} \quad A(x+y) = A(x) + A(y) \tag{2}$$

for all $x, y \in \Gamma$. We show that $1 \notin A(\Gamma)$. To prove this, assume that $A(x_0) = 1$ for some $x_0 \in \Gamma$. Then we conclude that $f(x_0) = 1 + f(x_0)$, which is immposible. Since $f(0) = 0$, evidently $A(0) = 0$. From the theorem proved by Ger [5] (since $A(0) = 0$) we conclude that A yields a homomorphism of groups Γ and $\mathbb{R}$ or there exist a subgroup Z of a group Γ such that A is of the form

$$A(x) := \begin{cases} 0 & \text{for } x \in Z, \\ \frac{1}{2} & \text{for } x \in \Gamma \setminus Z, \end{cases}$$

whence

$$f(x) := \begin{cases} \dfrac{A(x)}{1-A(x)} & \text{for } x \in \Gamma, \\ -1 & \text{for } x \in G \setminus \Gamma \end{cases}$$

or

$$f(x) := \begin{cases} 0 & \text{for } x \in Z, \\ 1 & \text{for } x \in \Gamma \setminus Z, \\ -1 & \text{for } x \in G \setminus \Gamma. \end{cases}$$

It is easy to check that each of the functions above yields a solution to the equation (E). Thus the proof has been completed.

The following remark gives the form of a constant solutions of equation (E).

Remark. *Let* $(G, \star)$ *be a group. The only constant solutions of* (E) *are* $f = -1$, $f = 0$, *and* $f = 1$.

To check this, assume that $f = c$ fulfils (E). Then

$$c^2 \neq 1 \Longrightarrow c = 2c\frac{1+c}{1-c^2},$$

i.e.

$$c \in \{-1, 1\} \quad \text{or} \quad c = 0 \quad \text{or} \quad c = 2\frac{1+c}{1-c^2},$$

whence

$$c \in \{-1, 0, 1\},$$

which was to be shown.

References

[1] J. Aczél. *Lectures on Functional Equations and Their Applications.* Academic Press, New York, 1966.

[2] A. Chéritat. Fractions rationnelles associatives et corps quadratiques, *Rev. Math. de l'Enseignement Supérieur*, **109**, 1025–1040, 1998-1999.

[3] K. Domańska. Cauchy type equations related to some singular associative operations. *Glasnik Matematički*, 31(51), 135–149, 1996.

[4] K. Domańska, R. Ger. Addition formulae with singularities. *Ann. Math. Silesianae*, **18**, 7–20, 2004.

[5] R. Ger. On some functional equations with a restricted domain, II. *Fund. Math.*, **98**, 249–272, 1978.

[6] R. Ger. O pewnych równaniach funkcyjnych z obciętą dziedziną. *Prace Naukowe Uniwersytetu Śląskiego*, Nr 132, Katowice, 1976.

[7] L. Losonczi. Local solutions of functional equations, *Glasnik Matematički*, 25(45), 57–67, 1990.