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A NEW TOLERANCE MODEL OF VIBRATIONS OF THIN MICROPERIODIC CYLINDRICAL SHELLS

The objects of consideration are thin linearly elastic Kirchhoff-Love-type circular cylindrical shells having a micro-periodic structure in circumferential direction (*uniperiodic shells*). At the same time the shells have constant structure in axial direction. The aim of this contribution is to formulate and discuss a new non-asymptotic averaged model for the analysis of selected dynamic problems for these shells. This, so-called, *general tolerance model* is derived by means of a certain extended version of the known tolerance modelling of micro-heterogeneous media. This version is based on a new notion of *weakly slowly-varying functions*. Contrary to the starting exact shell equations with highly oscillating, non-continuous and periodic coefficients, governing equations of the tolerance model have constant coefficients depending also on a period of inhomogeneity. Hence, the model makes it possible to investigate the effect of a cell size on the global shell dynamics (*the length-scale effect*). The differences between *the general tolerance model* proposed here and the corresponding *known standard tolerance model* derived by means of the more restrictive concept of *slowly-varying functions* are discussed.

Keywords: uniperiodic shells, mathematical modelling, weakly slowly-varying functions, dynamic problems, length-scale effect

1. Introduction

Thin linearly elastic Kirchhoff-Love-type cylindrical shells with a periodically micro-inhomogeneous structure in circumferential direction (*uniperiodic shells*) are analysed, cf. Fig. 1. At the same time, the shells have constant structure in axial direction.

The properties of such shells are described by highly oscillating and non-continuous periodic functions, so the exact equations of the shell theory are too

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complicated to apply to investigations of engineering problems. To obtain averaged equations with constant coefficients, various approximate modelling procedures for shells of this kind have been proposed. Periodic cylindrical shells (plates) are usually described using *homogenized models* derived by means of *asymptotic methods*, cf. [1, 2]. Unfortunately, in the models of this kind *the effect of a cell size* (called *the length-scale effect*) on the overall shell behaviour is neglected.

In order to analyse the length-scale effect in dynamic or/and stability problems, the new averaged non-asymptotic models of thin cylindrical shells with a periodic micro-heterogeneity either along two directions tangent to the shell midsurface (*biperiodic structure*) or along one direction (*uniperiodic structure*) have been proposed and discussed by Tomczyk in a series of publications and summarized as well as extended in [3]. These, so-called, *tolerance models* have been obtained by applying *the non-asymptotic tolerance averaging technique*, cf. [4, 5]. This technique based on the concept of *tolerance relations* between points and real numbers related to the accuracy of the performed measurements and calculations. The tolerance relations are determined by *the tolerance parameters*. Some applications of this method to the modelling of mechanical and thermomechanical problems for various periodic structures are shown in many works. The extended list of papers and books on this topic can be found in [3, 4, 5]. Governing equations of the tolerance models have coefficients which are constant or slowly varying and depend on a cell size.

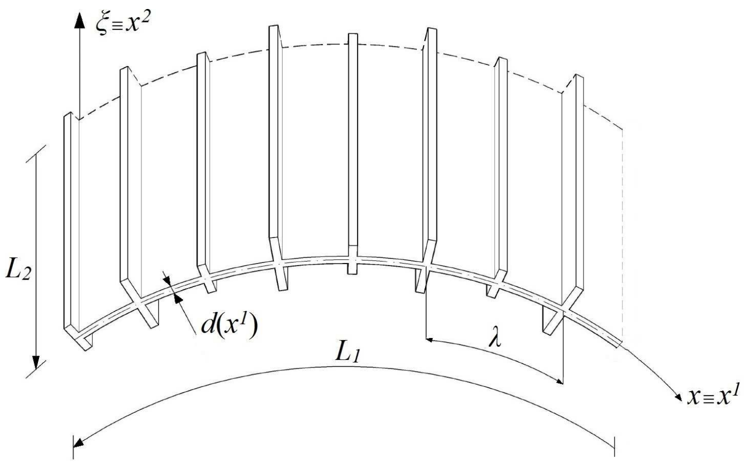

Fig. 1. Example of uniperiodic cylindrical shell

Rys. 1. Przykład walcowej powłoki uniperiodycznej

The aim of this contribution is to formulate and discuss a new mathematical non-asymptotic model for the analysis of selected dynamic problems for the uniperiodic shells under consideration. This model, called *the general tolerance* *model*, will be derived applying a certain extended version of the known tolerance modelling technique. This version, proposed by Tomczyk & Woźniak in [6], is based on a new concept of *weakly slowly-varying functions* which is a certain extension of the well-known concept of *slowly-varying functions*, cf. [4, 5].

The differences between *the general model* proposed here and the corresponding *known standard model* presented in [3] and derived by means of the more restrictive notion of *slowly-varying functions* will be discussed.

Moreover, we will compare *the general tolerance model* formulated here with the *general combined asymptotic-tolerance model* presented by Tomczyk in [7]. This model is derived by applying *the combined modelling* which includes two techniques: *the consistent asymptotic modelling procedure* given by Woźniak in [5] and the extended *tolerance modelling technique* based on the concept of *weakly slowly-varying functions*, cf. [6]. These two techniques are combined together into a new *combined modelling procedure*.

2. Formulation of the modelling problem

We assume that x^1 and x^2 are coordinates parametrizing the shell midsurface M in circumferential and axial directions, respectively. We denote $x \equiv x^1 \in \Omega \equiv (0, L_1)$ and $\xi \equiv x^2 \in \Xi \equiv (0, L_2)$, where L_1, L_2 are length dimensions of M, cf. Fig. 1. Let $O\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space R^3 and denote $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. A cylindrical shell midsurface M is given by $M \equiv \{\bar{\mathbf{x}} \in R^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}(x^1, x^2), (x^1, x^2) \in \Omega \times \Xi\}$, where $\bar{\mathbf{r}}(\cdot)$ is the smooth function such that $\partial\bar{\mathbf{r}}/\partial x^1 \cdot \partial\bar{\mathbf{r}}/\partial x^2 = 0$, $\partial\bar{\mathbf{r}}/\partial x^1 \cdot \partial\bar{\mathbf{r}}/\partial x^1 = 1$, $\partial\bar{\mathbf{r}}/\partial x^2 \cdot \partial\bar{\mathbf{r}}/\partial x^2 = 1$. It means that on M we have introduced *the orthonormal parametrization*.

Sub- and superscripts $\alpha, \beta, \ldots$ run over $1,2$ and are related to x^1, x^2, summation convention holds. Partial differentiation related to x^α is represented by ∂_α. Moreover, it is denoted $\partial_{\alpha\ldots\delta} \equiv \partial_\alpha \ldots \partial_\delta$. Let $a^{\alpha\beta}$ stand for the midsurface first metric tensor. Under orthonormal parametrization $a^{\alpha\beta}$ is the unit tensor. The time coordinate is denoted by $t \in \mathrm{I} \equiv [t_0, t_1]$. Let $d(x)$ and r stand for the shell thickness and the midsurface curvature radius, respectively.

The basic cell Δ and an arbitrary cell $\Delta(x)$ with the centre at point $x \in \Omega_\Delta$ are defined by means of: $\Delta \equiv [-\lambda/2, \lambda/2]$, $\Delta(x) \equiv x + \Delta$, $x \in \Omega_\Delta$, $\Omega_\Delta \equiv \{x \in \Omega : \Delta(x) \subset \Omega_\Delta\}$, where λ is a cell length dimension in $x \equiv x^1$-direction, cf. Fig. 1. *The microstructure length parameter* λ satisfies conditions: $\lambda / d_{max} >> 1$, $\lambda / r << 1$ and $\lambda / \mathrm{L}_1 << 1$.

Setting $z \equiv z^1 \in [-\lambda/2, \lambda/2]$, we assume that the cell Δ has a symmetry axis for $z = 0$. It is also assumed that inside the cell the geometrical, elastic and inertial properties of the shell are described by even functions of argument *z*.

Denote by $u_\alpha = u_\alpha(x,\xi,t)$, $w = w(x,\xi,t)$, $(x,\xi,t) \in \Omega \times \Xi \times \mathrm{I}$, the shell displacements in directions tangent and normal to M, respectively. Elastic properties of the shell are described by stiffness tensors $D^{\alpha\beta\gamma\delta}(x)$, $B^{\alpha\beta\gamma\delta}(x)$. Let $\mu(x)$ stand for a shell mass density per midsurface unit area. Let $f^{\alpha}(x,\xi,t)$, $f(x,\xi,t)$ be external forces per midsurface unit area, respectively tangent and normal to *M*.

The considerations are based on the well-known Kirchhoff-Love theory of thin elastic shells, cf. [8]. It is assumed that the behaviour of the shell under consideration is described by the action functional determined by lagrangian *L* being a highly oscillating function with respect to *x* and having the well-known form

$$
\begin{aligned}
L = -\frac{1}{2}(D^{\alpha\beta\gamma\delta}\partial_\beta u_\alpha \partial_\delta u_\gamma + \frac{2}{r} D^{\alpha\beta 11} w \partial_\beta u_\alpha + \frac{1}{r^2} D^{1111} ww + \\
+ B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta} w \partial_{\gamma\delta} w - \mu a^{\alpha\beta}\dot{u}_\alpha \dot{u}_\beta - \mu \dot{w}^2) + f^{\alpha} u_\alpha + fw \,.
\end{aligned}
\tag{1}
$$

Applying the principle of stationary action we obtain the following system of Euler-Lagrange equations

$$
\partial_\beta \frac{\partial L}{\partial(\partial_\beta u_\alpha)} - \frac{\partial L}{\partial u_\alpha} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{u}_\alpha} = 0 \,, \quad -\partial_{\alpha\beta}\frac{\partial L}{\partial(\partial_{\alpha\beta} w)} - \frac{\partial L}{\partial w} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = 0 \,. \tag{2}
$$

Combining (2) with (1) we arrive finally at the explicit form of *the fundamental equations of the shell theory under consideration*

$$
\begin{aligned}
&\partial_\beta (D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta (D^{\alpha\beta 11} w) - \mu a^{\alpha\beta}\ddot{u}_\beta + f^{\alpha} = 0 \,, \\
&r^{-1} D^{\alpha\beta 11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta} w) + r^{-2} D^{1111} w + \mu \ddot{w} - f = 0 \,.
\end{aligned}
\tag{3}
$$

Equations (3) coincide with the well-known governing equations of Kirchhoff-Love theory of thin elastic cylindrical shells, cf. [8]. For periodic shells, coefficients $D^{\alpha\beta\gamma\delta}(x), B^{\alpha\beta\gamma\delta}(x), \mu(x)$ of equations (3) are highly oscillating, noncontinuous and periodic functions in x. Applying *the extended version of the known tolerance modelling technique* proposed in [6], we obtain the averaged form of Lagrange function (1). Then, using the principle of stationary action, we arrive at the tolerance model equations with constant coefficients depending also on a cell size. To make the analysis more clear, in the next section we shall outline the basic concepts and the main assumptions of *the tolerance modelling approach*, following the monographs [5] and [6].

3. Modelling concepts and assumptions

The fundamental concepts of the extended tolerance modelling procedure under consideration are those of *two tolerance relations between points and real numbers determined by tolerance parameters, weakly slowly-varying functions, tolerance-periodic functions, fluctuation shape functions and the averaging operation.* It has to be emphasized that in the classical approach we deal with not *weakly slowly-varying* but with more restrictive *slowly-varying functions.*

Below, some of the mentioned above concepts are recalled.

Let $F(\cdot)$ be a function defined in $\overline{\Omega}=[0,L_1]$, which is continuous, bounded and differentiable in $\overline{\Omega}$ together with their derivatives up to the *R*-th order. Nonnegative integer *R* is assumed to be specified in every problem under consideration. Note, that function *F* can also depend on $\xi\in\overline{\Xi}=[0,L_2]$ and time coordinate *t* as parameters. Let $\delta\equiv(\lambda,\delta_0,\delta_1,..,\delta_R)$ be the set of tolerance parameters. The first of them is related to the distances between points in Ω, the second one is related to the distances between values of function $F(\cdot)$ and the *k*-th one to the distances between values of the *k*-th derivative of $F(\cdot)$, $k=1,..,R$. A function $F(\cdot)$ is called *weakly slowly-varying of the R-th kind* with respect to cell Δ and tolerance parameters δ, $F\in WSV_\delta^R(\Omega,\Delta)$, if and only if

$$(\forall(x,y)\in\Omega^2)[(x\overset{\lambda}{\approx}y)\Rightarrow F(x)\overset{\delta_0}{\approx}F(y)\ \text{ and }\ \partial_1^k F(x)\overset{\delta_k}{\approx}\partial_1^k F(y),\quad k=1,2,...,R], \tag{4}$$

where $\partial_1^k F(\cdot)$ stands for the *k*-th derivative of $F(\cdot)$ in Ω. Roughly speaking, *weakly slowly-varying* function $F(\cdot)$ can be treated as constant on the cell.

Let us recall that the known *slowly-varying function* $F(\cdot)$, $F\in SV_\delta^R(\Omega,\Delta)$, satisfies not only condition (4) but also the extra restriction

$$(\forall x\in\Omega)[\lambda\left|\partial_1^k F(x)\right|\overset{\delta_k}{\approx}0,\ \ k=1,2,...,R]. \tag{5}$$

An integrable and bounded function $f(\cdot)$ defined in $\overline{\Omega}=[0,L_1]$, which can also depend on $\xi\in\overline{\Xi}$ and time coordinate *t* as parameters, is called *tolerance-periodic of the R-th kind* with respect to cell Δ and tolerance parameters δ, $f\in TP_\delta^R(\Omega,\Delta)$, if it can be treated (together with its derivatives up to the *R-th* order) as periodic on an arbitrary cell.

Let $f(\cdot)$ be a function defined in $\overline{\Omega}=[0,L_1]$, which is integrable and bounded in every cell $\Delta(x)$, $x\in\Omega_\Delta$. By *the averaging of* $f(\cdot)$ we shall mean function $<f>(x)$ defined by

$$<f>(x)\equiv\frac{1}{\lambda}\int_{x-\lambda/2}^{x+\lambda/2}f(z)dz,\quad z\in\Delta(x),\quad x\in\Omega_\Delta\,. \tag{6}$$

If $f(\cdot)$ is a periodic function then $<f>$ is constant.

Let $h(\cdot)$ be a λ-periodic, highly oscillating function defined in $\overline{\Omega}=[0,L_1]$, which is continuous together with derivatives $\partial_1^k h, k=1,...,R-1,$ and has a continuous or a piecewise continuous bounded derivative $\partial_1^R h$. Function $h(\cdot)$ will be called *the fluctuation shape function,* $h(\cdot)\in FS^R(\Omega,\Delta)$, if it depends on λ as a parameter and satisfies conditions: $h\in O(\lambda^R)$, $\partial_1^k h\in O(\lambda^{R-k})$, $k=1,2,...,R$, $<\mu h>=0$, where $\mu(\cdot)$ is a shell mass density.

The tolerance modelling is based on two assumptions. The first assumption is called *the tolerance averaging approximation*. The second one is termed *the micro-macro decomposition.*

Let $f(\cdot)$ be an arbitrary integrable tolerance-periodic functions defined in $\overline{\Omega}=[0,L_1]$ and let $F(\cdot)\in WSV_\delta^1(\Omega,\Delta)$, $G(\cdot)\in WSV_\delta^2(\Omega,\Delta)$. *The tolerance averaging approximation has the form*

$$\begin{aligned}&<f\,\partial_1^R F>(x)=<f>\partial_1^R F(x)+O(\delta),\quad R=0,1\,,\qquad \partial_1^0 F\equiv F\,,\\&<f\,\partial_1^R G>(x)=<f>\partial_1^R G(x)+O(\delta)\,,\quad R=0,1,2\,,\quad \partial_1^0 G\equiv G\,.\end{aligned} \tag{7}$$

In the course of modelling, terms $O(\delta)$ will be neglected. Let us observe that the weakly slowly-varying functions can be regarded as invariant under averaging. Let us recall that the “classical” *slowly-varying functions* $F(\cdot)\in SV_\delta^1(\Omega,\Delta)$, $G(\cdot)\in SV_\delta^2(\Omega,\Delta)$ satisfy not only approximations (7) but also the extra approximate relations

$$\begin{aligned}&<f\,\partial_1(hF)>(x)=<f\,\partial_1 h>(x)F(x)+O(\delta),\\&<f\,\partial_1(gG)>(x)=<f\,\partial_1 g>(x)G(x)+O(\delta),\\&<f\,\partial_1^2(gG)>(x)=<f\,\partial_1^2 g>(x)G(x)+O(\delta)\,,\end{aligned} \tag{8}$$

where $h(\cdot)\in FS^1(\Omega,\Delta)$, $g(\cdot)\in FS^2(\Omega,\Delta)$.

The second fundamental assumption, called *the micro-macro decomposition,* states that the displacements fields occurring in the lagrangian under consideration can be decomposed into *unknown averaged* (*macroscopic*) *displacements* being *weakly slowly-varying functions* in periodicity direction and highly oscillating *fluctuations* represented by the known highly oscillating λ-periodic *fluctuation shape functions* multiplied by unknown *fluctuation amplitudes* (*microscopic variables*) being *weakly slowly-varying* in x.

4. Tolerance modelling

4.1. General tolerance model equations

The tolerance modelling procedure for Euler-Lagrange equations (2) is realized in two steps.

The first step is *the tolerance averaging of lagrangian* (1). To this end let us introduce *fluctuation shape functions* $h(x) \in FS^1(\Omega,\Delta)$ and $g(x) \in FS^2(\Omega,\Delta)$, $x \in \Omega$. These functions are assumed to be known in every problem under consideration. They depend on λ as parameter and have to satisfy conditions: $h \in O(\lambda),\ \lambda\partial_1 h \in O(\lambda)$, $g \in O(\lambda^2),\ \ \lambda\partial_1 g \in O(\lambda^2),\ \ \lambda^2\partial_{11} g \in O(\lambda^2)$, $<\mu h>=<\mu g>=0$, where $\mu(\cdot)$ is the shell mass density being a periodic function with respect to x. Taking into account that inside the cell the geometrical, elastic and inertial properties of the periodic shell under consideration are described by symmetric (i.e. even) functions of argument $z \in \Delta(x)$, we assume that $h(\cdot)$ is either even or odd function of z. This same restriction is imposed on function $g(\cdot)$.

Now, we have to introduce *the micro-macro decomposition* of displacements $u_\alpha(x,\xi,t) \in TP^1_\delta(\Omega,\Delta)$, $w(x,\xi,t) \in TP^2_\delta(\Omega,\Delta)$, $x \in \Omega$, $(\xi,t) \in \Xi \times \mathrm{I}$, which in the problem under consideration is assumed in the form

$$
\begin{aligned}
u_\alpha(x,\xi,t) &= u^0_\alpha(x,\xi,t) + h(x)U_\alpha(x,\xi,t)\,,\\
w(x,\xi,t) &= w^0(x,\xi,t) + g(x)W(x,\xi,t)\,,
\end{aligned}
\tag{9}
$$

where $u^0_\alpha(x,\xi,t), U_\alpha(x,\xi,t) \in WSV^1_\delta(\Omega,\Delta)$, $w^0(x,\xi,t),\ W(x,\xi,t) \in WSV^2_\delta(\Omega,\Delta)$.

Functions u^0_α, w^0, called *macrodisplacements,* and functions U_α, W, called *fluctuation amplitudes*, are *the new unknowns*.

Substituting the right-hand sides of micro-macro decomposition (9) into lagrangian (1) and then averaging the obtained result over the cell using operation (6) and *tolerance averaging approximation* (7), we arrive at function $<L_{hg}>$ called *the tolerance averaging of* L *in* $\Delta(x)$ *under micro-macro decomposition* (9). The obtained result has the form

$$
\begin{aligned}
&< L_{hg} > (\partial_\beta u^0_\alpha, u^0_\alpha, \partial_\beta U_\alpha, U_\alpha, \dot{u}^0_\alpha, \dot{U}_\alpha, \partial_{\alpha\beta} w^0, w^0, \partial_{\alpha\beta} W,\\
&\qquad \partial_\beta W, W, \dot{w}^0, \dot{W}) =\\
&= -\frac{1}{2}[< D^{\alpha\beta\gamma\delta} > \partial_\beta u^0_\alpha \partial_\delta u^0_\gamma + 2 < D^{\alpha\beta\gamma 1}\partial_1 h > \partial_\beta u^0_\alpha U_\gamma +\\
&+ 2\underline{< D^{\alpha\beta\gamma\delta} h >}\partial_\beta u^0_\alpha \partial_\delta U_\gamma +\\
&+ < D^{\alpha 11\gamma}(\partial_1 h)^2 > U_\gamma U_\alpha + \underline{< D^{\alpha\beta\gamma\delta}(h)^2 >}\partial_\beta U_\alpha \partial_\delta U_\gamma +\\
&+ 2r^{-1}(< D^{\alpha\beta 11} > \partial_\beta u^0_\alpha w^0 + < D^{\alpha 111}\partial_1 h^a > w^0 U_\alpha +\\
&+ \underline{< D^{\alpha\beta 11} g >}\partial_\beta u^0_\alpha W + \underline{< D^{\alpha 111}\partial_1 h\, g >} U_\alpha W +\\
&+ \underline{< D^{\alpha\beta 11} h >}\partial_\beta U_\alpha w^0 + \underline{< D^{\alpha\beta 11} h\, g >}\partial_\beta U_\alpha W) +\\
&+ r^{-2}(< D^{1111} > (w^0)^2 + 2\underline{< D^{1111} g >} w^0 W +\\
&+ \underline{< D^{1111}(g)^2 >}(W)^2) + < B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 +\\
&+ 2(< B^{\alpha\beta 11}\partial_{11} g > \partial_{\alpha\beta} w^0 W + \underline{< B^{\alpha\beta\gamma\delta} g >}\partial_{\alpha\beta} w^0 \partial_{\gamma\delta} W +\\
&+ \underline{< B^{11\gamma\delta} g\, \partial_{11} g >}\partial_{\gamma\delta} W\, W) + 4\underline{< B^{\alpha\beta 1\delta}\partial_1 g >}\partial_{\alpha\beta} w^0 \partial_\delta W +\\
&+ 4\underline{< B^{1\beta 1\delta}(\partial_1 g)^2 >}\partial_\beta W\, \partial_\delta W + < B^{1111}(\partial_{11} g)^2 > (W)^2 +\\
&+ \underline{< B^{\alpha\beta\gamma\delta}(g)^2 >}\partial_{\alpha\beta} W\, \partial_{\gamma\delta} W - < \mu > a^{\alpha\beta}\dot{u}^0_\alpha \dot{u}^0_\beta - < \mu > (\dot{w}^0)^2 +\\
&- \underline{< \mu(h)^2 >} a^{\alpha\beta}\dot{U}_\alpha \dot{U}_\beta - \underline{< \mu(g)^2 >}(\dot{W})^2] +\\
&+ < f^\alpha > u^0_\alpha + \underline{< f^\alpha h >} U_\alpha + < f > w^0 + \underline{< fg >} W\,.
\end{aligned}
\tag{10}
$$

The underlined terms in (10) depend on microstructure length parameter λ.

The second step in the tolerance modelling of starting equations (3) is to apply the principle of stationary action to action functional determined by averaged lagrangian $< L_{hg} >$. As a result we obtain the system of Euler-Lagrange equations for unknowns $u^0_\alpha, w^0, U_\alpha, W$, which explicit form can be written as

- the constitutive equations

$$
\begin{aligned}
N^{\alpha\beta} &= < D^{\alpha\beta\gamma\delta} > \partial_\delta u^0_\gamma + r^{-1}(< D^{\alpha\beta 11} > w^0 + \underline{< D^{\alpha\beta 11} g >} W) +\\
&\quad + < D^{\alpha\beta\gamma 1}\partial_1 h > U_\gamma + \underline{< D^{\alpha\beta\gamma\delta} h >}\partial_\delta U_\gamma,\\
M^{\alpha\beta} &= < B^{\alpha\beta\gamma\delta} > \partial_{\gamma\delta} w^0 + < B^{\alpha\beta 11}\partial_{11} g > W +\\
&\quad + 2\underline{< B^{\alpha\beta\gamma 1}\partial_1 g >}\partial_\gamma W + \underline{< B^{\alpha\beta\gamma\delta} g >}\partial_{\gamma\delta} W\,,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
H^{\beta} = {} & \langle \partial_1 h\, D^{\beta 1\gamma\delta} \rangle \partial_\delta u_\gamma^0 - \underline{\langle h\, D^{\alpha\beta\gamma\delta} \rangle} \partial_{\alpha\delta} u_\gamma^0 + \\
& + \langle D^{\beta 11\gamma} (\partial_1 h)^2 \rangle U_\gamma - \underline{\langle D^{\alpha\beta\gamma\delta} (h)^2 \rangle} \partial_{\alpha\delta} U_\gamma + \\
& + r^{-1} (\langle \partial_1 h D^{\beta 111} \rangle w^0 - \underline{\langle h\, D^{\alpha\beta 11} \rangle} \partial_\alpha w^0 + \\
& + \underline{\langle \partial_1 h\, D^{\beta 111} g \rangle} W - \underline{\langle h\, D^{\alpha\beta 11} g \rangle} \partial_\alpha W) , \\
G = {} & r^{-1} (\underline{\langle g\, D^{11\gamma\delta} \rangle} \partial_\delta u_\gamma^0 + \underline{\langle \partial_1 h\, D^{111\gamma} g \rangle} U_\gamma + \\
& + \underline{\langle h\, D^{11\gamma\delta} g \rangle} \partial_\delta U_\gamma) + r^{-2} \underline{\langle g\, D^{1111} \rangle} w^0 + \\
& + \langle \partial_{11} g\, B^{11\alpha\beta} \rangle \partial_{\alpha\beta} w^0 - 2 \underline{\langle \partial_1 g\, B^{\alpha\beta\gamma 1} \rangle} \partial_{\alpha\beta\gamma} w^0 + \\
& + \underline{\langle g\, B^{\alpha\beta\gamma\delta} \rangle} \partial_{\alpha\beta\gamma\delta} w^0 + (\langle (\partial_{11} g)^2 B^{1111} \rangle + \\
& + r^{-2} \underline{\langle (g)^2 D^{1111} \rangle}) W + (\underline{2 \langle \partial_{11} g\, B^{11\beta\delta} g^{B} \rangle} + \\
& - 4 \underline{\langle (\partial_1 g)^2 B^{1\beta 1\delta} \rangle}) \partial_{\beta\delta} W + \underline{\langle (g)^2 B^{\alpha\beta\gamma\delta} \rangle} \partial_{\alpha\beta\gamma\delta} W ,
\end{aligned} \tag{12}
$$

- the dynamic equilibrium equations

$$
\begin{aligned}
& \partial_\beta N^{\alpha\beta} - \langle \mu \rangle a^{\alpha\beta} \ddot{u}_\beta^0 + \langle f^{\alpha} \rangle = 0 , \quad \partial_{\alpha\beta} M^{\alpha\beta} + r^{-1} N^{11} + \langle \mu \rangle \ddot{w}^0 - \langle f \rangle = 0 , \\
& \underline{\langle \mu\, (h)^2 \rangle} a^{\alpha\beta} \ddot{U}_\alpha + H^{\beta} - \underline{\langle f^{\beta} h \rangle} = 0 , \quad \underline{\langle \mu\, (g)^2 \rangle} \ddot{W} + G - \underline{\langle fg \rangle} = 0 .
\end{aligned} \tag{13}
$$

In equations (11)-(13) the underlined terms depend on a cell size λ.

Equations (11)-(13) together with *micro-macro decomposition* (9) constitute *the general tolerance model of selected dynamic problems for the micro-heterogeneous uniperiodic shells under consideration*.

4.2. Discussion of results

The characteristic features of the derived *general tolerance model* are:

- In contrast to starting equations (3) with discontinuous, highly oscillating and periodic coefficients, the tolerance model equations (11)-(13) proposed here *have constant coefficients depending also on a cell size* (underlined terms). Hence, the tolerance model makes it possible to *describe the effect of a period length on the global shell behaviour*.
- Unknown macrodisplacements u_α^0, w^0 and fluctuation amplitudes U_α, W of the tolerance model equations must be *weakly slowly-varying functions* in periodicity direction. *This requirement can be verified only a posteriori* and *it determines the range of the physical applicability of the model.*

- The number and form of boundary/initial conditions for the basic unknowns of the tolerance model are the same as in the classical shell theory governed by equations (3).
- Decomposition (9) and hence also resulting tolerance model equations (11)-(13) are uniquely determined by the postulated *a priori* periodic *fluctuations shape functions* $h(x) \in FS^1(\Omega, \Delta)$, $h \in O(\lambda)$, and $g(x) \in FS^2(\Omega, \Delta)$, $g \in O(\lambda^2)$, representing oscillations inside a cell. These functions can be obtained as exact or approximate solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell, cf. [3]. It means that they represent either the principal modes of free periodic vibrations of the cell or physically reasonable approximation of these modes.
- The resulting equations involve terms with time and spatial derivatives of the fluctuation amplitudes. Hence, these equations describe certain *time-boundary-layer and space-boundary-layer phenomena* strictly related to the specific form of initial and boundary conditions imposed on unknown fluctuation amplitudes U_α, W.
- After neglecting in equations (11)-(13) the underlined terms, we obtain the asymptotic model of the shells under consideration. This model is not able to describe the length-scale effect on the overall shell dynamics being independent of a cell size. It is necessary to observe that now equations $(13)_{3,4}$ for the fluctuation amplitudes are linear algebraic equations.

4.3. Standard tolerance model equations

Let us compare *the general tolerance model* proposed here with the corresponding known *standard tolerance model* presented and discussed in [3], which was derived under assumption that the unknown functions $u^0_\alpha(x,\xi,t)$, $w^0(x,\xi,t)$, $U_\alpha(x,\xi,t), W(x,\xi,t)$ in micro-macro decomposition (9) are *slowly-varying*. We recall that *the slowly-varying functions* being a subclass of *the weakly slowly-varying functions* are defined by means of (4) and (5). For *the slowly-varying functions* approximate relations (7), (8) hold.

Following [3], *the standard tolerance model* consists of:

- micro-macro decomposition (9) in which *weakly slowly-varying functions* $u^0_\alpha, U_\alpha \in WSV^1_\delta(\Omega,\Delta)$, w^0, $W \in WSV^2_\delta(\Omega,\Delta)$ are replaced by *slowly-varying functions* $u^0_\alpha, U_\alpha \in SV^1_\delta(\Omega,\Delta)$, w^0, $W \in SV^2_\delta(\Omega,\Delta)$,
- constitutive equations (11) in which λ-depending terms $< D^{\alpha\beta\gamma\delta} h > \partial_\delta U_\gamma$ in $(11)_1$, and $< B^{\alpha\beta\gamma 1} \partial_1 g > \partial_\gamma W$, $< B^{\alpha\beta\gamma\delta} g > \partial_{\gamma\delta} W$ in $(11)_2$ are replaced respectively by $< D^{\alpha\beta\gamma 2} h > \partial_2 U_\gamma$ and $< B^{\alpha\beta 21} \partial_1 g > \partial_2 W$, $< B^{\alpha\beta 22} g > \partial_{22} W$,

- constitutive equations (12) in which λ-depending terms $< h D^{\alpha\beta\gamma\delta} > \partial_{\alpha\delta} u_\gamma^0$, $< D^{\alpha\beta\gamma\delta}(h)^2 > \partial_{\alpha\delta} U_\gamma$, $< h D^{\alpha\beta 11} > \partial_\alpha w^0$, $< h D^{\alpha\beta 11} g > \partial_\alpha W$ in $(12)_1$ and $< h D^{11\gamma\delta} g > \partial_\delta U_\gamma$, $< \partial_1 g B^{\alpha\beta\gamma 1} > \partial_{\alpha\beta\gamma} w^0$, $< g B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta\gamma\delta} w^0$, $< \partial_{11} g B^{11\beta\delta} > \partial_{\beta\delta} W$, $< (\partial_1 g)^2 B^{1\beta 1\delta} > \partial_{\beta\delta} W$, $< (g)^2 B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta\gamma\delta} W$ in $(12)_2$ are replaced respectively by $< h D^{2\beta\gamma\delta} > \partial_{2\delta} u_\gamma^0$, $< D^{2\beta\gamma 2}(h)^2 > \partial_{22} U_\gamma$, $< h D^{2\beta 11} > \partial_2 w^0$, $< h D^{2\beta 11} g > \partial_2 W$ and $< h D^{11\gamma 2} g > \partial_2 U_\gamma$, $< \partial_1 g B^{\alpha\beta 21} > \partial_{\alpha\beta 2} w^0$, $< g B^{\alpha\beta 22} > \partial_{\alpha\beta 22} w^0$, $< \partial_{11} g B^{1122} > \partial_{22} W$, $< (\partial_1 g)^2 B^{1212} > \partial_{22} W$, $< (g)^2 B^{2222} > \partial_{2222} W$,
- the dynamic equilibrium equations having form of equations (13).

It can be observed that the constitutive equations of the standard model do not involve derivatives of amplitude fluctuations U_α, W with respect to argument *x*. It arises from tolerance relations (8), which hold for the slowly-varying functions.

From comparison of both the general and the standard tolerance models it follows that the general model equations (11)-(13) contain a bigger number of terms depending on the microstructure size than the standard model equations. So, the general model proposed in this contribution allows us to investigate the length-scale effect in more detail.

It can be observed that for the standard model, the boundary conditions for unknown fluctuation amplitudes U_α, W should be defined only on boundaries $\xi = 0$, $\xi = L_2$ whereas in the framework of the general model the boundary conditions for U_α, W should be defined on all boundaries of the shell. It means that for open cylindrical shell, applying the general model we can investigate *the space-boundary-layer phenomena* near all boundaries of the shell whereas within the standard model we can analyse these phenomena only near boundaries $\xi = 0$, $\xi = L_2$.

4.4. Combined asymptotic-tolerance model equations

Let us compare *the general tolerance model* proposed here with the corresponding *general combined asymptotic-tolerance model* of uniperiodic shells under consideration which is presented and discussed in [7].

In the general case, the asymptotic and tolerance modelling procedures are discussed independently each other. In paper [7], these two techniques are combined together into a new *combined modelling procedure*.

Following [7], the combined model consists of:

- *Asymptotic* (*macroscopic*) *model equations* formulated by applying *the consistent asymptotic procedure*, cf. [5], and having constant coefficients being independent of a cell size. After eliminating fluctuation amplitudes $U_\alpha(x,\xi,t)$, $W(x,\xi,t)$ by means of $U_\gamma = -(G^{-1})_{\gamma\eta}[<\partial_1 hD^{1\eta\mu\vartheta}>\partial_\vartheta u_\mu^0 + r^{-1}<\partial_1 hD^{1\eta 11}> w^0]$ and $W = -E^{-1}<\partial_{11} gB^{11\gamma\delta}>\partial_{\gamma\delta} w^0$, *the asymptotic equations* are expressed only in macrodisplacements u_α^0, w^0

$$\begin{aligned}
&D_h^{\alpha\beta\gamma\delta}\partial_{\beta\delta}u_\gamma^0 + r^{-1}D_h^{\alpha\beta 11}\partial_\beta w^0 - <\mu> a^{\alpha\beta}\ddot{u}_\beta^0 + f^\alpha = 0\,,\\
&B_g^{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma\delta}w^0 + r^{-1}D_h^{11\gamma\delta}\partial_\delta u_\gamma^0 + r^{-2}D_h^{1111}w^0 + <\mu>\ddot{w}^0 - <f> = 0\,,
\end{aligned} \tag{14}$$

where $D_h^{\alpha\beta\gamma\delta} \equiv <D^{\alpha\beta\gamma\delta}> - <D^{\alpha\beta\eta 1}\partial_1 h>(G^{-1})_{\eta\zeta}<\partial_1 hD^{1\zeta\gamma\delta}>$ and $B_g^{\alpha\beta\gamma\delta} \equiv <B^{\alpha\beta\gamma\delta}> - <B^{\alpha\beta 11}\partial_{11}g>(E^{-1})<\partial_{11}gB^{11\gamma\delta}>$ with $G_{\alpha\gamma} = <D^{\alpha 1\gamma 1}(\partial_1 h)^2>$ and $E = <B^{1111}(\partial_{11}g)^2>$.

- *Microscopic model equations* imposed on the known solutions u_α^0, w^0 obtained in the framework of the asymptotic model and derived by applying an extended version of the tolerance modelling technique, cf. [6]. Setting $u_{0\alpha} = u_\alpha^0 + hU_\alpha$ and $w_0 = w^0 + gW$, we obtain the following form of *the superimposed microscopic model equations*

$$\begin{aligned}
&\underline{<D^{\alpha\beta\lambda\delta}c^2>}\partial_{\beta\gamma}Q_\delta - <D^{\alpha 1\gamma 1}(\partial_1 c)^2>Q_\gamma - \underline{<\mu c^2>}a^{\alpha\beta}\ddot{Q}_\beta + \underline{<f^\alpha h>} =\\
&= r^{-1}<D^{\alpha 111}\partial_1 c\, w_0> + <D^{\alpha\beta\gamma 1}\partial_1 c\partial_\beta u_{0\gamma}>\,,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\underline{<B^{\alpha\beta\gamma\delta}b^2>}\partial_{\alpha\beta\gamma\delta}V + [2\underline{<B^{\alpha\beta 11}b\partial_{11}b>} - 4\underline{<B^{\alpha 1\beta 1}(\partial_1 b)^2>}]\partial_{\alpha\beta}V +\\
&+ <B^{1111}(\partial_{11}b)^2>V + \underline{<\mu b^2>}\ddot{V} - \underline{<fg>} = -<B^{\alpha\beta 11}\partial_{11}b\partial_{\alpha\beta}w_0>.
\end{aligned} \tag{16}$$

where $Q_\alpha \in WSV_\delta^1(\Omega,\Delta)$, $V \in WSV_\delta^2(\Omega,\Delta)$ are *new unknown weakly slowly-varying fluctuation amplitudes* and $c(\cdot) \in FS^1(\Omega,\Delta)$, $b(\cdot) \in FS^2(\Omega,\Delta)$ are the *new known* periodic highly oscillating *fluctuation shape functions*. Coefficients of the tolerance model are *constant* and some of them *depend on a cell size* (underlined terms). The right-hand sides of (15), (16) are known under assumption that $u_{0\alpha}$, w_0 have been determined in the framework of asymptotic model.

- Decomposition of displacement fields $u_\alpha(x,\xi,t)$, $w(x,\xi,t)$ in $\Omega\times\Xi\times I$

$$\begin{aligned}
&u_\alpha(x,\xi,t) = u_\alpha^0(x,\xi,t) + h(x)U_\alpha(x,\xi,t) + c(x)Q_\alpha(x,\xi,t)\,,\\
&w(x,\xi,t) = w^0(x,\xi,t) + g(x)W(x,\xi,t) + b(x)V(x,\xi,t)\,,
\end{aligned}$$

where functions $u_{\alpha}^{0}, U_{\alpha}, w^{0}, W$ have to be obtained in the first step of combined modelling, i.e. in the framework of the asymptotic modelling.

From comparison of both the general tolerance and the general combined asymptotic-tolerance models it follows that tolerance model equations (11)-(13) proposed in this contribution contain a bigger number of terms depending on the microstructure size than the combined model equations (14)-(16) recalled here following [7]. Thus, the general tolerance model proposed in this paper allows us to investigate the length-scale effect in more detail.

It can be shown, that neglecting the external forces and under special conditions imposed on the fluctuation shape functions we can obtain superimposed microscopic model equations, which are independent of the solutions obtained in the framework of the macroscopic model, cf. [7]. It means, that *an important advantage of the combined model is that it makes it possible to separate the macroscopic description of some special problems from the microscopic description of these problems*.

5. Final remarks and conclusions

The tolerance modelling technique based on the notion of *weakly slowly-varying function*, cf. [6], is proposed as a tool to derive a new mathematical non-asymptotic averaged model for the analysis of selected dynamic problems for thin cylindrical shells with micro-periodic structure in circumferential direction.

Contrary to "exact" shell equations (3) with highly oscillating non-continuous periodic coefficients, the tolerance model equations (11)-(13) have constant coefficients depending also on a cell size. Hence, this model makes it possible *to describe the effect of a length scale on the global shell behaviour*.

The general tolerance model equations (11)-(13) formulated in this contribution contain a bigger number of terms depending on the microstructure size than *the standard tolerance model equations* presented in [3], which were derived applying the concept of *slowly-varying function*. Moreover, the tolerance model with *the weakly slowly-varying unknowns* proposed here allows us to investigate the length-scale effect in more detail than *the combined asymptotic-tolerance model* formulated in [7], which was also derived using the concept of *weakly slowly-varying function*.

The basic unknowns of the general tolerance model equations must be *the weakly slowly-varying functions* in periodicity direction. This requirement can be verified only *a posteriori*.

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NOWY MODEL TOLERANCYJNY DO ANALIZY DRGAŃ CIENKICH MIKROPERIODYCZNYCH POWŁOK WALCOWYCH

S t r e s z c z e n i e

Przedmiotem rozważań są cienkie liniowo-sprężyste powłoki walcowe typu Kirchhoffa-Love'a mające periodycznie mikro-niejednorodną strukturę w kierunku obwodowym. Powłoki takie nazywamy *uniperiodycznymi*. Celem pracy jest sformułowanie nowego, nieasymptotycznego, uśrednionego modelu służącego do analizy wybranych zagadnień dynamiki takich powłok. Przedstawiony *ogólny model tolerancyjny* wyprowadzony jest w oparciu o pewną zmodyfikowaną wersję znanej techniki tolerancyjnego modelowania struktur mikro-niejednorodnych. Wersja ta bazuje na nowym pojęciu *funkcji słabo wolno-zmiennej*. W przeciwieństwie do równań wyjściowych dla analizowanych powłok niejednorodnych mających współczynniki periodyczne, silnie oscylujące i nieciągłe, równania modelu tolerancyjnego mają stałe współczynniki. Ponadto, współczynniki te zależą od parametru długości mikrostruktury. Tym samym umożliwiają badanie efektu skali.

Słowa kluczowe: powłoka uniperiodyczna, modelowanie matematyczne, funkcja słabo wolno-zmienna, zagadnienia dynamiki, efekt skali

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