A HIERARCHY
OF MAXIMAL INTERSECTING TRIPLE SYSTEMS

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Abstract. We reach beyond the celebrated theorems of Erdős-Ko-Rado and Hilton-Milner, and a recent theorem of Han-Kohayakawa, and determine all maximal intersecting triples systems. It turns out that for each $n \geq 7$ there are exactly 15 pairwise non-isomorphic such systems (and 13 for $n = 6$). We present our result in terms of a hierarchy of Turán numbers $\mathrm{ex}^{(s)}(n; M_2^3)$, $s \geq 1$, where M_2^3 is a pair of disjoint triples. Moreover, owing to our unified approach, we provide short proofs of the above mentioned results (for triple systems only). The triangle C_3 is defined as $C_3 = \{\{x_1, y_3, x_2\}, \{x_1, y_2, x_3\}, \{x_2, y_1, x_3\}\}$. Along the way we show that the largest intersecting triple system H on $n \geq 6$ vertices, which is not a star and is triangle-free, consists of $\max\{10, n\}$ triples. This facilitates our main proof's philosophy which is to assume that H contains a copy of the triangle and analyze how the remaining edges of H intersect that copy.

Keywords: maximal intersecting family, 3-uniform hypergraph, triple system.

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1. INTRODUCTION

A hypergraph is a synonym for *set system* and in this context the sets are called *edges*. The elements of all the sets are called *vertices*. We often identify the edge set of a hypergraph H with the hypergraph itself but never forget about the underlying *vertex set* $V(H)$. A hypergraph is called *intersecting* if every two of its edges intersect. A hypergraph is *k-uniform*, a *k-graph*, for short, if every edge has size k.

Although in this paper we prove results about triple systems, or 3-uniform hypergraphs, we begin with some definitions and results valid for all k-graphs, $k \geq 2$.

The celebrated Erdös-Ko-Rado theorem [3] determines the maximum size of a k-uniform intersecting family. Since we formulate this result in terms of the Turán numbers, we need some more definitions and notation. Given a family of k-graphs $\mathcal{G}$, we call a k-graph H *$\mathcal{G}$-free* if for all $G \in \mathcal{G}$ we have $G \nsubseteq H$.

Definition 1.1. For a family of k-graphs $\mathcal{G}$ and an integer $n \geq 1$, the *Turán number* (*of the 1st order*) is defined as

$$\mathrm{ex}_k^{(1)}(n; \mathcal{G}) := \mathrm{ex}_k(n; \mathcal{G}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{G}\text{-free}\}.$$

Every n-vertex $\mathcal{G}$-free k-graph with $\mathrm{ex}_k(n; \mathcal{G})$ edges is called *extremal* (*1-extremal*) *for* $\mathcal{G}$. We denote by $\mathrm{Ex}_k(n; \mathcal{G}) = \mathrm{Ex}_k^{(1)}(n; \mathcal{G})$ the family of all n-vertex k-graphs which are extremal for $\mathcal{G}$.

In [9] the authors introduced a hierarchy of Turán numbers, where in each generation we consider only k-graphs which are not sub-k-graphs of extremal k-graphs from all previous generations. The next definition is iterative.

Definition 1.2. For a family of k-graphs $\mathcal{G}$ and integers $s, n \geq 1$, *the Turán number of the $(s+1)$-st order* is defined as

$$\mathrm{ex}_k^{(s+1)}(n; \mathcal{G}) = \max\{|E(H)| : |V(H)| = n, \ H \text{ is } \mathcal{G}\text{-free, and}$$
$$\text{for all } H' \in \mathrm{Ex}_k^{(1)}(n; \mathcal{G}) \cup \ldots \cup \mathrm{Ex}_k^{(s)}(n; \mathcal{G}), H \nsubseteq H'\},$$

if such a k-graph H exists. An n-vertex $\mathcal{G}$-free k-graph H is called $(s+1)$-*extremal for* $\mathcal{G}$ if $|E(H)| = \mathrm{ex}_k^{(s+1)}(n; \mathcal{G})$ and for any $H' \in \mathrm{Ex}_k^{(1)}(n; \mathcal{G}) \cup \ldots \cup \mathrm{Ex}_k^{(s)}(n; \mathcal{G}), H \nsubseteq H'$; we denote by $\mathrm{Ex}_k^{(s+1)}(n; \mathcal{G})$ the family of n-vertex k-graphs which are $(s+1)$-extremal for $\mathcal{G}$.

We will often write $\mathrm{ex}_k^{(s)}(n; G)$ for $\mathrm{ex}_k^{(s)}(n; \{G\})$ and $\mathrm{Ex}_k^{(s)}(n; G)$ for $\mathrm{Ex}_k^{(s)}(n; \{G\})$.

A *star* is a hypergraph with a vertex, called its center, contained in all the edges. Obviously, a star is intersecting. An n-vertex, k-uniform star with $\binom{n-1}{k-1}$ edges is called *full* and denoted by S_n^k. Let M_2^k be a k-graph consisting of two disjoint edges.

Theorem 1.3 ([3]). *For $n \geq 2k$,*

$$\mathrm{ex}_k(n; M_2^k) = \binom{n-1}{k-1}.$$

Moreover, for $n \geq 2k+1$, $\mathrm{Ex}_k(n; M_2^k) = \{S_n^k\}$.

A historically first example of a Turán number of the 2nd order is due to Hilton and Milner [8] (see [4] for a simple proof). They determined the maximum size of an intersecting k-graph which is not a star.

Theorem 1.4 ([8]). *For $n \geq 2k+1$,*

$$\mathrm{ex}_k^{(2)}(n; M_2^k) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, for $k = 3$, $\mathrm{Ex}_3^{(2)}(n; M_2^3) = \{H_1(n), H_2(n)\}$, while for $k \geq 4$, $\mathrm{Ex}_k^{(2)}(n; M_2^k) = \{H_1^k(n)\}$.

The two 2-extremal 3-graphs $H_1(n)$ and $H_2(n)$ appearing in Theorem 1.4 are defined later in this section. For the definitions of $H_1^k(n)$ for arbitrary $k \geq 4$, see [8].

Recently, the third order Turán number for M_2^k has been established for arbitrary k by Han and Kohayakawa in [6].

Theorem 1.5 ([6]). *For $k \geq 3$ and $n \geq 2k + 1$,*

$$\mathrm{ex}_k^{(3)}(n; M_2^k) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2.$$

Han and Kohayakawa have also determined the 3-extremal k-graphs which are not shown here. Below we define $H_3(n)$, the only 3-extremal 3-graph for M_2^3 and $n \geq 7$.

A natural question arises if this process terminates. In other words, is the number of maximal intersecting k-graphs finite, that is, independent of n, the number of vertices? This question has been answered positively already in [1] (see also [11]) but no extremal hypergraphs were given.

In this paper we produce explicitly the entire spectrum of maximal, intersecting 3-graphs and arrange them by means of the ordered Turán numbers for the matching M_2^3 (Proposition 1.6 and Theorem 1.7 below). In particular, we find all, pairwise non-isomorphic, maximal, intersecting 3-graphs on six vertices (Proposition 1.6). Although, for $n \geq 7$, the Turán numbers of the first, second, and third order are already known (and are stated above), for the sake of unification, we include them into our main result. In addition, we determine the complete Turán hierarchy for non-intersecting 3-graphs which are triangle-free (Corollary 2.5 in Section 2.2). For $k = 3$ we suppress the superscript ^{3} from the notation of 3-graphs, i.e., $M_2^3 = M_2$ and $S_n^3 = S_n$.

Before stating our results, we need to define several specific 3-graphs which turn out to be extremal. In our description, we will put an emphasis on the vertex covers. A subset of vertices T of a hypergraph H is called *a vertex cover* if it has nonempty intersection with every edge of H. We denote by $\tau(H)$ the size of the smallest vertex cover of H. Clearly, every edge of an intersecting hypergraph is its vertex cover, so, for an intersecting k-graph H we have $1 \leq \tau(H) \leq k$.

For a subset $A \subset V$, set

$$\mathrm{up}(A) = \left\{ f \in \binom{V}{k} : f \supset A \right\}.$$

Observe that if T is a vertex cover of a *maximal* intersecting k-graph H, then $\mathrm{up}(T) \subseteq H$.

Let $x, y, z, v, w, u \in V$ be six different vertices of V, $|V| = n$. We define

$$H_1(n) = \mathrm{up}(\{x, y\}) \cup \mathrm{up}(\{x, z\}) \cup \mathrm{up}(\{x, v\}) \cup \{\{y, z, v\}\}$$

and

$$H_2(n) = \mathrm{up}(\{x, y\}) \cup \mathrm{up}(\{x, z\}) \cup \mathrm{up}(\{y, z\}).$$

Note that for $i = 1, 2$, $M_2 \not\subset H_i(n)$ and $|H_i(n)| = 3n - 8$. Next, let

$$H_3(n) = \mathrm{up}(\{x, y\}) \cup \mathrm{up}(\{x, z\}) \cup \{\{x, v, w\}, \{y, z, w\}, \{y, z, v\}\}.$$

Note that $M_2 \not\subset H_3(n)$ and $|H_3(n)| = 2n - 2$. Further, let

$$H_4(n) = \mathrm{up}(\{x, y\}) \cup \{\{x, v, z\}, \{x, w, z\}, \{x, v, w\}, \{y, z, w\}, \{y, z, v\}, \{y, v, w\}\},$$

$$H_5(n) = \mathrm{up}(\{x, y\}) \cup \{\{x, v, z\}, \{x, w, u\}, \{x, v, w\}, \{y, z, w\}, \{y, u, v\}, \{y, v, w\}\},$$

and

$$H_6(n) = \mathrm{up}(\{x, y\}) \cup \{\{x, v, z\}, \{x, w, u\}, \{x, v, w\}, \{y, z, w\}, \{y, u, v\}, \{x, z, u\}\}.$$

Note that for $i = 4, 5, 6$, $M_2 \not\subset H_i(n)$ and $|H_i(n)| = n + 4$.

Observe also that $\tau(H_i(n)) = 2$ for $i = 1, \ldots, 6$. The minimal vertex covers can be easily identified, as they are exactly the 2-element sets which are the arguments of the operator $\mathrm{up}(\cdot)$ appearing in all 6 definitions above. For instance, $H_1(n)$ has three minimal vertex covers, $\{x, y\}, \{x, z\}, \{x, v\}$, while $H_6(n)$ has just one, $\{x, y\}$.

Next, we define five more intersecting 3-graphs, $H_i(n)$, $i = 7, \ldots, 11$, all with just 10 edges and spanned on 6 vertices. As now $\tau(H_i(n)) = 3$, the remaining $n - 6$ vertices are isolated. Below we use notation $H \cup sK_1$ to designate the 3-graph obtained from a 3-graph H by adding s isolated vertices. We find it convenient and in line with the forthcoming proof to base their description on the notion of a triangle whose copy they all contain. Let $U = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. We call the cycle

$$C_3 = \{\{x_i, y_j, x_k\} : \{i, j, k\} = \{1, 2, 3\}\}$$

a triangle. Further, let

$$A_1 = \{\{x_i, y_i, y_j\} : \{i, j\} \subset \{1, 2, 3\}\},$$

$$A_2 = \{\{x_i, x_j, y_j\} : \{i, j\} \subset \{1, 2, 3\}\},$$

$$A_3 = (A_1 \setminus \{x_1, y_1, y_2\}) \cup \{x_2, x_3, y_3\},$$

and

$$A_4 = (A_1 \setminus \{\{x_1, y_1, y_2\}, \{x_1, y_1, y_3\}\}) \cup \{\{x_2, x_3, y_3\}, \{x_2, x_3, y_2\}\}.$$

Note that $A_4 = (A_3 \setminus \{x_1, y_1, y_3\}) \cup \{x_2, x_3, y_2\}$.

We define the following 3-graphs on U:

$$H_7(6) = C_3 \cup \{\{x_1, x_2, x_3\}\} \cup A_1,$$

and, for $i = 8, 9, 10, 11$,

$$H_i(6) = C_3 \cup \{\{y_1, y_2, y_3\}\} \cup A_{i-7}.$$

Finally, for $i = 7, \ldots, 11$, and $n \geq 7$, set

$$H_i(n) = H_i(6) \cup (n - 6)K_1.$$

Observe that the 3-graphs S_n and $H_i(n)$, $i = 1, \ldots, 11$, are all *maximal* with respect to being intersecting, that is, adding a new edge always results in the appearance of a copy of M_2. Consequently, they are mutually not sub-3-graphs of each other.

We suppress the subscript $_3$ in all Turán related notation. It is already known (see Theorem 1.3) that for each $n \geq 6$, the full star S_n is a 1-extremal intersecting 3-graph for M_2. We are now ready to identify all Turán numbers $\mathrm{ex}^{(s)}(n; M_2)$ together with the sets of s-extremal 3-graphs $\mathrm{Ex}^{(s)}(n; M_2)$, $s \geq 1$. Let us fix the vertex set V, $|V| = n$. For $n \leq 5$ every 3-graph is intersecting and thus $\mathrm{ex}^{(1)}(n; M_2) = \binom{n}{3}$, the only 1-extremal 3-graph is the clique K_n, and the higher order Turán numbers $\mathrm{ex}^{(s)}(n; M_2)$, $s \geq 2$, do not exist.

If $n = 6$, each triple in $\binom{V}{3}$ intersects all other triples except its complement. Therefore, we may arrange all 20 triples into 10 pairs (an edge and its complement) and from each such a pair choose arbitrarily one triple to get a maximal intersecting 3-graph, consisting of 10 edges. This yields 2^{10} 3-graphs, among which we found 13 pairwise non-isomorphic ones, as specified in Proposition 1.6 below.

Proposition 1.6. *We have* $\mathrm{ex}^{(1)}(6; M_2) = 10$ *and*

$$\mathrm{Ex}^{(1)}(6; M_2) = \{S_6, K_5 \cup K_1, \ H_i(6), \ i = 1, \ldots, 11\},$$

where the $H_i(6)$*'s are defined above.*

As every intersecting 3-graph on 6 vertices is a sub-3-graph of one of the above 13 extremal 3-graphs, there are no higher order Turán numbers for $n = 6$.

Things change dramatically for $n \geq 7$. First notice that maximal intersecting 3-graphs on $n \geq 7$ vertices can be obtained from any of the 13 6-vertex 3-graphs appearing in Proposition 1.6 by adding all triples containing any of their vertex covers. In this way we obtain 3-graphs S_n, $K_5 \cup (n-5)K_1$, and $H_i(n)$, $i = 1, \ldots, 11$. As it turns out there are only two other maximal intersecting 3-graphs for $n \geq 7$.

Let F_7 be the Fano plane, that is a 3-graph on 7 vertices obtained from the triangle C_3 by adding one new vertex z and four new edges: $\{x_i, z, y_i\}$, $i = 1, 2, 3$, and $\{y_1, y_2, y_3\}$. Further, let F_{10} be a 3-graph obtained from the triangle C_3 by adding one more vertex z and 7 new edges: $\{x_1, x_2, x_3\}, \{x_1, x_2, z\}, \{x_1, z, x_3\}, \{z, x_2, x_3\}$, and $\{x_i, y_i, z\}$, $i = 1, 2, 3$.

Theorem 1.7. *For* $n \geq 7$,

1. $\mathrm{ex}^{(1)}(n; M_2) = \binom{n-1}{2}$ *and* $\mathrm{Ex}^{(1)}(n; M_2) = \{S_n\}$,
2. $\mathrm{ex}^{(2)}(n; M_2) = 3n - 8$ *and* $\mathrm{Ex}^{(2)}(n; M_2) = \{H_1(n), H_2(n)\}$,
3. $\mathrm{ex}^{(3)}(n; M_2) = 2n - 2$ *and* $\mathrm{Ex}^{(3)}(n; M_2) = \{H_3(n)\}$,
4. $\mathrm{ex}^{(4)}(n; M_2) = n + 4$ *and* $\mathrm{Ex}^{(4)}(n; M_2) = \{H_4(n), H_5(n), H_6(n)\}$,
5. $\mathrm{ex}^{(5)}(n; M_2) = 10$ *and*

$$\mathrm{Ex}^{(5)}(n; M_2) = \{K_5 \cup (n-5)K_1, F_{10} \cup (n-7)K_1, H_i(n) : i = 7, \ldots, 11\},$$

6. $\mathrm{ex}^{(6)}(n; M_2) = 7$ *and* $\mathrm{Ex}^{(6)}(n; M_2) = \{F_7 \cup (n-7)K_1\}$.

The Turán numbers $\mathrm{ex}^{(s)}(n; M_2)$ *do not exist for* $s \geq 7$.

2. PROOFS

In this section we present all our proofs. We begin with some general simple observations about the structure of maximal intersecting hypergraphs. In Subsection 2.2 we determine all Turán numbers $\mathrm{ex}^{(s)}(n; \{M_2, C_3\})$, $n \geq 6$, $s \geq 1$, and accompanying them s-extremal 3-graphs for the pair $\{M_2, C_3\}$ (see Corollary 2.5). The remaining two subsections contain the proofs of Proposition 1.6 and Theorem 1.7, respectively.

2.1. THE STRUCTURE OF MAXIMAL INTERSECTING HYPERGRAPHS

Recall that a subset of vertices T of a hypergraph H is called *a vertex cover* if it has nonempty intersection with every edge of H and that $\tau(H)$ stands for the size of the smallest vertex cover of H. For $k \geq 2$ and $n \geq 2k$, let H be an n-vertex, maximal intersecting k-graph. Clearly, every edge of H is its vertex cover, so $1 \leq \tau(H) \leq k$. We have already mentioned that if T is a vertex cover of H then, by maximality, $\mathrm{up}(T) \subseteq H$. As an immediate consequence, we deduce the following useful observation.

Fact 2.1. *For $k \geq 2$ and $n \geq 2k$, let H be an n-vertex, maximal intersecting k-graph. Then the family of all vertex covers of H is intersecting itself.*

Proof. Suppose T_1 and T_2 are two disjoint vertex covers of H. Then, since $n \geq 2k$, there are $e_i \in \mathrm{up}(T_i) \subseteq H$, $i = 1, 2$, such that $e_i \cap e_2 = \emptyset$, a contradiction. □

Our next observation will be of great help in the proof of the main theorem in Subsection 2.4. We call a subset $U \subseteq V(H)$ *a heart of* H if every two edges of H intersect on U, that is, if for all $e, f \in H$, we have $e \cap f \cap U \neq \emptyset$. *The induced* sub-$k$-graph $H[U]$ consists of all edges of H which are contained in U, that is, $H[U] = \{e \in H : e \subset U\}$. Trivially, for every $U \subseteq V(H)$, $H[U]$ is intersecting as well. It turns out that every reasonably large heart of H is also maximal.

Fact 2.2. *For $k \geq 2$ and $n \geq 2k$, let H be an n-vertex, maximal intersecting k-graph. If U is a heart of H, $|U| \geq 2k$, then $H[U]$ is maximal intersecting k-graph itself.*

Proof. Suppose not. Then there exists a k-element set $T \subset U$ and an edge $e \in H$ such that $T \cap e = \emptyset$ but $H[U] \cup \{T\}$ is still intersecting, i.e. T is a vertex cover of $H[U]$. Now, $e \setminus U$ is a vertex-cover of H, and so, $\mathrm{up}(e \setminus U) \subseteq H$. In particular, since $|U| \geq 2k$, there is an edge $f \in \mathrm{up}(e \setminus U)$ such that $f \subset U$, that is, $\in H[U]$, and $f \cap T = \emptyset$. This is, however, a contradiction with the assumption that T is a vertex cover of $H[U]$. □

2.2. TRIANGLE-FREE INTERSECTING 3-GRAPHS

Recall that a triangle C_3 consists of a vertex set $U = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and the edge set

$$C_3 = \{\{x_i, y_j, x_k\} : \{i, j, k\} = \{1, 2, 3\}\}.$$

Thus, the vertices x_1, x_2, x_3 are of degree two in C_3, while y_1, y_2, y_3 are of degree one.

The Turán numbers for C_3 were determined in [5] for $n \geq 75$ and in [2] for all n.

Theorem 2.3 ([2]). *For $n \geq 6$, $\mathrm{ex}^{(1)}(n; C) = \binom{n-1}{2}$. Moreover, for $n \geq 8$, $\mathrm{Ex}^{(1)}(n; C_3) = \{S_n\}$, for $n = 7$, $\mathrm{Ex}^{(1)}(7; C_3) = \{S_7, \mathrm{up}(\{u,v\}) \cup \binom{V \setminus \{u,v\}}{3}\}$, and for $n = 6$, $\mathrm{Ex}^{(1)}(6; C_3) = \{S_6, K_5 \cup K_1\}$.* $\square$

Define $H_0(n)$ as a 3-graph obtained from a copy of K_4 on the set of vertices $\{x, y, z, v\}$, by adding to it all the edges of the form $\{x, y, w\}$ where $w \notin \{x, y, z, v\}$, namely,

$$H_0(n) = \mathrm{up}(\{x, y\}) \cup \{\{x, z, v\}, \{y, z, v\}\}.$$

Note that $|H_0(n)| = n$, $H_0(n) \subset G_i(n)$ for $i = 1, \ldots, 5$, and $H_0(n)$ is $\{M_2, C_3\}$-free. The next lemma plays an important role in the proof of Theorem 1.7.

Lemma 2.4. *For $n \geq 6$, if H is an n-vertex $\{M_2, C_3\}$-free 3-graph not contained in the star S_n, then $H \subseteq K_5 \cup (n-5)K_1$ or $H \subset H_0(n)$.*

Proof. Let H be a $\{M_2, C_3\}$-free 3-graph H on the set of vertices V, $|V| = n$, which is not a star. We will show that H is a sub-3-graph of either $K_5 \cup (n-5)K_1$ or $H_0(n)$.

Let P_2 denote a 3-graph consisting of two edges sharing exactly one vertex. We may assume that $P_2 \subset H$, because otherwise, every two edges of H would intersect in exactly two vertices and, consequently, $H \subseteq \mathrm{up}(\{x, y\})$, for some two vertices $x, y \in V$, or $H \subseteq K_4 \cup (n-4)K_1$, implying that $H \subseteq H_0(n)$. Let us set $P_2 = \{e_1, e_2\}$, $e_1 \cap e_2 = \{x\}$, $U = V(P_2)$, and $W = V \setminus U$, $|W| = n - 5$. If all the edges of H are contained in U, then $H \subseteq K_5 \cup (n-5)K_1$. Therefore, in the rest of the proof we will be assuming that there exists an edge $f \in H$ with $f \cap W \neq \emptyset$.

As H is intersecting, every edge $f \in H$ must, in particular, intersect e_1 and e_2. Therefore, since $C_3 \nsubseteq H$, every edge $f \in H$ with $f \cap W \neq \emptyset$ contains vertex x. But $H \nsubseteq S_n$ and hence there exists an edge $h \in H$ such that $x \notin h$. As explained above, $h \subset U$. Without loss of generality we may assume that $|h \cap e_i| = i$ for $i = 1, 2$ and let $h \cap e_1 = \{y\}$, $h \cap e_2 = \{z, v\}$. Then the edges h and e_1 form another copy of P_2 and using the same argument as above, every edge $f \in H$ with $f \cap W \neq \emptyset$ must contain vertex y. Consequently, all the edges of H satisfying $f \cap W \neq \emptyset$ are of the form $\{x, y, w\}$, where w is an arbitrary vertex of W. One can check that adding to H any triple $e \in \binom{U}{3}$, except for $\{x, y, z\}$ and $\{x, y, v\}$, creates, together with an edge $\{x, y, w\}$, $w \in W$, either a triangle or a pair of disjoint edges. Hence, $H \subseteq H_0(n)$. $\square$

An immediate corollary of Theorem 2.3 and Lemma 2.4 gives the Turán numbers for the pair $\{M_2, C_3\}$. Note that the 3-graph $\mathrm{up}(\{u, v\}) \cup \binom{V \setminus \{u,v\}}{3}$ on 7 vertices contains M_2 and therefore is "disqualified" here.

Corollary 2.5. *The complete Turán hierarchy for the pair $\{M_2, C_3\}$ is as follows:*

1. *For $n \leq 5$,*

$$\mathrm{ex}^{(1)}(n; \{M_2, C_3\}) = \binom{n}{3}, \qquad \mathrm{Ex}^{(1)}(n; \{M_2, C_3\}) = \{K_n\},$$

and $\mathrm{ex}^{(s)}(n; \{M_2, C_3\})$ does not exist for $s \geq 2$.

2. *For $n = 6$,*

$$\text{ex}^{(1)}(6; \{M_2, C_3\}) = 10, \quad \text{Ex}^{(1)}(6; \{M_2, C_3\}) = \{S_6, K_5 \cup K_1\},$$

$$\text{ex}^{(2)}(6; \{M_2, C_3\}) = 6, \quad \text{Ex}^{(2)}(6; \{M_2, C_3\}) = \{H_0(6)\},$$

and $\text{ex}^{(s)}(6; \{M_2, C_3\})$ does not exist for $s \geq 3$.

3. *For $n \geq 7$,*

$$\text{ex}^{(1)}(n; \{M_2, C_3\}) = \binom{n-1}{2}, \quad \text{Ex}^{(1)}(n; \{M_2, C_3\}) = \{S_n\}.$$

4. *For $n = 10$,*

$$\text{Ex}^{(2)}(10; \{M_2, C_3\}) = \{K_5 \cup 5K_1, H_0(10)\}$$

and $\text{ex}^{(s)}(10; \{M_2, C_3\})$ does not exist for $s \geq 3$.

5. *For $n \geq 7$, $n \neq 10$,*

$$\text{ex}^{(2)}(n; \{M_2, C_3\}) = \max\{10, n\}, \quad \text{ex}^{(3)}(n; \{M_2, C_3\}) = \min\{10, n\},$$

$$\text{Ex}^{(2)}(n; \{M_2, C_3\}) \cup \text{Ex}^{(3)}(n; \{M_2, C_3\}) = \{K_5 \cup (n-5)K_1, H_0(n)\},$$

and $\text{ex}^{(s)}(n; \{M_2, C_3\})$ does not exist for $s \geq 4$.

2.3. PROOF OF PROPOSITION 1.6

To prove Proposition 1.6, we need to show that among all 1024 labeled intersecting 3-graphs on 6 vertices there are exactly 13 isomorphism types listed therein. We already know from Theorem 2.3, that only two of these 3-graphs are C_3-free, namely, S_6 and $K_5 \cup K_1$.

Not without a reason, we classify the remaining 3-graphs H, that is, those containing C_3, with respect to the number of vertex covers of size 2. Since every edge in an intersecting 3-graph is its vertex cover, the minimum size of a vertex cover is either 2 or 3 (1 is impossible due to the presence of C_3). Let us call a cover set of size 2, simply a *2-cover*.

Let C be a copy of C_3 on vertex set $U = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and with edge set $C = \{\{x_i, y_j, x_k\} : \{i, j, k\} = \{1, 2, 3\}\}$. Note that there are six different 2-covers of C: $T_i = \{x_i, y_i\}$, $i = 1, 2, 3$, $T_4 = \{x_1, x_2\}$, $T_5 = \{x_2, x_3\}$ and $T_6 = \{x_1, x_3\}$. Thus, the 2-covers of H must be among these six. But, by Fact 2.1, there are no disjoint 2-covers in H, so there are at most three 2-covers in H. Recall also that if a pair T is a 2-cover in a maximal intersecting 3-graph H, then $H \supseteq \text{up}(T)$.

Case 1. there are three 2-covers in H. Up to isomorphism, there are only two possibilities: either T_4, T_1, T_6 are the 2-covers in H or T_4, T_5, T_6 are the 2-covers in H. In each case, there are exactly ten triples belonging to C or containing at least one of these 2-covers, so there are no more triples in H. Then, it is easy to check that H is isomorphic to $H_1(6)$ (in the first case) or H is isomorphic to $H_2(6)$ (the second case).

Case 2. there are exactly two 2-covers in H. Without loss of generality either T_1 and T_4 or T_4 and T_6 are the unique 2-covers in H. In both cases H must contain all triples

that contains at least one of these sets. Therefore, in the first case H contains the following triples: $T_1 \cup \{v\}$, where $v \in \{x_2, x_3, y_2, y_3\}$ and $T_4 \cup \{v\}$ where $v \in \{x_3, y_2\}$. Moreover, since T_6 is not a 2-cover of H, there must be in H a triple disjoint from T_6 but touching both, T_1 and T_4, and all the edges of C. There is only one such triple, namely $\{x_2, y_1, y_2\}$. We have $|H| = 10$ and $H \cong H_3(6)$.

If T_4, T_6 are the unique 2-covers in H, then, as above, H contains all the triples containing T_4 or T_6, namely $T_4 \cup \{v\}$, where $v \in \{x_3, y_1, y_2\}$, and $T_6 \cup \{v\}$, where $v \in \{y_1, y_3\}$. Again, since neither T_1 nor T_5 is a 2-cover in H, H must contain two intersecting triples disjoint from T_1 and T_5, respectively, but touching both 2-sets, T_4 and T_6, and all the edges of C. Up to isomorphism there is only one possibility for this: $\{x_2, x_3, y_3\} \in H$ and $\{x_1, y_1, y_3\} \in H$. As before we have $|H| = 10$ and $H \cong H_3(6)$.

Case 3. there is exactly one 2-cover in H. We claim that H is isomorphic to one of $H_4(6), H_5(6)$, and $H_6(6)$. Note that all three have a similar structure: there is one 2-cover $\{x, y\}$, and so each consists of all four edges containing it, plus the edges adjacent to x but not y and vice versa. Let $L_i(x)$ be the set of pairs making an edge with x but not with y in $H_i(6)$, $i = 4, 5, 6$, and we define $L_i(y)$ analogously. Referring to the definitions of $H_i(6)$, $i = 4, 5, 6$, in Introduction, we see that $L_4(x) = L_4(y)$ is the (graph) triangle on z, v, w; $L_5(x)$ is the path $zvwu$, while $L_5(y)$ is the path $zwvu$, so these two paths share the middle pair; finally, $L_6(x)$ is the (graph) 4-cycle $zvwuz$, while $L_6(y)$ is a (graph) matching consisting of the two diagonals of that 4-cycle, zw and uv.

Up to isomorphism there are only two subcases. Either T_1 or T_4 is the unique 2-cover of H. Assume first it is T_1. Then we do not need to worry about T_2, T_3, and T_5 as they are all disjoint from T_1 (recall that up$(T_1) \subset H$). To prevent T_4 and T_6 from being also 2-covers of H, there must be an edge or edges in H disjoint from those two pairs, but intersecting T_1.

Assume first that $\{y_1, y_2, y_3\} \in H$ is such an edge. It takes care of both, T_4 and T_6. As H is maximal, it must also contain two more edges, say $\{x_1, x_3, y_3\}$ and $\{x_1, x_2, y_2\}$ (there are 3 more options here, in which either of these two edges is replaced by its complement; we leave their analysis to the reader). So, there are 10 edges altogether. We see that the pairs making an edge with x_1, but not with y_1, form the 4-cycle $x_2y_2x_3y_3x_2$, and there are only two pairs, x_2x_3 and y_2y_3 making an edge with y_1, but not with x_1. Thus, $H \cong H_6(6)$ (with $x := x_1$ and $y := y_1$).

If $\{y_1, y_2, y_3\} \notin H$, then its complement $\{x_1, x_2, x_3\} \in H$. The only edges which "exclude" T_4 and T_6 are $\{x_3, y_1, y_3\}$ and $\{x_2, y_1, y_2\}$, respectively. So, again, we have 10 edges in H, but this time the pairs making an edge with x_1, but not with y_1, form the path $y_3x_2x_3y_2$, while in the opposite case, it is the path $y_2x_2x_3y_3$. Thus, $H \cong H_5(6)$.

Assume now that the unique 2-cover of H is T_4. We need to "exclude" four other 2-covers of C, namely, T_1, T_2, T_5, and T_6, from being present in H. There are four subcases. In the first one, let $\{x_1, y_1, y_3\} \in H$ and $\{x_2, y_2, y_3\} \in H$ (the remaining 3 cases come from negating one or both clauses in this conjunction). The first of these two edges excludes T_2 and T_5, while the other does the same to T_1 and T_6. By maximality there are two more edges in H, say $\{x_1, y_1, y_2\}$ and $\{x_2, y_2, y_1\}$ (again, we

skip 3 more cases with involving the complements). A similar analysis of the graph links of $x := x_1$ and $y := x_2$ leads to a conclusion that, again, $H \cong H_5(6)$.

Consider now the subcase when $\{x_1, y_1, y_3\} \notin H$ and $\{x_2, y_2, y_3\} \notin H$. Then, the complements $\{x_2, x_3, y_2\} \in H$ and $\{x_1, x_3, y_1\} \in H$ exclude T_1 and T_2, respectively. In addition, we must also have $\{x_1, y_1, y_2\} \in H$ and $\{x_2, y_1, y_2\} \in H$ which take care of T_5 and T_6, respectively. A similar analysis reveals that $H \cong H_4(6)$.

The remaining two subcases are symmetrical, so we consider only one of them. Let $\{x_2, x_3, y_2\} \in H$ and $\{x_2, y_2, y_3\} \in H$. These two edges, together with $\{x_1, y_1, y_2\}$ and $\{x_1, x_3, y_3\}$, exclude all four forbidden 2-covers, T_1, T_6, T_5, and T_2. A quick look at the links of x_1 and x_2 shows that this time $H \cong H_5(6)$.

Case 4. there is no 2-cover in H. This means that for each 2-cover of C, T_i, $i = 1, \ldots, 6$, there is an edge in H disjoint from it. A tedious case by case analysis of the 2^7 remaining choices between triples and their complements (we have already made three choices by implanting the triangle C in H) leads always to one of the 3-graphs $H_i(6)$, $i = 7, \ldots, 11$. We omit the details.

2.4. PROOF OF THEOREM 1.7

Let H be a *maximal M_2-free* 3-graph with $V(H) = V$ and $|V| = n \geq 7$, not contained in a star S_n and $K_5 \cup (n-5)K_1$. Then by Lemma 2.4 we have $C_3 \subset H$ (note that since $H_0(n)$ is not maximal, $H \nsubseteq H_0(n)$). We say that a copy C of the triangle, $C \subset H$, is *a triangular heart* of H if $V(C)$ is a heart of H (see the definition of heart in Subsection 2.1). Let $C \subset H$ be a copy of C_3 in H. Set $U = V(C) = \{x_1, x_2, x_3, y_1, y_2, y_3\} \subset V$, and, $C = \{\{x_i, y_j, x_k\} : \{i, j, k\} = \{1, 2, 3\}\}$. Further, let $W = V \setminus U$, $|W| = n - 6$. Since H is intersecting, every edge of H intersects U on at least 2 vertices.

Case 1. H has no triangular heart. Then there exist two edges $h_1, h_2 \in H$ with $h_1 \cap h_2 \cap U = \emptyset$. Without loss of generality let $h_1 = \{x_1, y_1, w\}$, where $w \in W$. We start with the case $h_2 = \{x_2, y_2, w\}$ (the case $h_2 = \{x_3, y_3, w\}$ is symmetrical). There exists only one 2-cover of the edge set $C \cup \{h_1, h_2\}$, namely $T = \{x_1, x_2\}$. Therefore, all the edges $h \in H$ such that $h \cap (W \setminus \{w\}) \neq \emptyset$ contain T. There are only two triples which are disjoint from T and intersect all the edges of $C \cup \{h_1, h_2\}$, namely $h_3 = \{x_3, y_3, w\}$ and $h_4 = \{y_1, y_2, y_3\}$.

If for $i = 3, 4$, $h_i \notin H$, then the triangle $C' = \{\{x_1, w, y_1\}, \{y_1, x_2, x_3\}, \{x_3, y_2, x_1\}\}$ is a triangular heart of H, a contradiction. Therefore at least one of the edges, h_3 or h_4 belongs to H. If both $h_3 \in H$ and $h_4 \in H$, then $H[U \cup \{w\}]$ is the Fano plane F_7 which is a maximal intersecting family. Moreover, there are no 2-covers in F_7, so there are no other edges in H, and we conclude that $H = F_7 \cup (n-7)K_1$.

Next, let $h_3 \in H$ and $h_4 \notin H$. Then, since H is maximal, it contains four more edges, $\{x_1, x_2, x_3\}$, $\{x_1, x_2, w\}$, $\{x_1, x_3, w\}$ and $\{x_2, x_3, w\}$, and so, $H = F_{10} \cup (n-7)K_1$. Otherwise $h_4 \in H$ and, since $h_3 \notin H$, there are four more edges in H, $\{x_1, y_1, y_2\}$, $\{x_1, y_1, x_2\}$, $\{x_2, y_2, y_1\}$ and $\{x_2, y_2, x_1\}$. Again, $H = F_{10} \cup (n-7)K_1$.

Now we move to the case when for $i = 2, 3$, $\{x_i, y_i, w\} \notin H$. Then, since H is an intersecting family not containing a triangular heart, we must have $h_2 = \{x_2, x_3, w\} \in H$. This time there are two intersecting 2-covers of $C \cup \{h_1, h_2\}$, $T_1 = \{x_1, x_2\}$ and

$T_2 = \{x_1, x_3\}$. Like above, since there is no triangular heart in H, there must be in H two edges h_3 and h_4 such that $h_3 \cap T_1 = \emptyset$ and $h_4 \cap T_2 = \emptyset$. We have no other choice but set $h_3 = \{x_3, y_3, y_1\}$ and $h_4 = \{x_2, y_2, y_1\}$. So, there are only three more edges in H, $\{x_1, x_2, x_3\}$, $\{x_1, x_2, y_1\}$ and $\{x_1, x_3, y_1\}$. Thus, $H = F_{10} \cup (n-7)K_1$ again. As $|F_7| = 7$ and $|F_{10}| = 10$, these two 3-graphs do not play any role in establishing the first four Turán numbers for M_2.

Case 2. all the edges of H intersect each other on $U = V(C)$, that is, C is a triangular heart of H. By Fact 2.2, the induced sub-3-graph $H[U]$ is maximal. As $H[U] \supset C$, by Proposition 1.6, $H[U]$ is isomorphic to one of the 3-graphs $H_i(6)$, $i = 1, \ldots, 11$.

Since H is maximal, it consists of all triples containing any 2-cover of $H[U]$ and a vertex outside U. Hence, if $H[U] \cong H_i(6)$, then $H_i(n) \cong H_i(n)$, $i = 1, \ldots, 11$. This, in view of Lemma 2.4 and the "heartless" case 1, proves all parts of Theorem 1.7. □

3. CONCLUDING REMARKS

Upon completing this project, we realized that the maximal intersecting 3-graphs with $\tau = 3$, can be fished out from a huge family of so called *1-special* 3-graphs described in [7] (see Theorem 5 therein). However, the authors of [7] admitted that their family contains several isomorphic 3-graphs and do not provide any proof. Also recently, we noticed that independently of us, Kostochka and Mubayi [10] (see Theorem 8 therein) determined all maximal intersecting 3-graphs with more than 10 edges.

Although, both these results together can be, in principle, used to derive the main results of this paper, we feel that our streamlined and unified approach, as well as the statement in terms of the hierarchy of Turán numbers might still be interesting. Moreover, in [10] the authors attempted to describe all maximal, intersecting k-graphs for $k \geq 4$. Their result is, however, restricted to k-graphs with large number of vertices and large number of edges. We believe that our approach has the potential to be generalized to all k-graphs.

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