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Second-order optimality conditions in nonsmooth vector optimization*

by

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Abstract: In this paper, we introduce new classes of nonsmooth second-order cone-convex functions and respective generalizations in terms of first and second-order directional derivative. These classes encapsulate several already existing classes of cone-convex functions and their weaker variants. Second-order KKT type sufficient optimality conditions and duality results for a nonsmooth vector optimization problem are proved using these functions. The results have been supported by examples.

Keywords: vector optimization; cones; nonsmooth second-order cone-convexity; second-order optimality; duality

1. Introduction

It is well-known that second-order optimality conditions have important applications in sensitivity analysis and optimal algorithms for example penalty methods (see Auslender, 1979; Facchinei and Lucidi, 1998). There is a need for studying second-order optimality conditions for nonsmooth vector optimization problems because, first, the differentiability condition does not always hold. Secondly, as pointed by Cominetti and Correa (1990), there are many techniques commonly used in optimization theory that generate nonsmoothness even when the problems are differentiable. For example, in duality theory, sensitivity and stability analysis, decomposition techniques, penalty methods and many more (see Auslender, 1979; Yuan et al., 2010).

We have a rich literature that deals with vector optimization problems containing twice differentiable data (wherein the objective function and constraints

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are twice differentiable) with both natural (cone is $\mathbb{R}^n_+$) and unnatural ordering cones. Generalizations to second-order convex function (Mond, 1974) like second-order (F, ρ) convex (Aghezzaf, 2003), second-order (F, α, ρ, d) convex (Ahmad and Husain, 2006), second-order cone-convex (Suneja, Sharma and Vani, 2008), genearalized higher order (F, ρ, θ, m, h) (Kumar and Sharma, 2017) and many others along with their weaker notions have been defined for twice differentiable functions and used to study second-order duality results for multiobjective and vector optimization problems. Mangasarian (1975) first formulated the second-order dual involving second-order derivatives for a nonlinear programming problem and established second-order duality results under certain inclusion conditions. By introducing two additional parameters, Hanson (1993) formulated a second-order dual similar to that of Mangasarian (1975) and established duality results under the assumption of second-order type I invexity. Mishra (1997) deduced second-order duality results for multiobjective programming problem using classes of second-order pseudo-type I, second-order quasi-type I and related functions. Suneja, Sharma and Vani (2008) defined second-order cone-convex, second-order cone-pseudoconvex and second-order cone-quasiconvex functions and used them to prove second-order duality results for vector optimization problem over cones with twice differentiable data.

In this paper, in the absence of twice differentiability, we have extended the classes of second-order cone-convex and related functions from Suneja, Sharma and Vani (2008) to the nonsmooth setting. We have defined new classes of nonsmooth second-order cone-convex, nonsmooth second-order (strictly, strongly) cone-pseudoconvex and nonsmooth second-order cone-quasiconvex functions in terms of second-order directional derivative. Interrelations among the above functions have been studied and illustrated by examples. We have also discussed the conditions, under which these functions reduce to the functions already existing in the literature. Using these functions, we have proven second-order KKT type sufficient optimality conditions and Mond-Weir type duality results for nonsmooth vector optimization problem over cones.

2. Notations and definitions

Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed $(K \cap (-K) = \{0\})$ cone with vertex at the origin, such that $intK \neq \emptyset$, where $intK$ denotes interior of K. The positive dual cone K^+ and strict positive dual cone K^{s+} are defined as follows:

$$K^+ := \{y \in \mathbb{R}^m : z^T y \geq 0, \ \ \forall\, z \in K\}$$

and

$$K^{s+} := \{y \in \mathbb{R}^m : z^T y > 0, \ \ \forall\, z \in K_0 = K \setminus \{0\}\}.$$

Since the cone under consideration is closed and convex, by bipolar theorem there is $K = (K^+)^+$. In this case,

$$x \in K \Longleftrightarrow \lambda^T x \geq 0, \ \ \forall\, \lambda \in K^+.$$

As given by Flores–Baźan, Hadjisavvas and Vera (2007), we have

$$x \in intK \Longleftrightarrow \lambda^T x > 0, \ \ \forall\, \lambda \in K^+ \setminus \{0\}.$$

Let $S \subseteq \mathbb{R}^n$ be a non-empty open subset and $f = (f_1, f_2, \ldots, f_m)^T : S \to \mathbb{R}^m$ be a vector valued function. We recall the definitions of first and second-order directionally differentiable functions, which are weaker notions as compared to that of differentiability, and twice differentiability, respectively.

Definition 2.1 *The first-order directional derivative of $f_i : S \to \mathbb{R}$ at $x \in S$ in the direction $d \in \mathbb{R}^n$ is defined as an element of $\mathbb{R}$ given by:*

$$f_i'(x, d) := \lim_{t \to 0^+} \frac{f_i(x + td) - f_i(x)}{t}.$$

If $f_i'(x, d)$ exists and is finite, then function f_i is called first-order directionally differentiable at x in the direction d. The function f_i is said to be first-order directionally differentiable on S if the derivative $f_i'(x, d)$ exists for each $x \in S$ and direction $d \in \mathbb{R}^n$.

Definition 2.2 (Demyanov and Pernyi, 1974) *Suppose $f_i : S \to \mathbb{R}$ is first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$. The second-order directional derivative of f_i at x in the direction d is defined as an element of $\mathbb{R}$, given by:*

$$f_i''(x, d) := \lim_{t \to 0^+} \frac{2(f_i(x + td) - f_i(x) - t f_i'(x, d))}{t^2}.$$

If $f_i''(x, d)$ exists and is finite, then function f_i is called second-order directionally differentiable at x in the direction d. The function f_i is said to be second-order directionally differentiable on S if it is first-order directionally differentiable on S and the derivative $f_i''(x, d)$ exists for each $x \in S$ and direction $d \in \mathbb{R}^n$.

Remark 2.1 *f is said to be first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$ if each f_i is first-order directionally differentiable at x in the direction d. The first-order directional derivative of f at x in the direction d is defined to be the vector:*

$$(f_1'(x, d), f_2'(x, d), \ldots, f_m'(x, d))^T.$$

Remark 2.2 *Suppose f is first-order directionally differentiable at $x \in S$ in the direction $d \in \mathbb{R}^n$. f is said to be second-order directionally differentiable at x in the direction d if each f_i is second-order directionally differentiable at x in the direction d. The second-order directional derivative of f at x in the direction d is defined to be the vector:*

$$(f_1''(x,d), f_2''(x,d), \ldots, f_m''(x,d))^T.$$

Remark 2.3 *In the absence of twice differentiability, the idea of using first and second-order directional derivatives stems from the observation that if f is twice continuously differentiable at $\bar{x}$, then*

$$f'(\bar{x}, x-\bar{x}) = \nabla f(\bar{x})(x-\bar{x})$$

and

$$f''(\bar{x}, x-\bar{x}) = (x-\bar{x})^T \nabla^2 f(\bar{x})(x-\bar{x})$$

where $(x-\bar{x})^T \nabla^2 f(\bar{x})(x-\bar{x})$ denotes the vector

$$\left((x-\bar{x})^T \nabla^2 f_1(\bar{x})(x-\bar{x}), \ldots, (x-\bar{x})^T \nabla^2 f_m(\bar{x})(x-\bar{x})\right)^T.$$

For each $i = 1, 2, \ldots, m$, $\nabla^2 f_i(\bar{x})$ is the $n \times n$ Hessian matrix of f_i at $\bar{x}$ and $\nabla f(\bar{x})$ is the $m \times n$ Jacobian matrix of f at $\bar{x}$.

Now, we introduce new classes of nonsmooth second-order cone-convex functions and their weaker variants involving first and second-order directional derivatives. These will be used to study second-order optimality conditions and duality results for the nonsmooth vector optimization problem. Let $\bar{x} \in S$, where S is a non-empty open subset of $\mathbb{R}^n$, $K \subseteq \mathbb{R}^m$, be a closed convex pointed cone with $intK \neq \emptyset$ and $f = (f_1, f_2, \ldots, f_m) : S \to \mathbb{R}^m$ be first and second-order directionally differentiable vector valued function.

Definition 2.3 *f is said to be nonsmooth second-order K-convex at $\bar{x}$, if there exists a real valued function $\omega(.,.) : S \times S \longrightarrow [0, \infty)$ such that for all $x \in S$*

$$f(x) - f(\bar{x}) - f'(\bar{x}, x-\bar{x}) - \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x}) \in K.$$

Definition 2.4 *f is said to be nonsmooth second-order K-pseudoconvex at $\bar{x}$, if there exists a real valued function $\omega(.,.) : S \times S \longrightarrow [0, \infty)$ such that for all $x \in S$*

$$\begin{aligned} &-[f'(\bar{x}, x-\bar{x}) + 2\omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \notin intK \\ \Longrightarrow &-[f(x) - f(\bar{x}) + \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \notin intK. \end{aligned}$$

Definition 2.5 *f is said to be nonsmooth second-order K-quasiconvex at $\bar{x}$, if there exists a real valued function $\omega(.,.) : S \times S \longrightarrow [0,\infty)$ such that for all $x \in S$*

$$[f(x) - f(\bar{x}) + \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \notin intK$$
$$\Longrightarrow -[f'(\bar{x}, x-\bar{x}) + 2\omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \in K.$$

Definition 2.6 *f is said to be nonsmooth second-order strongly K-pseudoconvex at $\bar{x}$, if there exists a real valued function $\omega(.,.) : S \times S \longrightarrow [0,\infty)$ such that for all $x \in S$*

$$-[f'(\bar{x}, x-\bar{x}) + 2\omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \notin intK$$
$$\Longrightarrow [f(x) - f(\bar{x}) + \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \in K.$$

Definition 2.7 *f is said to be nonsmooth second-order strictly K-pseudoconvex at $\bar{x}$, if there exists a real valued function $\omega(.,.) : S \times S \longrightarrow [0,\infty)$ such that for all $x \in S$*

$$-[f(x) - f(\bar{x}) + \omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \in K_0$$
$$\Longrightarrow -[f'(\bar{x}, x-\bar{x}) + 2\omega(x,\bar{x}) f''(\bar{x}, x-\bar{x})] \in intK.$$

Remark 2.4 *Following are few important reductions of the new classes defined above:*

(*i*) *Suppose f is twice continuously differentiable at $\bar{x}$ and $\omega(.,.) \equiv \frac{1}{2}$. Then, nonsmooth second-order K-convex function and its generalizations reduce to the second-order K-convex function and the corresponding generalizations, defined by Suneja, Sharma and Vani (2008).*

(*ii*) *Let $m = 1$, $K = \mathbb{R}_+$. Suppose f is twice continuously differentiable at $\bar{x}$ and $\omega(.,.) \equiv \frac{1}{2}$. Then, nonsmooth second-order K-convex function and its generalizations reduce to the second-order convex function and the corresponding generalizations, defined by Mond (1974) and Mond and Weir (1981–1983).*

(*iii*) *If $\omega(.,.) \equiv 0$, then nonsmooth second-order (strongly, strictly) K-pseudoconvex function reduces to the (strongly, strictly) pseudoconvex with respect to K and nonsmooth second-order K-quasiconvex function reduces to quasiconvex with respect to K, introduced by Aggarwal (1998).*

(*iv*) *Suppose f is differentiable at $\bar{x}$ and $\omega(.,\bar{x}) \equiv 0$, then nonsmooth second-order K-convex function and its generalizations reduce to the K-convex function and the corresponding generalizations, defined by Giorgi and Guerraggio (1996).*

Remark 2.5 *The second-order (cone)-convex functions and its generalizations for twice differentiable functions have been introduced using Jacobian and Hessian matrix. Using the relation mentioned in the Remark 2.3, it can be observed* *that the first and second-order directional derivatives help in direct extension of the above already existing cone-convexity and second-order cone-convexity concepts in nonlinear and vector optimization problems.*

Remark 2.6 *Clearly, every nonsmooth second-order K-convex function with respect to $\omega(.,.)$ is nonsmooth second-order K-pseudoconvex function with respect to same $\omega(.,.)$ but converse is not true as can be seen from the following example.*

Example 2.1 *Let $S = (-3, 3) \subseteq \mathbb{R}$. Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ as*

$$f_1(x) = \begin{cases} \dfrac{1}{x+1}, & x > 0 \\ \dfrac{1}{-x+1}, & x \leqslant 0 \end{cases}, f_2(x) = \begin{cases} \dfrac{x}{x^2+1}, & x > 0 \\ x^2, & x \leqslant 0 \end{cases}.$$

Let $\bar{x} = 0$. Then,

$$f'(0, x) = \begin{cases} (-x, x)^T, & x > 0 \\ (x, 0)^T, & x \leqslant 0 \end{cases}$$

$$f''(0, x) = \begin{cases} (2x^2, 0)^T, & x > 0 \\ (2x^2, 2x^2)^T, & x \leqslant 0 \end{cases}.$$

Let $K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leqslant 0, x_2 \leqslant x_1\}$ and $\omega : S \times S \longrightarrow (0, \infty)$ be defined as

$$\omega(x, \bar{x}) = \begin{cases} \left(\dfrac{x^2 + x + 2}{(x+1)(x^2+1)x} \right) + \bar{x}^2, & x > 0 \\ 1 + \bar{x}^2, & x \leq 0 \end{cases}.$$

Now, f is nonsmooth second-order K-pseudoconvex at $\bar{x} = 0$ with respect to $\omega(.,.)$ as

$$intK \ni -[f(x) - f(0) + \omega(x, 0) f''(0, x)] =$$

$$\begin{cases} \left(\dfrac{-1}{x+1} + 1 - \dfrac{2(x^2+x+2)x}{(x+1)(x^2+1)}, -\frac{x}{x^2+1} \right)^T, & x > 0 \\ \left(\dfrac{-1}{1-x} + 1 - 2x^2, -3x^2 \right)^T, x \leq 0 \end{cases}$$

implies that

$$x < 0 \Longrightarrow intK \ni -[f'(0, x) + 2\omega(x, 0) f''(0, x)] =$$

$$\begin{cases} \left(x - \dfrac{4(x^2+x+2)x}{(x+1)(x^2+1)}, -x \right)^T, & x > 0 \\ \left(-x - 4x^2, -4x^2 \right)^T, & x \leq 0 \end{cases}.$$

However, f is not nonsmooth second-order K-convex at $\bar{x}$ with respect to $\omega(.,.)$ as for $x = -2$

$$f(x) - f(0) - f'(0,x) - \omega(x,0)f''(0,x) = \left(-\tfrac{20}{3}, -4\right)^T \notin K.$$

REMARK 2.7 *Every nonsmooth second-order strictly K-pseudoconvex function with respect to $\omega(.,.)$ is nonsmooth second-order K-pseudoconvex with respect to same $\omega(.,.)$, but converse is not true as can be seen from the following example.*

EXAMPLE 2.2 *Let $S = (-8, 8) \subseteq \mathbb{R}$. Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ as*

$$f_1(x) = \begin{cases} 0, & x \geq 0 \\ x^2, & x < 0 \end{cases} \text{ and } f_2(x) = x^2.$$

Let $\bar{x} = 0$. Then,

$$f'(0,x) = (0,0)^T \text{ and } f''(0,x) = \begin{cases} (0, 2x^2)^T, & x \geq 0 \\ (2x^2, 2x^2)^T, & x < 0 \end{cases}.$$

Let $K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leq 0, x_1 \geq x_2\}$ and $\omega : S \times S \longrightarrow [0, \infty)$ be a constant real valued function with $\omega(.,.) \equiv 1$. Now, f is nonsmooth second-order K-pseudoconvex at $\bar{x} = 0$ with respect to $\omega(.,.)$ as

$$intK \ni -[f(x) - f(0) + \omega(x,0)f''(0,x)] =$$
$$\begin{cases} \left(0, -3x^2\right)^T, & x \geq 0 \\ \left(-3x^2, -3x^2\right)^T, & x < 0 \end{cases}$$

implies that

$$x > 0 \Longrightarrow intK \ni -[f'(0,x) + 2\omega(x,0)f''(0,x)] =$$
$$\begin{cases} \left(0, -4x^2\right)^T, & x \geq 0 \\ \left(-4x^2, -4x^2\right)^T, & x < 0 \end{cases}.$$

However, f is not nonsmooth second-order strictly K-pseudoconvex at $\bar{x} = 0$ with respect to $w(.,.)$ as for $x < 0$,

$$K_0 \ni -[f(x) - f(0) + \omega(x,0)f''(x,0)] =$$
$$\begin{cases} \left(0, -3x^2\right)^T, & x \geq 0 \\ \left(-3x^2, -3x^2\right)^T, & x < 0 \end{cases}$$

but

$$-[f'(0,x) + 2\omega(x,0)f''(0,x)] =$$
$$\begin{cases} \left(0, -4x^2\right)^T, & x \geq 0 \\ \left(-4x^2, -4x^2\right)^T, & x < 0 \end{cases} \notin intK.$$

Remark 2.8 *Every nonsmooth second-order strongly K-pseudoconvex function with respect to $\omega(.,.)$ is nonsmooth second-order K-pseudoconvex with respect to the same $\omega(.,.)$, but the converse is not true. Let us consider the last example again.*

Example 2.3 *We have shown in the last example that f is nonsmooth second-order K-pseudoconvex at $\bar{x} = 0$ with respect to $\omega(.,.)$. However, f is not nonsmooth second-order strongly K-pseudoconvex at $\bar{x} = 0$ with respect to the same $\omega(.,.)$, as for $x < 0$,*

$$-[f'(0,x) + 2\omega(x,0)f''(0,x)] = \begin{cases} \left(0, -4x^2\right)^T, & x \geq 0 \\ \left(-4x^2, -4x^2\right)^T, & x < 0 \end{cases} \notin intK$$

but

$$[f(x) - f(0) + \omega(x,0)f''(x,0)] = \begin{cases} \left(0, 3x^2\right)^T, & x \geq 0 \\ \left(3x^2, 3x^2\right)^T, & x < 0 \end{cases} \notin K.$$

3. Second-order optimality conditions

We consider the following nonsmooth vector optimization problem:

$$\begin{aligned} &K\text{-Minimize} \quad f(x) \\ &\text{subject to} \quad -g(x) \in Q, \end{aligned} \qquad \text{(VOP)}$$

where S is a non-empty open subset of $\mathbb{R}^n$ and $f = (f_1, f_2, \ldots, f_m)^T : S \to \mathbb{R}^m$, $g = (g_1, g_2, \ldots, g_p)^T : S \to \mathbb{R}^p$ are first and second-order directionally differentiable vector valued functions on S. K and Q are closed convex pointed cones with non-empty interiors in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, and $S_0 = \{x \in S : -g(x) \in Q\}$ denotes the set of all feasible solutions of (VOP).

Vector optimization problems do not possess an optimal solution in the sense that such a solution optimizes all the objective functions simultaneously. The solution concepts for vector optimization problem (VOP) are defined as follows:

Definition 3.1 (Coladas, Li and Wang, 1994) *A point $\bar{x} \in S_0$ is called a*

(*i*) *weak minimum of (VOP) if for all $x \in S_0, f(\bar{x}) - f(x) \notin intK$.*
(*ii*) *minimum of (VOP) if for all $x \in S_0, f(\bar{x}) - f(x) \notin K_0 = K \setminus \{0\}$.*
(*iii*) *strong minimum of (VOP) if for all $x \in S_0, f(x) - f(\bar{x}) \in K$.*

We begin with proving second-order KKT type sufficient optimality conditions for a feasible point to be weak minimum of (VOP) using nonsmooth second-order cone-convexity. The optimality conditions involve first and second-order directional derivatives, which allows us to employ nonsmooth second-order cone-convexity concepts and its generalizations in proving the result. These optimality conditions help to locate (weak, strong) minimizer for vector optimization problem, where the functions involved are not differentiable, but possess first- and second–order directional derivative. Thus, the results are applicable for a broader class of functions.

THEOREM 3.1 *Let f be nonsmooth second-order K-convex and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to $\omega(.,.)$. Suppose there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that for all $x \in S_0$,*

$$\bar{\lambda}^T f'(\bar{x}, x-\bar{x}) + \bar{\mu}^T g'(\bar{x}, x-\bar{x}) + 2\omega(x,\bar{x})[\bar{\mu}^T g''(\bar{x}, x-\bar{x})] \geq 0, \tag{1}$$

$$\bar{\mu}^T g(\bar{x}) - \omega(x,\bar{x})\bar{\mu}^T g''(\bar{x}, x-\bar{x}) \geq 0, \text{ and} \tag{2}$$

$$\omega(x,\bar{x})\bar{\lambda}^T f''(\bar{x}, x-\bar{x}) = 0. \tag{3}$$

Then, $\bar{x}$ is a weak minimum of (VOP).

PROOF Let, if possible, $\bar{x}$ be not a weak minimum of (VOP). Then, there exists $u \in S_0$ such that, $f(\bar{x}) - f(u) \in intK$. As $\bar{\lambda} \in K^+ \setminus \{0\}$, we get

$$\bar{\lambda}^T[f(\bar{x}) - f(u)] > 0. \tag{4}$$

Since f is nonsmooth second-order K-convex at $\bar{x}$ with respect to $\omega(.,.)$ and $\bar{\lambda} \in K^+ \setminus \{0\}$, we obtain

$$\bar{\lambda}^T[f(u) - f(\bar{x}) - f'(\bar{x}, u-\bar{x}) - \omega(u,\bar{x})f''(\bar{x}, u-\bar{x})] \geq 0. \tag{5}$$

Adding (4) and (5), we get

$$-\bar{\lambda}^T f'(\bar{x}, u-\bar{x}) - \omega(u,\bar{x})\bar{\lambda}^T f''(\bar{x}, u-\bar{x}) > 0.$$

Using the above equation, (1) and (3), we obtain

$$\bar{\mu}^T g'(\bar{x}, u-\bar{x}) + 2\omega(u,\bar{x})\bar{\mu}^T g''(\bar{x}, u-\bar{x}) > 0. \tag{6}$$

Nonsmooth second-order cone-convexity of g at $\bar{x}$ with respect to $\omega(.,.)$, together with $\bar{\mu} \in Q^+$ implies

$$\bar{\mu}^T[g(u) - g(\bar{x}) - g'(\bar{x}, u-\bar{x}) - \omega(u,\bar{x})g''(\bar{x}, u-\bar{x})] \geq 0. \tag{7}$$

By adding (6) and (7), we get

$$\bar{\mu}^T[g(u) - g(\bar{x}) + \omega(u,\bar{x})g''(\bar{x}, u - \bar{x})] > 0.$$

Using (2), we obtain that $\bar{\mu}^T g(u) > 0$, which is a contradiction to $u \in S_0$. Hence, $\bar{x}$ is a weak minimum of (VOP). ■

Along the similar lines, we can prove the following second-order KKT type sufficient optimality conditions for a feasible point to be a minimum and strong minimum of (VOP).

Theorem 3.2 *Let f be nonsmooth second-order K-convex and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to $\omega(.,.)$. Suppose there exist $\bar{\lambda} \in K^{s+}, \bar{\mu} \in Q^+$ such that for all $x \in S_0$, (1), (2) and (3) hold. Then, $\bar{x}$ is a minimum of (VOP).*

Theorem 3.3 *Let f be nonsmooth second-order K-convex and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to $\omega(.,.)$. Suppose there exist $\bar{\mu} \in Q^+$ such that for all $x \in S_0$, (1) and (2) hold and (1) and (3) hold for every $\lambda \in K^+$. Then, $\bar{x}$ is a strong minimum of (VOP).*

Next, we prove the KKT type sufficient optimality conditions for (VOP) under the weaker assumption of nonsmooth second-order K-pseudoconvexity and nonsmooth second-order Q-quasiconvexity.

Theorem 3.4 *Let f be nonsmooth second-order K-pseudoconvex and g be nonsmooth second-order Q-quasiconvex at $\bar{x} \in S_0$ with respect to $\omega(.,.)$. Suppose there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that for all $x \in S_0$, (1), (2) and (3) hold. Then, $\bar{x}$ is a weak minimum of (VOP).*

Proof Let $x \in S_0$. Then, $\bar{\mu}^T g(x) \leq 0$. Using (2), we get

$$\bar{\mu}^T g(x) - \bar{\mu}^T g(\bar{x}) + \omega(x,\bar{x})\bar{\mu}^T g''(\bar{x}, x - \bar{x}) \leq 0.$$

If $\bar{\mu} \neq 0$, then

$$g(x) - g(\bar{x}) + \omega(x,\bar{x})g''(\bar{x}, x - \bar{x}) \notin intQ.$$

Since g is nonsmooth second-order Q-quasiconvex at $\bar{x}$ with respect to $\omega(.,.)$, we obtain

$$\begin{aligned} &- [g'(\bar{x}, x - \bar{x}) + 2\omega(x,\bar{x})g''(\bar{x}, x - \bar{x})] \in Q \\ \Longrightarrow &\bar{\mu}^T g'(\bar{x}, x - \bar{x}) + 2\omega(x,\bar{x})\bar{\mu}^T g''(\bar{x}, x - \bar{x}) \leq 0. \end{aligned}$$

The above inequality also holds for $\bar{\mu} = 0$. From (1), we get

$$\bar{\lambda}^T f'(\bar{x}, x - \bar{x}) \geq 0.$$

Using (3), the above can be written as

$$\bar{\lambda}^T f'(\bar{x}, x - \bar{x}) + 2\omega(x, \bar{x})\bar{\lambda}^T f''(\bar{x}, x - \bar{x}) \geq 0.$$

Since $\bar{\lambda} \neq 0$, we get

$$-[f'(\bar{x}, x - \bar{x}) + 2\omega(x, \bar{x}) f''(\bar{x}, x - \bar{x})] \notin intK.$$

Now f is nonsmooth second-order K-pseudoconvex at $\bar{x}$ with respect to $\omega(.,.)$, therefore

$$-[f(x) - f(\bar{x}) + \omega(x, \bar{x}) f''(\bar{x}, x - \bar{x})] \notin intK.$$

From (3) we get,

$$f(\bar{x}) - f(x) \notin intK.$$

As $x \in S_0$ is arbitrary, we get

$$f(\bar{x}) - f(x) \notin intK, \; \forall\, x \in S_0.$$

Hence, $\bar{x}$ is a weak minimum of (VOP). ■

Remark 3.1 *In the above theorem, if we assume f to be nonsmooth second-order strictly K-pseudoconvex (nonsmooth second-order strongly K-pseudoconvex), then $\bar{x}$ will be a minimum (strong minimum) of (VOP).*

We conclude this section with an example to illustrate Theorem 3.4.

Example 3.1 *Let*

$$S = (-2, 2) \subseteq \mathbb{R}, K = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_2 \leqslant 0, x_2 \leqslant x_1\}$$

and

$$Q = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, x_1 \geq x_2\}.$$

Define $f = (f_1, f_2)^T : S \longrightarrow \mathbb{R}^2$ *and* $g = (g_1, g_2)^T : S \longrightarrow \mathbb{R}^2$ *as*

$$f_1(x) = f_2(x) = \begin{cases} \dfrac{1}{x+1}, & x \geqslant 0 \\ \dfrac{1}{-x+1}, & x < 0 \end{cases}$$

$$g_1(x) = \begin{cases} 0, & x \geqslant 0 \\ x^2, & x < 0 \end{cases} \text{ and } g_2(x) = \begin{cases} x^2, & x \geqslant 0 \\ -x^2, & x < 0 \end{cases}.$$

The feasible set of the corresponding problem (VOP) is $S_0 = [0, 2)$. *Let* $\bar{x} = 0$. *Then,*

$$f'(0, x) = \begin{cases} (-x, -x)^T, & x \geqslant 0 \\ (x, x)^T, & x < 0 \end{cases} \text{ and } f''(0, x) = (2x^2, 2x^2)^T.$$

$$g'(0, x) = (0, 0)^T \text{ and } g''(0, x) = \begin{cases} (0, 2x^2)^T, & x \geqslant 0 \\ (2x^2, -2x^2)^T, & x < 0 \end{cases}.$$

Let $\omega : S \times S \longrightarrow [0, \infty)$ *be a real valued function with* $\omega(., .) \equiv 1$. *Now,* f *is nonsmooth second-order* K*-pseudoconvex at* $\bar{x} = 0$ *with respect to* $\omega(., .)$ *as*

$$-[f'(0, x) + 2\omega(x, 0)f''(0, x)] =$$
$$\begin{cases} \left(x - 4x^2, x - 4x^2\right)^T, & x \geqslant 0 \\ \left(-x - 4x^2, -x - 4x^2\right)^T, & x < 0 \end{cases} \notin intK$$

$\Longrightarrow x \in (-2, 2)$ *and for all such* x,

$$-[f(x) - f(0) + \omega(x, 0)f''(0, x)] =$$
$$\begin{cases} (\frac{-1}{x+1} + 1 - 2x^2, \frac{-1}{x+1} + 1 - 2x^2)^T, x \geqslant 0 \\ (\frac{1}{x-1} + 1 - 2x^2, \frac{1}{x-1} + 1 - 2x^2)^T, x < 0 \end{cases} \notin intK.$$

Also, g *is nonsmooth second-order* Q*-quasiconvex at* $\bar{x} = 0$ *with respect to* $\omega(., .)$ *as*

$$[g(x) - g(0) + \omega(x, 0)g''(0, x)] = \begin{cases} (0, 3x^2)^T, & x \geqslant 0 \\ (3x^2, -3x^2)^T, & x < 0 \end{cases} \notin intQ$$

$$\Longrightarrow x \geqslant 0$$

$$\Longrightarrow -[g'(0, x) + 2\omega(x, 0)g''(0, x)] = \begin{cases} (0, -4x^2)^T, & x \geqslant 0 \\ (-4x^2, 4x^2)^T, & x < 0 \end{cases} \in Q.$$

Here, $K^+ = \{(x_1, x_2) : x_2 \leqslant -x_1 \leqslant 0\}, Q^+ = \{(x_1, x_2) : -x_1 \leqslant x_2 \leqslant 0\}$.
For $\lambda = (1, -1) \in K^+ \setminus \{0\}$ *and* $\mu = (1, 0) \in Q^+$, *the following conditions hold for all* $x \in S_0$:

$$\lambda^T f'(0, x) + \mu^T [g'(0, x) + 2\omega(x, 0)g''(0, x)] = \begin{cases} 0, & x \geqslant 0 \\ 4x^2, & x < 0 \end{cases} \geqslant 0,$$

$$\mu^T g(0) - \omega(x, 0)\mu^T g''(0, x) = \begin{cases} 0, & x \geqslant 0 \\ -2x^2, & x < 0 \end{cases} \geqslant 0,$$

$$\omega(x, 0)\lambda^T f''(0, x) = 0.$$

Therefore, by Theorem 3.4, $\bar{x} = 0$ *is a weak minimum of (VOP).*

4. Second-order duality

Aggarwal (1998) associated first-order dual with (VOP) in terms of first-order directional derivative and proved duality results under the assumption of pseudoconvexity and quasiconvexity with respect to cone. Suneja, Sharma and Vani (2008) formulated a second-order dual involving first and second-order derivatives for vector optimization problem over cones and established duality results using second-order cone-convex functions and its weaker notions.

In the absence of second-order derivatives, we formulate the following second-order Mond-Weir type Dual (MD) in terms of first and second-order directional derivatives and establish duality results using nonsmooth second-order strongly cone-pseudoconvexity and nonsmooth second-order cone-quasiconvexity:

$$K\text{-Maximize} \quad f(u) \tag{MD}$$

$$\text{subject to} \quad \lambda^T f'(u, x-u) + \mu^T g'(u, x-u)$$
$$+ 2\xi[\lambda^T f''(u, x-u) + \mu^T g''(u, x-u)] \geq 0, \forall x \in S_0, \tag{8}$$
$$\mu^T g(u) - \xi \mu^T g''(u, x-u) \geq 0, \quad \forall x \in S_0, \tag{9}$$
$$\xi \lambda^T f''(u, x-u) \leq 0, \quad \forall x \in S_0, \tag{10}$$

$u \in S, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+, \xi \in \mathbb{R}_+$. In general, ξ can be regarded as a function.

Let D be the feasible set of (MD).

DEFINITION 4.1 *A point $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{\xi}) \in D$ is called weakly efficient solution (weak maximum) of (MD) if for all $(u, \lambda, \mu, \xi) \in D$, $f(u) - f(\bar{u}) \notin intK$.*

THEOREM 4.1 (WEAK DUALITY) *Let $x \in S_0$ and $(u, \lambda, \mu, \xi) \in D$. Suppose f is nonsmooth second-order strongly K-pseudoconvex and g is nonsmooth second-order Q-quasiconvex at u with respect to ξ. Then, $f(u) - f(x) \notin intK$.*

PROOF Since $x \in S_0$, we get $\mu^T g(x) \leqslant 0$. This, along with equation (9), implies

$$\mu^T g(x) - \mu^T g(u) + \xi(x, u) \mu^T g''(u, x-u) \leq 0.$$

If $\mu \neq 0$, then

$$g(x) - g(u) + \xi(x, u) g''(u, x-u) \notin intQ.$$

As g is nonsmooth second-order Q-quasiconvex at u with respect to $\xi(.,.)$, we obtain

$$- [g'(u, x-u) + 2\xi(x, u) g''(u, x-u)] \in Q$$
$$\Longrightarrow \mu^T g'(u, x-u) + 2\xi(x, u) \mu^T g''(u, x-u) \leq 0.$$

The above inequality also holds for $\mu = 0$. From (8), we get

$$\lambda^T f'(u, x-u) + 2\xi(x,u)\lambda^T f''(u, x-u) \geq 0.$$

As $\lambda \neq 0$, the above implies

$$-[f'(u, x-u) + 2\xi(x,u) f''(u, x-u)] \notin intK.$$

Also f is nonsmooth second-order strongly K-pseudoconvex at u with respect to $\xi(.,.)$, therefore

$$\begin{aligned} &[f(x) - f(u) + \xi(x,u) f''(u, x-u)] \in K \\ \Longrightarrow &\lambda^T [f(x) - f(u) + \xi(x,u) f''(u, x-u)] \geq 0. \end{aligned}$$

From (10), we get $\lambda^T [f(x) - f(u)] \geq 0$. Hence, $f(u) - f(x) \notin intK$. ■

To prove the Strong Duality result, we use the first-order KKT type necessary optimality conditions, derived by Aggarwal (1998) under the following regularity condition.

Definition 4.2 (Aggarwal, 1998) *The function g is said to satisfy regularity condition at $\bar{x} \in S$ if*

$$g'(\bar{x}; S - \bar{x}) + \{\alpha g(\bar{x}) \mid \alpha \geq 0\} + Q = \mathbb{R}^p. \tag{11}$$

Theorem 4.2 (Aggarwal, 1998) *Let $\bar{x}$ be a weak minimum of (VOP). If $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike and $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition* (11) *holds at $\bar{x}$, then there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that*

$$\lambda^T f'(\bar{x}, x-\bar{x}) + \mu^T g'(\bar{x}, x-\bar{x}) \geq 0, \forall\, x \in S, \tag{12}$$

$$\mu^T g(\bar{x}) = 0. \tag{13}$$

Theorem 4.3 (Strong Duality) *Let $\bar{x}$ be a weak minimum of (VOP). Assume that $f'(\bar{x}, x-\bar{x})$ is K-subconvexlike and $g'(\bar{x}, x-\bar{x})$ is Q-subconvexlike on S and the regularity condition* (11) *holds at $\bar{x}$. Then, there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is feasible for the dual problem (MD) and the objective function values of (VOP) and (MD) are equal. Moreover, if the conditions of Weak Duality Theorem 4.1 hold for all $(u, \lambda, \mu, \xi) \in D$, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is a weak maximum of (MD).*

Proof Since $\bar{x}$ is a weak minimum of (VOP), by Theorem 4.2 there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that (12) and (13) are satisfied. Then, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is feasible for the dual problem (MD) and objective function values of (VOP) and (MD) are equal. Let, if possible, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ be not a weak maximum of (MD), then there exists $(u, \lambda, \mu, \xi) \in D$ such that $f(u) - f(\bar{x}) \in intK$, which is a contradiction to Weak Duality Theorem 4.1. Hence, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi} = 0)$ is a weak maximum of (MD). ■

REMARK 4.1 *The second-order KKT type sufficient optimality conditions involve ω with second-order directional derivatives of both the objective function f as well as constraint function g. Thus, we need to assume that f and g satisfy nonsmooth second-order cone-convexity assumptions (and their generalizations) with respect to the same ω.*

We can also consider the general case, wherein f satisfies nonsmooth second-order cone-convexity assumptions with respect to $\phi(.,.)$ and g satisfies nonsmooth second-order cone-convexity assumptions with respect to $\omega(.,.)$. The modified second-order KKT type sufficient optimality conditions will be as follows:

THEOREM 4.4 *Let f be nonsmooth second-order K-convex at $\bar{x}$ with respect to $\phi(.,.)$ and g be nonsmooth second-order Q-convex at $\bar{x} \in S_0$ with respect to $\omega(.,.)$.*

Suppose there exist $\bar{\lambda} \in K^+ \setminus \{0\}, \bar{\mu} \in Q^+$ such that for all $x \in S_0$,

$$\bar{\lambda}^T f'(\bar{x}, x-\bar{x}) + \bar{\mu}^T g'(\bar{x}, x-\bar{x}) + 2\omega(x,\bar{x})[\bar{\mu}^T g''(\bar{x}, x-\bar{x})] \geq 0, \tag{14}$$

$$\bar{\mu}^T g(\bar{x}) - \omega(x,\bar{x})\bar{\mu}^T g''(\bar{x}, x-\bar{x}) \geq 0, \text{ and} \tag{15}$$

$$\phi(x,\bar{x})\bar{\lambda}^T f''(\bar{x}, x-\bar{x}) = 0. \tag{16}$$

Then, $\bar{x}$ is a weak minimum of (VOP).

Along the lines of Hanson (1993), the second-order Mond-Weir dual will then be reformulated as

$$K\text{-Maximize} \quad f(u) \tag{MD}$$

$$\text{subject to} \quad \lambda^T f'(u, x-u) + \mu^T g'(u, x-u) + 2\xi[\lambda^T f''(u, x-u) + \mu^T g''(u, x-u)] \geq 0, \forall x \in S_0, \tag{17}$$

$$\mu^T g(u) - \xi \mu^T g''(u, x-u) \geq 0, \quad \forall x \in S_0, \tag{18}$$

$$\psi \lambda^T f''(u, x-u) \leq 0, \quad \forall x \in S_0, \tag{19}$$

$u \in S, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+, \xi, \psi \in \mathbb{R}_+$. *In general, ξ and ψ can be regarded as functions.*

Here the number of parameters in the dual has increased.

5. Conclusion

We have introduced new classes of nonsmooth second-order cone-convex and related functions in terms of second-order directional derivative. Second-order

KKT type sufficient optimality conditions and Mond-Weir type duality results for (VOP) are proved using these functions. As first-order (second-order) differentiable functions are also first-order (second-order) directionally differentiable, so the results obtained by us can be applied to a wider class of problems. It will be interesting to derive aforesaid results in the higher-order setting.

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