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PERIODIC SOLUTIONS IN MULTIVARIATE INVARIANCE ARGUMENTS

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Abstract. Inspired by the recent results of A.E. Abbas we determine continuous multivariate utility functions invariant with respect to a wide family of transformations related to the shift transformations.

Keywords: multivariate utility function, invariance, periodic solution, additive function, exponential function.

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1. INTRODUCTION

A fundamental step in decision analysis is the construction of a utility function representing a preference relation over lotteries. There are several approaches to this problem. One of them is based on the notion of invariance of a utility function. The notion has been introduced by Pfanzagl [8] and developed in [1–5]. Let us recall that a utility function U is invariant with respect to a family transformations Γ provided that, for every member γ of Γ, U and $U \circ \gamma$ represent the same preference relation over lotteries. Since every utility function over lotteries is uniquely determined by its values on a set of degenerate lotteries and this set can be identified with the set of outcomes, it is enough to consider the problem of invariance on the set of all outcomes. Recently, Abbas [2] extended the notion of invariance onto multivariate utility functions. One of the main problems considered in [2] concerns n-attribute utility functions defined on a Cartesian product $\bigtimes_{j=1}^{n} I_j$ of non-degenerate intervals $I_1, \ldots, I_n$ and invariant with respect to a family of transformations $\left\{\gamma_{(t_1,\ldots,t_n)} : \bigtimes_{j=1}^{n} I_j \to \bigtimes_{j=1}^{n} I_j | \, (t_1, \ldots, t_n) \in \bigtimes_{j=1}^{n} T_j\right\}$, where $T_1, \ldots, T_n$ are non-degenerate intervals and

$$\gamma_{(t_1,\ldots,t_n)}(x_1, \ldots, x_n) = (g_1^{-1}(g_1(x_1) + \beta_1(t_1)), \ldots, g_n^{-1}(g_n(x_n) + \beta_n(t_n))) \tag{1.1}$$

for $(x_1,\dots,x_n)\in\bigtimes_{j=1}^n I_j$, $(t_1,\dots,t_n)\in\bigtimes_{j=1}^n T_j$, where $g_j\colon I_j\to\mathbb{R}$ are continuous bijections and $\beta_j\colon T_j\to\mathbb{R}$ for $j\in\{1,\dots,n\}$. This family contains a wide class of transformations that can be converted into shift transformations (see [2] for suitable examples). It is also remarkable that transformations of the form (1.1) commute, i.e.

$$\gamma_{(t_1,\dots,t_n)}\circ\gamma_{(s_1,\dots,s_n)}=\gamma_{(s_1,\dots,s_n)}\circ\gamma_{(t_1,\dots,t_n)}\quad\text{for}\quad(s_1,\dots,s_n),(t_1,\dots,t_n)\in\bigtimes_{j=1}^n T_j.$$

The following results deriving the forms of multivariate utility functions invariant with respect to a family of transformations $\{\gamma_{(t_1,\dots,t_n)}|\,(t_1,\dots,t_n)\in\bigtimes_{j=1}^n T_j\}$ given by (1.1) has been proved in [2].

Theorem 1.1. *Assume that $I_1,\dots,I_n$ and $T_1,\dots,T_n$ are non-degenerated intervals, $g_j:I_j\to\mathbb{R}$ for $j\in\{1,\dots,n\}$ are continuous bijections and $\beta_j:T_j\to\mathbb{R}$ for $j\in\{1,\dots,n\}$ are nonconstant continuous functions. A continuous multivariate utility function $U:\bigtimes_{j=1}^n I_j\to\mathbb{R}$ is invariant with respect to a family of transformations $\{\gamma_{(t_1,\dots,t_n)}|(t_1,\dots,t_n)\in\bigtimes_{j=1}^n T_j\}$ of the form* (1.1) *if and only if either*

$$U(x_1,\dots,x_n)=\sum_{j=1}^n c_jg_j(x_j)+c\quad\textit{for}\quad(x_1,\dots,x_n)\in\bigtimes_{j=1}^n I_j$$

with some $c,c_1,\dots,c_n\in\mathbb{R}$, or

$$U(x_1,\dots,x_n)=a\prod_{j=1}^n e^{c_jg_j(x_j)}+c\quad\textit{for}\quad(x_1,\dots,x_n)\in\bigtimes_{j=1}^n I_j$$

with some $a,c,c_1,\dots,c_n\in\mathbb{R}$.

In the above result it is explicitly stated that the functions β_j for $j\in\{1,\dots,n\}$ are nonconstant, which implies that $\beta_j(T_j)$ for $j\in\{1,\dots,n\}$ are intervals of positive length. Such an assumption is widely used in utility theory, since otherwise, the functional form of the utility would not be derived and the work would have no motivation in utility theory. More precisely, utility functions invariant with respect to a single value of parameter may depend on arbitrary periodic functions. Such solutions of the invariance problem are not really useful for utility theory because they do not help to identify the utility function. Nevertheless, the question concerning periodic solutions of the problem could be considered as the interesting one from the mathematical point of view. In fact the periodicity result for fixed constants was discussed by A.E. Abbas [1] in a univariate case and by A.E. Abbas and J. Aczél [3] in a multivariate case. In both papers the nonconstant cases have been considered as well.

In this paper we explore the case where some of the functions β_j are constant. We determine all continuous utility functions $U:\bigtimes_{j=1}^n I_j\to\mathbb{R}$ invariant with respect to a family of transformations $\{\gamma_{(t_1,\dots,t_n)}|\,(t_1,\dots,t_n)\in\bigtimes_{j=1}^n T_j\}$ of the form (1.1) under the following assumption:

(A) $I_1,\dots,I_n$ and $T_1,\dots,T_n$ are non-degenerate intervals, $g_j:I_j\to\mathbb{R}$ for $j\in\{1,\dots,n\}$ are continuous bijections, $\beta_j:T_j\to\mathbb{R}$ for $j\in\{1,\dots,m\}$ $(1\le m\le n)$ are constant, say $\beta_j\equiv\delta_j$ for $j\in\{1,\dots,m\}$ with some $\delta_j\in\mathbb{R}$, and functions $\beta_j:T_j\to\mathbb{R}$ for $j\in\{m+1,\dots,n\}$ (if any) are nonconstant and continuous.

Our main result reads as follows.

Theorem 1.2. *A continuous utility function* $U : \times_{j=1}^{n} I_j \to \mathbb{R}$ *is invariant with respect to a family of transformations* $\{\gamma_{(t_1,\ldots,t_n)} | (t_1,\ldots,t_n) \in \times_{j=1}^{n} T_j\}$ *of the form* (1.1) *satisfying* (A) *if and only if there exists a function* $p : \mathbb{R}^m \to \mathbb{R}$ *satisfying condition*

$$p(x_1+\delta_1,\ldots,x_m+\delta_m) = p(x_1,\ldots,x_m) \quad \textit{for} \quad (x_1,\ldots,x_m) \in \bigtimes_{j=1}^{n} I_j \tag{1.2}$$

(*that is a* $(\delta_1,\ldots,\delta_m)$*-periodic function*) *such that either*

$$U(x_1,\ldots,x_n) = \sum_{j=1}^{n} c_j g_j(x_j) + p(g_1(x_1),\ldots,g_m(x_m)) \quad \textit{for} \quad (x_1,\ldots,x_n) \in \bigtimes_{j=1}^{n} I_j \tag{1.3}$$

with some $c_1,\ldots,c_n \in \mathbb{R}$*, or*

$$U(x_1,\ldots,x_n) = p(g_1(x_1),\ldots,g_m(x_m)) \prod_{j=1}^{n} e^{c_j g_j(x_j)} + c \ \ \textit{for} \ \ (x_1,\ldots,x_n) \in \bigtimes_{j=1}^{n} I_j \tag{1.4}$$

with some $c, c_1,\ldots,c_n \in \mathbb{R}$.

2. PROOF OF THEOREM 1.2

According to the fundamental property of utility functions, two such functions U and V represent the same preference relation over lotteries if and only if $U = kV + l$ for some $k \in (0,\infty)$ and $l \in \mathbb{R}$. Hence, a utility function $U : \times_{j=1}^{n} I_j \to \mathbb{R}$ is invariant with respect to a family Γ if and only if for every $(t_1,\ldots,t_n) \in \times_{j=1}^{n} T_j$ there exist $k(t_1,\ldots,t_n) \in (0,\infty)$ and $l(t_1,\ldots,t_n) \in \mathbb{R}$ such that

$$\begin{gathered} U(g_1^{-1}(g_1(x_1)+\delta_1),\ldots,g_m^{-1}(g_m(x_m)+\delta_m), g_{m+1}^{-1}(g_{m+1}(x_{m+1})+\beta_{m+1}(t_{m+1})), \\ \ldots, g_n^{-1}(g_n(x_n)+\beta_n(t_n))) = k(t_1,\ldots,t_n)U(x_1,\ldots,x_n) + l(t_1,\ldots,t_n) \end{gathered} \tag{2.1}$$

for $(x_1,\ldots,x_n) \in \mathbb{R}^n$.

A straightforward calculation shows that if U is of the form (1.3), then (2.1) holds with $k(t_1,\ldots,t_n) = 1$ and $l(t_1,\ldots,t_n) = \sum_{j=1}^{m} c_j\delta_j + \sum_{j=m+1}^{n} c_j\beta_j(t_j)$ for $(t_1,\ldots,t_n) \in \times_{j=1}^{n} T_j$ (we adopt the convention $\sum_{j=n+1}^{n} = 0$). In the case of (1.4), we get (2.1) with $k(t_1,\ldots,t_n) = \prod_{j=1}^{m} e^{c_j\delta_j} \prod_{j=m+1}^{n} e^{c_j\beta_j(t_j)}$ and $l(t_1,\ldots,t_n) = c(1-k(t_1,\ldots,t_n))$ for $(t_1,\ldots,t_n) \in \times_{j=1}^{n} T_j$ (where $\prod_{j=n+1}^{n}=1$). Therefore, if U is of the form (1.3) or (1.4) with p satisfying (1.2), then U is invariant with respect to Γ.

Assume that a utility function U is invariant with respect to the family of transformations Γ. Then (2.1) holds with some functions $k : \times_{j=1}^{n} T_j \to (0,\infty)$ and

$l : \bigtimes_{j=1}^{n} T_j \to \mathbb{R}$. Inserting into (2.1) $g_j^{-1}(x_j)$ in the place of x_j for $j \in \{1, \dots, n\}$ and taking

$$F(z_1, \dots, z_n) = U(g_1^{-1}(z_1), \dots, g_n^{-1}(z_n)) \quad \text{for} \quad (z_1, \dots, z_n) \in \mathbb{R}^n, \tag{2.2}$$

we obtain

$$\begin{aligned} &F(x_1 + \delta_1, \dots, x_m + \delta_m, x_{m+1} + \beta_{m+1}(t_{m+1}), \dots, x_n + \beta_n(t_n)) = \\ &= k(t_1, \dots, t_n)F(x_1, \dots, x_n) + l(t_1, \dots, t_n) \\ &\text{for } \ (x_1, \dots, x_n) \in \mathbb{R}^n, (t_1, \dots, t_n) \in \bigtimes_{j=1}^{n} T_j. \end{aligned} \tag{2.3}$$

If $m = n$, then (2.3) becomes

$$F(x_1 + \delta_1, \dots, x_n + \delta_n) = kF(x_1, \dots, x_n) + l \quad \text{for} \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{2.4}$$

where $k := k(t_1, \dots, t_n)$ and $l := l(t_1, \dots, t_n)$ with a fixed $(t_1, \dots, t_n) \in \bigtimes_{j=1}^{n} T_j$. If $k = 1$, then taking $c_1, \dots, c_n \in \mathbb{R}$ with $\sum_{j=1}^{n} c_j\delta_j = l$ and $p : \mathbb{R}^n \to \mathbb{R}$ of the form

$$p(x_1, \dots, x_n) = F(x_1, \dots, x_n) - \sum_{j=1}^{n} c_j x_j \quad \text{for} \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{2.5}$$

making use of (2.4), we obtain (1.2). Furthermore, from (2.2) and (2.5) it follows (1.3). If $k \neq 1$, then taking $c_1, \dots, c_n \in \mathbb{R}$ such that $\sum_{j=1}^{n} c_j\delta_j = \ln k$ and $p : \mathbb{R}^n \to \mathbb{R}$ of the form

$$p(x_1, \dots, x_n) = \left(F(x_1, \dots, x_n) - \frac{l}{1-k}\right) e^{-\sum_{j=1}^{n} c_j x_j} \quad \text{for} \quad (x_1, \dots, x_n) \in \mathbb{R}^n,$$

in view of (2.4), we get (1.2). So, taking into account (2.2), we obtain (1.4) with $c := \frac{l}{1-k}$.

Now, assume that $m < n$. If F is constant then, in view of (2.2), we get (1.4) with $p \equiv 0$. Assume that F is nonconstant. Fix $(t_1^\circ, \dots, t_m^\circ) \in \mathbb{R}^m$ and put

$$\tilde{k}(t_{m+1}, \dots, t_n) = k(t_1^\circ, \dots, t_m^\circ, t_{m+1}, \dots, t_n) \quad \text{for} \quad (t_{m+1}, \dots, t_n) \in \bigtimes_{j=m+1}^{n} T_j$$

and

$$\tilde{l}(t_{m+1}, \dots, t_n) = l(t_1^\circ, \dots, t_m^\circ, t_{m+1}, \dots, t_n) \quad \text{for} \quad (t_{m+1}, \dots, t_n) \in \bigtimes_{j=m+1}^{n} T_j.$$

Then, by (2.3), we get

$$\begin{aligned} F(x_1 + \delta_1, \dots, x_m + \delta_m, x_{m+1} + \beta_{m+1}(t_{m+1}), \dots, x_n + \beta_n(t_n)) = \\ = \tilde{k}(t_{m+1}, \dots, t_n)F(x_1, \dots, x_n) + \tilde{l}(t_{m+1}, \dots, t_n) \end{aligned} \tag{2.6}$$

for $(x_1,\dots,x_n)\in\mathbb{R}^n$, $(t_{m+1},\dots,t_n)\in\bigtimes_{j=m+1}^n T_j$.

As F is nonconstant, taking $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathbb{R}^n$ with $F(x_1,\dots,x_n)\neq F(y_1,\dots,y_n)$, in view of (2.6), we obtain

$$\tilde{k}(t_{m+1},\dots,t_n)=K(\beta_{m+1}(t_{m+1}),\dots,\beta_n(t_n))\quad\text{for}\quad(t_{m+1},\dots,t_n)\in\bigtimes_{j=m+1}^n T_j,\tag{2.7}$$

where

$$K(z_{m+1},\dots,z_n)=\frac{F(x_1+\delta_1,\dots,x_m+\delta_m,x_{m+1}+z_{m+1},\dots,x_n+z_n)}{F(x_1,\dots,x_n)-F(y_1,\dots,y_n)}-\\ -\frac{F(y_1+\delta_1,\dots,y_m+\delta_m,y_{m+1}+z_{m+1},\dots,y_n+z_n)}{F(x_1,\dots,x_n)-F(y_1,\dots,y_n)}\ \text{ for }\ (z_{m+1},\dots,z_n)\in\mathbb{R}^{n-m}.$$

Furthermore, from (2.6) and (2.7), we deduce that

$$\tilde{l}(t_{m+1},\dots,t_n)=L(\beta_{m+1}(t_{m+1}),\dots,\beta_n(t_n))\quad\text{for}\quad(t_{m+1},\dots,t_n)\in\bigtimes_{j=m+1}^n T_j,\tag{2.8}$$

where

$$L(z_{m+1},\dots,z_n)=F(x_1+\delta_1,\dots,x_m+\delta_m,x_{m+1}+z_{m+1},\dots,x_n+z_n)-\\ -K(z_{m+1},\dots,z_n)F(x_1,\dots,x_n)\quad\text{for}\quad(z_{m+1},\dots,z_n)\in\mathbb{R}^{n-m}$$

with a fixed $(x_1,\dots,x_n)\in\mathbb{R}^n$.

Now, (2.6)–(2.8) imply that

$$\begin{aligned}&F(x_1+\delta_1,\dots,x_m+\delta_m,x_{m+1}+z_{m+1},\dots,x_n+z_n)=\\&=K(z_{m+1},\dots,z_n)F(x_1,\dots,x_n)+L(z_{m+1},\dots,z_n)\\&\text{for }\ (x_1,\dots,x_n)\in\mathbb{R}^n,\quad(z_{m+1},\dots,z_n)\in\bigtimes_{j=m+1}^n\beta_j(T_j).\end{aligned}\tag{2.9}$$

Given $\mathbf{x}=(x_1,\dots,x_m)\in\mathbb{R}^m$, put

$$F_{\mathbf{x}}(u_{m+1},\dots,u_n):=F(x_1+\delta_1,\dots,x_m+\delta_m,u_{m+1},\dots,u_n)\text{ for }(u_{m+1},\dots,u_n)\in\mathbb{R}^{n-m}\tag{2.10}$$

and

$$G_{\mathbf{x}}(u_{m+1},\dots,u_n):=F(x_1,\dots,x_m,u_{m+1},\dots,u_n)\text{ for }(u_{m+1},\dots,u_n)\in\mathbb{R}^{n-m}.\tag{2.11}$$

Then, in view of (2.9), for every $\mathbf{x}\in\mathbb{R}^m$, we obtain

$$F_{\mathbf{x}}(x_{m+1}+z_{m+1},\dots,x_n+z_n)=K(z_{m+1},\dots,z_n)G_{\mathbf{x}}(x_{m+1},\dots,x_n)+L(z_{m+1},\dots,z_n)\tag{2.12}$$

for $(x_{m+1},\dots,x_n)\in\mathbb{R}^{n-m}$, $(z_{m+1},\dots,z_n)\in\bigtimes_{j=m+1}^n\beta_j(T_j)$.

In particular, (2.12) holds for every $(x_{m+1},\ldots,x_n)\in\mathbb{R}^{n-m}$ and $(z_{m+1},\ldots,z_n)\in \bigtimes_{j=m+1}^{n}\operatorname{int}\beta_j(T_j)$. Since β_j for $j\in\{m+1,\ldots,n\}$ are nonconstant continuous functions and T_j for $j\in\{m+1,\ldots,n\}$ are non-degenerate intervals, the set $\mathbb{R}^{n-m}\times\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j)$ is nonempty, open and connected. Thus, applying [6, Theorem 1 and Proposition 2], we conclude that, for every $\mathbf{x}\in\mathbb{R}^m$, one of the following three possibilities holds:

(i) there exist $c_{\mathbf{x}}$, $d_{\mathbf{x}}\in\mathbb{R}$ such that $F_{\mathbf{x}}\equiv d_{\mathbf{x}}$, $G_{\mathbf{x}}\equiv c_{\mathbf{x}}$ and

$$L(z_{m+1},\ldots,z_n)=d_{\mathbf{x}}-c_{\mathbf{x}}K(z_{m+1},\ldots,z_n)\text{ for }(z_{m+1},\ldots,z_n)\in\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j);\tag{2.13}$$

(ii) there exist a nonzero additive function $A_{\mathbf{x}}:\mathbb{R}^{n-m}\to\mathbb{R}$ (that is $A_{\mathbf{x}}(u+v)=A_{\mathbf{x}}(u)+A_{\mathbf{x}}(v)$ for $u,v\in\mathbb{R}^{n-m}$), $b_{\mathbf{x}}$, $c_{\mathbf{x}}\in\mathbb{R}$ and $d_{\mathbf{x}}\in\mathbb{R}\setminus\{0\}$ such that

$$F_{\mathbf{x}}(x_{m+1},\ldots,x_n)=A_{\mathbf{x}}(x_{m+1},\ldots,x_n)+b_{\mathbf{x}}+c_{\mathbf{x}}\quad\text{for}\quad(x_{m+1},\ldots,x_n)\in\mathbb{R}^{n-m},\tag{2.14}$$

$$G_{\mathbf{x}}(x_{m+1},\ldots,x_n)=\frac{1}{d_{\mathbf{x}}}(A_{\mathbf{x}}(x_{m+1},\ldots,x_n)+c_{\mathbf{x}})\quad\text{for}\quad(x_{m+1},\ldots,x_n)\in\mathbb{R}^{n-m},\tag{2.15}$$

$$K(z_{m+1},\ldots,z_n)=d_{\mathbf{x}}\quad\text{for}\quad(z_{m+1},\ldots,z_n)\in\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j),\tag{2.16}$$

$$L(z_{m+1},\ldots,z_n)=A_{\mathbf{x}}(z_{m+1},\ldots,z_n)+b_{\mathbf{x}}\text{ for }(z_{m+1},\ldots,z_n)\in\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j);\tag{2.17}$$

(iii) there exist a nonconstant exponential function $E_{\mathbf{x}}:\mathbb{R}^{n-m}\to\mathbb{R}$ (that is $E_{\mathbf{x}}(u+v)=E_{\mathbf{x}}(u)E_{\mathbf{x}}(v)$ for $u,v\in\mathbb{R}^{n-m}$), $a_{\mathbf{x}}$, $b_{\mathbf{x}}\in\mathbb{R}\setminus\{0\}$ and $c_{\mathbf{x}}$, $d_{\mathbf{x}}\in\mathbb{R}$ such that

$$\begin{aligned}F_{\mathbf{x}}(x_{m+1},\ldots,x_n)&=a_{\mathbf{x}}b_{\mathbf{x}}E_{\mathbf{x}}(x_{m+1},\ldots,x_n)+d_{\mathbf{x}}\quad\text{for}\quad(x_{m+1},\ldots,x_n)\in\mathbb{R}^{n-m},\\ G_{\mathbf{x}}(x_{m+1},\ldots,x_n)&=b_{\mathbf{x}}E_{\mathbf{x}}(x_{m+1},\ldots,x_n)+c_{\mathbf{x}}\quad\text{for}\quad(x_{m+1},\ldots,x_n)\in\mathbb{R}^{n-m},\end{aligned}\tag{2.18}$$

$$K(z_{m+1},\ldots,z_n)=a_{\mathbf{x}}E_{\mathbf{x}}(z_{m+1},\ldots,z_n)\quad\text{for}\quad(z_{m+1},\ldots,z_n)\in\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j),\tag{2.19}$$

$$\begin{aligned}&L(z_{m+1},\ldots,z_n)=\\&=-a_{\mathbf{x}}c_{\mathbf{x}}E_{\mathbf{x}}(z_{m+1},\ldots,z_n)+d_{\mathbf{x}}\text{ for }(z_{m+1},\ldots,z_n)\in\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j).\end{aligned}\tag{2.20}$$

Note that if, for some $\mathbf{x}\in\mathbb{R}^m$, (i) or (iii) holds, then

$$L(z_{m+1},\ldots,z_n)+c_{\mathbf{x}}K(z_{m+1},\ldots,z_n)=d_{\mathbf{x}}\quad\text{for}\quad(z_{m+1},\ldots,z_n)\in\bigtimes_{j=m+1}^{n}\operatorname{int}\,\beta_j(T_j).$$

Thus, if also (ii) is valid for some $\mathbf{y} \in \mathbb{R}^m$, we have

$$d_{\mathbf{x}} = L(z_{m+1}, \ldots, z_n) + c_{\mathbf{x}} K(z_{m+1}, \ldots, z_n) = A_y(z_{m+1}, \ldots, z_n) + b_{\mathbf{y}} + c_{\mathbf{x}} d_{\mathbf{y}}$$

for $(z_{m+1}, \ldots, z_n) \in \bigtimes_{j=m+1}^{n} \mathrm{int}\beta_j(T_j)$.

Since A_y is a nonconstant function, this means that either (ii) holds for every $\mathbf{x} \in \mathbb{R}^m$ or, for every $\mathbf{x} \in \mathbb{R}^m$, one of the conditions (i), (iii) holds. Therefore the following three cases are possible:

1. (i) holds for every $\mathbf{x} \in \mathbb{R}^m$,
2. (ii) holds for every $\mathbf{x} \in \mathbb{R}^m$,
3. (iii) holds for some $\mathbf{x}_\circ \in \mathbb{R}^m$.

Case 1. Since F is nonconstant, making use of (2.11), we have

$$G_{(x_1,\ldots,x_m)}(x_{m+1}, \ldots, x_n) = F(x_1, \ldots, x_n) \neq F(y_1, \ldots, y_n) = G_{(y_1,\ldots,y_m)}(y_{m+1}, \ldots, y_n)$$

for some $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, whence $c_{(x_1,\ldots,x_m)} \neq c_{(y_1,\ldots,y_m)}$. Moreover, according to (2.13), we have

$$d_{(x_1,\ldots,x_m)} - c_{(x_1,\ldots,x_m)} K(z_{m+1}, \ldots, z_n) = d_{(y_1,\ldots,y_m)} - c_{(y_1,\ldots,y_m)} K(z_{m+1}, \ldots, z_n)$$

for $(z_{m+1}, \ldots, z_n) \in \bigtimes_{j=m+1}^{n} \mathrm{int}\ \beta_j(T_j)$.

Thus

$$K(z_{m+1}, \ldots, z_n) = \frac{d_{(x_1,\ldots,x_m)} - d_{(y_1,\ldots,y_m)}}{c_{(x_1,\ldots,x_m)} - c_{(y_1,\ldots,y_m)}} \quad \text{for} \quad (z_{m+1}, \ldots, z_n) \in \bigtimes_{j=m+1}^{n} \mathrm{int}\ \beta_j(T_j).$$

Hence K is constant and, in view of (2.13), so is L. Furthermore, by (2.11), we get

$$F(x_1, \ldots, x_n) = G_{(x_1,\ldots,x_m)}(x_{m+1}, \ldots, x_n) = c_{(x_1,\ldots,x_m)} \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{R}^m. \tag{2.21}$$

Therefore, taking into account (2.9), we obtain

$$c_{(x_1+\delta_1,\ldots,x_m+\delta_m)} = K c_{(x_1,\ldots,x_m)} + L \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

So, arguing as in the case of (2.4), we conclude that either $K = 1$ and

$$c_{(x_1,\ldots,x_m)} = \sum_{j=1}^{m} c_j x_j + p(x_1, \ldots, x_m) \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{R}^m$$

with some $c_1, \ldots, c_m \in \mathbb{R}$ and $p : \mathbb{R}^m \to \mathbb{R}$ satisfying (1.2), or $K \neq 1$ and

$$c_{(x_1,\ldots,x_m)} = p(x_1, \ldots, x_m) e^{\sum_{j=1}^{m} c_j x_j} + \frac{L}{1-K} \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{R}^m$$

with some $c_1, \ldots, c_m \in \mathbb{R}$ and $p : \mathbb{R}^m \to \mathbb{R}$ satisfying (1.2). Thus, in view of (2.2) and (2.21), we get (1.3) and (1.4), respectively with $c_j = 0$ for $j \in \{m+1, \ldots, n\}$ and $c = \frac{L}{1-K}$ in the second case.

Case 2. From (2.16) it follows that $d := d_{\mathbf{x}} = d_{\mathbf{y}}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Furthermore, by (2.17), for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, we have

$$A_{\mathbf{x}}(z_{m+1}, \ldots, z_n) + b_{\mathbf{x}} = A_{\mathbf{y}}(z_{m+1}, \ldots, z_n) + b_{\mathbf{y}} \text{ for } (z_{m+1}, \ldots, z_n) \in \bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j).$$

Thus, as $\bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j)$ is a nonempty open set, for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, we get $A := A_{\mathbf{x}} = A_{\mathbf{y}}$ and $b := b_{\mathbf{x}} = b_{\mathbf{y}}$ (cf. [7, p. 328]). So, making use of (2.14) and (2.15), for every $\mathbf{x} \in \mathbb{R}^m$, we obtain

$$F_{\mathbf{x}}(x_{m+1}, \ldots, x_n) = A(x_{m+1}, \ldots, x_n) + b + c_{\mathbf{x}} \quad \text{for} \quad (x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n-m} \tag{2.22}$$

and

$$G_{\mathbf{x}}(x_{m+1}, \ldots, x_n) = \frac{1}{d}(A(x_{m+1}, \ldots, x_n) + c_{\mathbf{x}}) \quad \text{for} \quad (x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n-m}. \tag{2.23}$$

Hence, in view of (2.10) and (2.11), we get

$$\begin{aligned} &\frac{1}{d}(A(x_{m+1}, \ldots, x_n) + c_{(x_1+\delta_1, \ldots, x_m+\delta_m)}) = G_{(x_1+\delta_1, \ldots, x_m+\delta_m)}(x_{m+1}, \ldots, x_n) = \\ &= F(x_1 + \delta_1, \ldots, x_m + \delta_m, x_{m+1}, \ldots, x_n) = F_{(x_1, \ldots, x_m)}(x_{m+1}, \ldots, x_n) = \\ &= A(x_{m+1}, \ldots, x_n) + b + c_{(x_1, \ldots, x_m)} \quad \text{for} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \end{aligned}$$

Therefore

$$\left(\frac{1}{d} - 1\right) A(x_{m+1}, \ldots, x_n) = c_{(x_1, \ldots, x_m)} - \frac{1}{d} c_{(x_1+\delta_1, \ldots, x_m+\delta_m)} + b \text{ for } (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Since A is nonconstant and the right hand side of the latter equality does not depend on $(x_{m+1}, \ldots, x_n)$, this means that $d = 1$ and

$$c_{(x_1+\delta_1, \ldots, x_m+\delta_m)} = c_{(x_1, \ldots, x_m)} + b \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{R}^m. \tag{2.24}$$

Let $c_1, \ldots, c_m \in \mathbb{R}$ be such that $\sum_{j=1}^{m} c_j \delta_j = b$ and let $p : \mathbb{R}^m \to \mathbb{R}$ be given by

$$p(x_1, \ldots, x_m) = c_{(x_1, \ldots, x_m)} - \sum_{j=1}^{m} c_j x_j \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{R}^m. \tag{2.25}$$

Then, in view of (2.24), p satisfies (1.2). Furthermore, as U is continuous, for every $\mathbf{x} \in \mathbb{R}$, so is $F_{\mathbf{x}}$. Thus, by (2.22), A is continuous, whence (cf. [7, p. 130]) there exist $c_{m+1}, \ldots, c_n \in \mathbb{R}$ such that

$$A(x_{m+1}, \ldots, x_n) = \sum_{j=m+1}^{n} c_j x_j \quad \text{for} \quad (x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n-m}.$$

Therefore, since $d = 1$, taking into account (2.11), (2.23) and (2.25), we obtain

$$F(x_1, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j + p(x_1, \ldots, x_m) \quad \text{for} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Hence, in view of (2.2), U is of the form (1.3).
Case 3. Let $S_{(\mathrm{i})}$, $S_{(\mathrm{ii})}$ and $S_{(\mathrm{iii})}$ denote the sets of all elements of $\mathbb{R}^m$ for which (i), (ii) and (iii) is valid, respectively. Then, as we have already noted, in this case $S_{(\mathrm{ii})} = \emptyset$. Hence $S_{(\mathrm{i})} \cup S_{(\mathrm{iii})} = \mathbb{R}^m$. Since $\mathbf{x}_\circ \in S_{(\mathrm{iii})}$, according to (2.19) and (2.20), we get

$$K(z_{m+1}, \dots, z_n) = aE(z_{m+1}, \dots, z_n) \quad \text{for} \quad (z_{m+1}, \dots, z_n) \in \bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j) \tag{2.26}$$

and

$$L(z_{m+1}, \dots, z_n) = -acE(z_{m+1}, \dots, z_n) + d \quad \text{for} \quad (z_{m+1}, \dots, z_n) \in \bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j), \tag{2.27}$$

where $E := E_{x_\circ}$, $a := a_{x_\circ}$, $c := c_{x_\circ}$ and $d := d_{x_\circ}$. Moreover, in view of (2.2) and (2.11), the continuity of U and $g_{m+1}, \dots, g_n$ implies the continuity of $G_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{R}^m$. Thus, as $a_{\mathbf{x}} \neq 0$ for $\mathbf{x} \in \mathbb{R}^m$, from (2.18) it follows that, for every $\mathbf{x} \in \mathbb{R}^m$, $E_{\mathbf{x}}$ is continuous. In particular E is continuous and nonconstant, so (cf. [7, p. 311])

$$E(x_{m+1}, \dots, x_n) = e^{\sum_{j=m+1}^{n} c_j x_j} \quad \text{for} \quad (x_{m+1}, \dots, x_n) \in \mathbb{R}^{n-m} \tag{2.28}$$

with some $c_{m+1}, \dots, c_n \in \mathbb{R}$ such that $\sum_{j=m+1}^{n} c_i^2 > 0$. Furthermore, in view of (2.13), (2.19), (2.20), (2.26) and (2.27), for every $\mathbf{x} \in \mathbb{R}^m$, we have

$$d_{\mathbf{x}} = L(z_{m+1}, \dots, z_n) + c_{\mathbf{x}} K(z_{m+1}, \dots, z_n) = a(c_{\mathbf{x}} - c)E(z_{m+1}, \dots, z_n) + d$$

for $(z_{m+1}, \dots, z_n) \in \bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j)$, that is

$$a(c_{\mathbf{x}} - c)E(z_{m+1}, \dots, z_n) = d_{\mathbf{x}} - d \quad \text{for} \quad (z_{m+1}, \dots, z_n) \in \bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j).$$

Since E is nonconstant and $a \neq 0$, this means that, for every $\mathbf{x} \in \mathbb{R}^m$, we get $c_{\mathbf{x}} = c$ and $d_{\mathbf{x}} = d$. Therefore, taking into account (2.11), (2.18) and (2.28), we obtain

$$F(x_1, \dots, x_n) = G_{(x_1, \dots, x_m)}(x_{m+1}, \dots, x_n) = B(x_1, \dots, x_m)e^{\sum_{j=m+1}^{n} c_j x_j} + c \tag{2.29}$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$, where

$$B(x_1, \dots, x_m) = \begin{cases} b_{(x_1, \dots, x_m)} & \text{whenever} \quad (x_1, \dots, x_m) \in S_{(\mathrm{iii})}, \\ 0 & \text{whenever} \quad (x_1, \dots, x_m) \in S_{(\mathrm{i})}. \end{cases}$$

Consequently, making use of (2.9), (2.26), (2.27) and (2.29), we conclude that

$$e^{\sum_{j=m+1}^{n} c_i(x_i + z_i)}(B(x_1 + \delta_1, \dots, x_m + \delta_m) - aB(x_1, \dots, x_m)) = d - c$$

for $(x_1, \dots, x_n) \in \mathbb{R}^n$, $(z_{m+1}, \dots, z_n) \in \bigtimes_{j=m+1}^{n} \operatorname{int} \beta_j(T_j)$.

As $\sum_{j=m+1}^{n} c_i^2 > 0$, this implies that $d = c$ and

$$B(x_1 + \delta_1, \dots, x_m + \delta_m) = aB(x_1, \dots, x_m) \quad \text{for} \quad (x_1, \dots, x_m) \in \mathbb{R}^m.$$

Thus

$$B(x_1, \dots, x_m) = p(x_1, \dots, x_m) e^{\sum_{j=1}^{m} c_j x_j} \quad \text{for} \quad (x_1, \dots, x_m) \in \mathbb{R}^m \tag{2.30}$$

with some $c_1, \dots, c_m \in \mathbb{R}$ and $p : \mathbb{R}^m \to \mathbb{R}$ satisfying (1.2). In fact, if $a = 1$ then (2.30) holds with $c_j = 0$ for $j \in \{1, \dots, m\}$ and $p := B$. If $a \neq 1$ then it is enough to take $c_1, \dots, c_m \in \mathbb{R}$ such that $\sum_{j=1}^{m} c_j \delta_j = \ln a$ and $p : \mathbb{R}^m \to \mathbb{R}$ of the form

$$p(x_1, \dots, x_m) = B(x_1, \dots, x_m) e^{-\sum_{j=1}^{m} c_j x_j} \quad \text{for} \quad (x_1, \dots, x_m) \in \mathbb{R}^m.$$

Finally, from (2.2), (2.29) and (2.30), it follows (1.4).

3. CONCLUSION

In a recent paper [2] A.E. Abbas determined multivariate utility functions invariant with respect to a family of transformations of the form (1.1) under the assumption that $\beta_1, \dots, \beta_n$ are nonconstant functions. This family contains a wide class of transformations that can be converted into shift transformations. In our work we have considered the case where some of the functions $\beta_1, \dots, \beta_n$ are constant. We have proved that in such a case the invariant utility functions contain the periodic components, depending in fact on an arbitrary function. Therefore a class of all utility functions invariant with respect to the family of such transformations substantially differs from that obtained in [2]. Results of this type have been already discussed in [1] and [3] in a univariate and a multivariate case, respectively. It is known that, as the presented solutions depend on arbitrary periodic functions, they are not really useful for utility theorists. Nevertheless, one can consider them as interesting from the mathematical point of view.

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