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# ON THE NONOSCILLATORY BEHAVIOR OF SOLUTIONS OF THREE CLASSES OF FRACTIONAL DIFFERENCE EQUATIONS

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**Abstract.** In this paper, we study the nonoscillatory behavior of three classes of fractional difference equations. The investigations are presented in three different folds. Unlike most existing nonoscillation results which have been established by employing Riccati transformation technique, we employ herein an easily verifiable approach based on the fractional Taylor's difference formula, some features of discrete fractional calculus and mathematical inequalities. The theoretical findings are demonstrated by examples. We end the paper by a concluding remark.

**Keywords:** Caputo difference operator, nonoscillation criteria, fractional difference equation, mathematical inequalities.

**Mathematics Subject Classification:** 34A08, 39A21.

## 1. INTRODUCTORY BACKGROUND

In alignment with the extensive interest in the study of fractional differential equations (FDEs) which has demonstrated high potential for real life applications [17, 24, 27], the determination of oscillation or nonoscillation of solutions for FDEs has also received significant attention amongst researchers. The recent literature has witnessed the appearance of many papers which have reported the oscillatory behavior of different types of FDEs; see the papers [1, 7, 15, 20, 21, 25, 29–31] and the references quoted therein. On the other hand, the study of oscillation/nonoscillation of fractional difference equations (FdEs), which are the discrete analogue of FDEs, has comparably gained less consideration. In particular, few results have been released concerning the existence of nonoscillatory solutions of FdEs; we refer herein to some relevant oscillation results for FdEs [2, 3, 5, 11–13].

 

As in the case of integer order difference equations versus integer order differential equations, the use and importance of FdEs as an approximations to FDEs afford a powerful method for the analysis of electrical, mechanical, thermal and other systems in which there are a recurrence of identical sections. Indeed, it has been realized that the study of the behavior of electrical wave filters, multistage amplifiers and insulator strings has been greatly facilitated by FdEs [19,26]. Oscillation/nonoscillation of physical waves which could be adequately described by FdEs are amongst the most attractive topic for scientists and engineers [10,28].

In this paper, we investigate the nonoscillatory solutions of the fractional difference equations of the following form

$$\begin{cases} {}_C\Delta^{\alpha} y(t) = e(t+\alpha) + f(t+\alpha, x(t+\alpha)), \quad t \in \mathbb{N}_{1-\alpha}, \\ y(0) = c_0, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, ${}_C\Delta^{\alpha}$ is a Caputo like discrete fractional difference, $f : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $t$ and $x$, and satisfies $xf(t,x) > 0$ for $x \neq 0$, $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$ and $e$ is a positive sequence. In [21], the authors studied the asymptotic behavior of nonoscillatory solutions of the fractional differential equations of the form

$${}_C D_a^{\alpha} y = e(t) + f(t, x).$$

Strongly motivated by the idea in [21], in this study, we will carry on our investigation for the following particular cases of Eq. (1.1):

$$y(t) = \Delta(r(t)|\Delta x(t)|^{\delta-1}\Delta x(t)), \quad \delta \geq 1, \tag{1.2}$$

$$y(t) = \Delta x(t), \tag{1.3}$$

$$y(t) = x(t), \tag{1.4}$$

where $r$ is a positive sequence. The investigations are presented in three different folds. Unlike most established results in the literature which mainly depend on the employment of Riccati transformation, our approach is based on the fractional Taylor's difference formula, some features of the newly defined discrete fractional calculus and mathematical inequalities.

The rest of the paper is organized as follows: Section 1 presents descriptive introduction that gives background on FDEs, states the prominence of FdEs and introduces the targeted problems. Section 2 assembles essential preliminaries needed prior to proceeding to the main results. Section 3 is devoted to the main theorems which provide sufficient conditions for the nonoscillatory behavior of solutions of the proposed problems. We provide examples as an application in Section 4. At the end, a short remark is concluded.

## 2. FUNDAMENTAL PRELIMINARIES

In this section, we assemble basic definitions and lemmas on discrete fractional calculus. The presented identities and statements serve as essential prerequisites for the proofs of the main results.

**Definition 2.1.** By a solution of Eq. (1.1), we mean a real-valued sequence $x(t)$ satisfying Eq. (1.1) for $t \in \mathbb{N}_{t_0}$ with $t_0 \in \mathbb{N}_1$. A solution $x(t)$ of Eq. (1.1) is called oscillatory if for every positive integers $T_0 > t_0$ there exists $t \geq T_0$ such that $x(t)\, x(t+1) \leq 0$, otherwise it is called non-oscillatory.

**Definition 2.2** ([19])**.** The generalized falling function is defined by

$$t^{(r)} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)},$$

for any $t$, $r \in \mathbb{R}$ for which the right hand-side is defined. Here $\Gamma$ denotes the Euler's gamma function. We also use the standard extensions of the domain of this rising function by defining it to be zero whenever the numerator is well defined, but the denominator is not defined.

**Definition 2.3** ([8])**.** Let $\nu > 0$. The $\nu$-th fractional sum of $x$ is defined by

$$\Delta_a^{-\nu} x(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} x(s), \tag{2.1}$$

where $x$ is defined for $s = a\ mod(1)$ and $\Delta^{-\nu} x$ is defined for $t = (a+\nu)\ mod(1)$. In (2.1), it is to note that the fractional sum operator $\Delta^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$.

**Definition 2.4** ([6])**.** Let $\mu > 0$ and $m-1 < \mu < m$, where $m$ denotes a positive integer, $m = \lceil \mu \rceil$, and $\lceil . \rceil$ is the ceiling of a number. Set $\nu = m - \mu$. The $\mu$-th fractional Caputo like difference is defined as

$${}_C\Delta^{\mu} x(t) = \Delta^{-\nu}(\Delta^m x(t)) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} (\Delta^m x)(s), \tag{2.2}$$

where $\Delta^m$ is the $m-$th order forward difference operator. The fractional Caputo like difference ${}_C\Delta^{\mu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a-\mu}$.

**Lemma 2.5** ([22])**.** *If $X$ and $Y$ are nonnegative numbers, then we have*

$$X^k - (1-k)Y^k - kXY^{k-1} \leq 0, \quad \text{for } 0 < k < 1,$$

*where the equality holds if and only if $X = Y$.*

**Lemma 2.6** ([14])**.** *Assume that $\beta > 1$ and $\gamma > 0$, then*

$$\left[t^{(-\gamma)}\right]^{\beta} < \frac{\Gamma(1+\beta\gamma)}{\Gamma^{\beta}(1+\gamma)} t^{(-\beta\gamma)}$$

*for $t \in \mathbb{N}_1$.*

**Lemma 2.7** ([18])**.** *Let $a \in \mathbb{R}$, $\mu \in \mathbb{R} \backslash \{\ldots, -2, -1, 0\}$, $\nu > 0$ and $(t-a)^{(\mu)} : \mathbb{N}_{a+\mu} \to \mathbb{R}$. Then,*

$$\Delta_{a+\mu}^{-\nu} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{(\mu+\nu)} \ \text{ for } t \in \mathbb{N}_{a+\mu+\nu}. \tag{2.3}$$

**Lemma 2.8.** *Initial value problem* (*IVP*) (1.1) *is equivalent to the following summation equation*

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}[e(s+\alpha) + f(s+\alpha, x(s+\alpha))], \quad t \in \mathbb{N}_1. \tag{2.4}$$

*Proof.* The proof is similar that of Lemma 2.2 in [14]. □

**Lemma 2.9.** *Assume* $p > 1,\ 0 < \alpha \leq 1,\ p(\alpha-1)+1 > 0$ *and* $\gamma = 2 - \alpha - \frac{1}{p}$. *Then one has*

$$(t-s-1)^{(p\alpha-p)} \geq (t-s-1+\alpha+p\,(1-\alpha)-1)^{(p\alpha-p)}$$

*and*

$$(s)^{(p\gamma-p)} \geq (s+p\,(\alpha-1)+1)^{(p\gamma-p)},$$

*where*

$$t \in \mathbb{N}_1 \quad and \quad s \in \{1-(p\alpha-p), 2-(p\alpha-p), \ldots, t-2-(p\alpha-p)\} \cdot$$

*Proof.* The proof is similar that of Corollary 2.9 in [16]. □

**Lemma 2.10** (Discrete Gronwall's inequality, [23]). *Let* $x$ *and* $m$ *be nonnegative sequences and* $c$ *be a nonnegative constant. If*

$$x\,(t) \leq c + \sum_{s=0}^{t} m\,(s)\, x\,(s) \quad for\ t \geq 0.$$

*Then, it holds*

$$x\,(t) \leq c \exp\left(\sum_{s=0}^{t} m\,(s)\right) \quad for\ t \geq 0.$$

## 3. MAIN RESULTS

This section is devoted to the main results. The results are stated and proved in three separate subsections.

### 3.1. NONOSCILLATION SOLUTIONS OF EQ. (1.1) WITH (1.2)

Consider the equation

$$\begin{cases} {}_C\Delta^{\alpha+1}\big(r(t)|\Delta x|^{\delta-1}\Delta x\big) = e(t+\alpha) + f(t+\alpha, x\,(t+\alpha)), & t \in \mathbb{N}_{1-\alpha}, \\ \Delta(r(t)|\Delta x|^{\delta-1}\Delta x)|_{t=0} = c_0. \end{cases} \tag{3.1}$$

It is assumed that the function $f$ satisfies

$$xf(t,x) \leq t^{(\gamma-1)} h(t)|x|^{\beta+1}, \quad x \neq 0 \tag{3.2}$$

for some function $h : (t_1, \infty) \to R^+$ and real numbers $\gamma > 0$ and $0 < \beta < \delta$. For the sake of simplification, we define

$$R(t) := \sum_{s=1}^{t-1} r^{-1/\delta}(s)$$

and

$$g_1(t) := \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)} m^{\beta/(\beta-\delta)}(s+\alpha) h^{\delta/(\delta-\beta)}(s+\alpha), \quad (3.3)$$

where $t_1 \in \mathbb{N}_1$ and $m$ is a positive sequence.

**Theorem 3.1.** *Let $q$ be a conjugate number of $p > 1$ (that is, $q = p/(p-1)$), $p(\alpha-1)+1 > 0$, and $\gamma = 2 - \alpha - \frac{1}{p}$. Suppose that for any positive integer $t_1$, we have*

$$\sum_{s=t_1-\alpha}^{\infty} (s+\alpha)^q R^{q\delta}(s+\alpha) m^q(s+\alpha) < \infty, \quad (3.4)$$

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{s=t_1}^{t-1} g_1(s) < \infty, \quad (3.5)$$

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t} \sum_{u=t_1}^{t-1} \sum_{s=1-\alpha}^{u-\alpha} (u-s-1)^{(\alpha-1)} e(s+\alpha) > -\infty, \\ &\limsup_{t\to\infty} \frac{1}{t} \sum_{u=t_1}^{t-1} \sum_{s=1-\alpha}^{u-\alpha} (u-s-1)^{(\alpha-1)} e(s+\alpha) < \infty. \end{aligned} \quad (3.6)$$

*Then every nonoscillatory solution $x$ of (3.1) satisfies*

$$|x(t)| = O(t^{1/\delta} R(t)), \quad t \to \infty.$$

*Proof.* Let $x$ be a nonoscillatory solution of (3.1), say $x(t) > 0$ for all $t \in \mathbb{N}_{t_1}$, where $t_1$ is a positive integer. Let $F(t) := f(t, x(t))$ and $t \in \mathbb{N}_{t_1}$. Then by Lemma 2.8, we have

$$\begin{aligned} y(t) \le c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)). \end{aligned}$$

By using (3.2), we get

$$y(t) \le c_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} \, |F(s+\alpha)| + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} h(s+\alpha) x^{\beta}(s+\alpha)$$

and

$$\begin{aligned} y(t) \le c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} \Big\{ (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} \\ &\cdot \big[ h(s+\alpha) x^{\beta}(s+\alpha) - m(s+\alpha) x^{\delta}(s+\alpha) \big] \Big\} \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x^{\delta}(s+\alpha). \end{aligned} \tag{3.7}$$

Setting $X = h^{\delta/\beta}(s+\alpha)x^{\delta}(s+\alpha)$, $Y = \left(\frac{\delta}{\beta} m(s+\alpha) h^{-\delta/\beta}(s+\alpha)\right)^{\delta/\beta-\delta}$ and $k = \frac{\beta}{\delta}$, then using Lemma 2.5, we deduce that

$$h(s+\alpha)x^{\beta}(s+\alpha) - m(s+\alpha)x^{\delta}(s+\alpha) \le \lambda_1 m^{\beta/(\beta-\delta)}(s+\alpha) h^{\delta/\delta-\beta}(s+\alpha),$$

where $\lambda_1 = (1-\beta/\delta)\big(\beta/\delta\big)^{\frac{\beta}{\beta-\delta}}$. Thus, (3.7) becomes

$$\begin{aligned} y(t) \le c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} g_1(t) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x^{\delta}(s+\alpha), \end{aligned}$$

where $g_1$ is defined by (3.3). Summing from $t_1$ to $t-1$ and interchanging the order of the last summation, we have

$$r(t)(\Delta x(t))^{\delta} \le w(t), \tag{3.8}$$

where

$$\begin{aligned} w(t) &:= r(t_1)(\Delta x(t_1))^{\delta} + |c_0|\,(t-t_1) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{u=t_1}^{t-1} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t_1-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{u=t_1}^{t-1} \sum_{s=1-\alpha}^{u-\alpha} (u-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} \sum_{s=t_1}^{t-1} g_1(s) \\ &+ \frac{1}{\Gamma(1+\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha-1} (t-s-1)^{(\alpha)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x^{\delta}(s+\alpha). \end{aligned}$$

Summing (3.8) from $t_1$ to $t-1$ and noting that $w(t)$ is an increasing function of $t$, we have

$$\begin{aligned} x(t) &\le d_1 + \sum_{s=t_1}^{t-1} r^{-1/\delta}(s) w^{1/\delta}(s), \\ &\le d_1 + w^{1/\delta}(t) \sum_{s=1}^{t-1} r^{-1/\delta}(s) = d_1 + w^{1/\delta}(t) R(t), \end{aligned} \tag{3.9}$$

where $d_1 = x(t_1)$. By applying the elementary inequality $(A+B)^{\delta} \le 2^{\delta-1}(A^{\delta}+B^{\delta})$, we have

$$\frac{x^{\delta}(t)}{R^{\delta}(t)} \le 2^{\delta-1} d_1^{\delta} + 2^{\delta-1} w(t).$$

On other hand, by applying the Hölder inequality, Lemma 2.6 and Lemma 2.9, we have

$$\begin{aligned} &\sum_{s=t_1-\alpha}^{t-\alpha-1} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x^{\delta}(s+\alpha) \\ &\le \left[ \sum_{s=t_1-\alpha}^{t-\alpha-1} \Big( (t-s-1)^{(\alpha-1)} \Big)^p \Big( (s+\alpha)^{(\gamma-1)} \Big)^p \right]^{1/p} \\ &\quad \cdot \left[ \sum_{s=t_1-\alpha}^{t-\alpha-1} m^q(s+\alpha) x^{\delta q}(s+\alpha) \right]^{1/q} \\ &< \left[ \Big( \frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)} \Big) \Big( \frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)} \Big) \right]^{1/p} \\ &\quad \cdot \left[ \sum_{s=-1-(p\alpha-p)}^{t-(p\alpha-p+1)} \Big( (t-s-1)^{(p\alpha-p)} \Big) \Big( (s)^{(p\gamma-p)} \Big) \right]^{1/p} \\ &\quad \cdot \left[ \sum_{s=t_1-\alpha}^{t-\alpha-1} m^q(s+\alpha) x^{\delta q}(s+\alpha) \right]^{1/q}. \end{aligned}$$

From Definition 2.3 and then using Lemma 2.7, we get

$$\begin{aligned}
&\sum_{s=t_1-\alpha}^{t-\alpha-1}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}m(s+\alpha)x^{\delta}(s+\alpha)\\
&\le\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\Big(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\Big)\right]^{1/p}\\
&\quad\cdot\left[\Delta_{-1-(p\alpha-p)}^{-(p\alpha-p+1)}(t)^{(p\gamma-p)}\right]\left[\sum_{s=t_1-\alpha}^{t-\alpha-1}m^q(s+\alpha)x^{\delta q}(s+\alpha)\right]^{1/q}\\
&=\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\Big(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\Big)\right]^{1/p}\\
&\quad\cdot\left[\frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]}(t)^{(p\alpha-p+1+p\gamma-p)}\right]^{1/p}\\
&\quad\cdot\left[\sum_{s=t_1-\alpha}^{t-\alpha-1}m^q(s+\alpha)x^{q}(s+\alpha)\right]^{1/q},
\end{aligned}$$

or

$$\begin{aligned}
&\sum_{s=t_1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}m(s+\alpha)x^{\delta}(s+\alpha)\\
&\le M_1\left[\sum_{s=t_1-\alpha}^{t-\alpha-1}m^q(s+\alpha)x^{\delta q}(s+\alpha)\right]^{1/q},
\end{aligned}$$

where by the definition of $\gamma$, $p\gamma-p+p\alpha-p+1=0$ and

$$\begin{aligned}
M_1=&\left[\Big(\frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)}\Big)\Big(\frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)}\Big)\right]^{1/p}\\
&\cdot\left[\frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]}\right]^{1/p}.
\end{aligned}$$

Thus (3.9) becomes

$$\frac{x^{\delta}(t)}{R^{\delta}(t)}\le At+\frac{M_1t}{\Gamma(\alpha)}\left[\sum_{s=t_1-\alpha}^{t-\alpha-1}m^q(s+\alpha)x^{\delta q}(s+\alpha)\right]^{1/q},$$

where, in view of (3.5) and (3.6), $A$ is an upper bound for $2^{\delta-1}$ times

$$\frac{r(t_1)\Delta x(t_1)}{t} + |c_0| + d_1^\delta + \frac{1}{t\Gamma(\alpha)} \sum_{u=t_1}^{t-1} \sum_{s=1-\alpha}^{t_1-1-\alpha} (u-s-1)^{(\alpha-1)} |F(s+\alpha)|$$
$$+ \frac{1}{t\Gamma(\alpha)} \sum_{u=t_1}^{t-1} \sum_{s=1-\alpha}^{u-\alpha} (u-s-1)^{(\alpha-1)} e(s+\alpha) + \sum_{s=t_1}^{t-1} \frac{\lambda_1}{t\Gamma(\alpha)} g_1(s).$$

Therefore, we have

$$\begin{aligned}\left(\frac{x(t)}{t^{1/\delta}R(t)}\right)^{q\delta} \le & \left(A + \frac{M_1}{\Gamma(\alpha)} \left( \sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q R^{q\delta}(s+\alpha) m^q(s+\alpha) \right.\right. \\ & \left.\left. \cdot \left[\frac{x(s+\alpha)}{(s+\alpha)^{1/\delta}R(s+\alpha)}\right]^{\delta q}\right)^{1/q}\right)^q \\ \le & 2^{q-1}A^q + 2^{q-1}\frac{M_1^q}{\Gamma^q(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q R^{q\delta}(s+\alpha) m^q(s+\alpha) \\ & \cdot \left[\frac{x(s+\alpha)}{(s+\alpha)^{1/\delta}R(s+\alpha)}\right]^{\delta q}.\end{aligned}$$

Finally, if we apply the Lemma 2.10, we have

$$\left(\frac{x(t)}{t^{1/\delta}R(t)}\right)^q \le A^q 2^{2(q-1)} \frac{M_1^q}{\Gamma^q(\alpha)} \exp\left(\sum_{s=t_1-\alpha}^{t-\alpha-1} (s+\alpha)^q R^{q\delta}(s+\alpha) m^q(s+\alpha)\right).$$

By using (3.4), we have

$$\limsup_{t\to\infty} \frac{x(t)}{t^{1/\delta}R(t)} < \infty.$$

This completes the proof. □

**Remark 3.2.** If $x(t)$ is eventually negative, then we can set $y = -x$ to see that $y$ satisfies (1.1) with $e(t)$ replaced by $-e(t)$ and $f(t,x)$ by $-f(t,-y)$. It follows in the similiar manner that

$$\limsup_{t\to\infty} \frac{-x(t)}{t^{1/\delta}R(t)} < \infty.$$

### 3.2. NONOSCILLATION SOLUTIONS OF EQ. (1.1) WITH (1.3)

Consider the equation

$$\begin{cases} {}_C\Delta^{\alpha+1}x(t) = e(t+\alpha) + f(t+\alpha, x(t+\alpha)), & t \in \mathbb{N}_{1-\alpha}, \\ \Delta x(t)|_{t=0} = c_0. \end{cases} \tag{3.10}$$

It is assumed that the function $f$ satisfies

$$xf(t,x) \leq t^{(\gamma-1)}h(t)|x|^{\lambda+1}, \quad x \neq 0, \tag{3.11}$$

for some function $h : (t_1, \infty) \to R^+$ and real numbers $\gamma > 0$ and $0 < \lambda < 1$. For clarity, we define

$$g_2(t) := \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}(s+\alpha)^{(\gamma-1)}m^{\lambda/(\lambda-1)}(s+\alpha)h^{1/(1-\lambda)}(s+\alpha), \tag{3.12}$$

where $m$ is a positive sequence.

**Theorem 3.3.** *Let $q$ be a conjugate number of $p > 1$, (that is, $q = p/(p-1)$), $p(\alpha-1)+1 > 0$, and $\gamma = 2-\alpha-\frac{1}{p}$. Suppose that for any positive integer $t_1$, we have*

$$\sum_{s=t_1-\alpha}^{\infty} (s+\alpha)^q m^q(s+\alpha) < \infty, \tag{3.13}$$

$$\limsup_{t\to\infty} g_2(t) < \infty, \tag{3.14}$$

$$\begin{aligned} \liminf_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}e(s+\alpha) > -\infty, \\ \limsup_{t\to\infty} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}e(s+\alpha) < \infty. \end{aligned} \tag{3.15}$$

*Then every nonoscillatory solution $x(t)$ of* (3.10) *satisfies*

$$|x(t)| = O(t), \ \text{as}\, t \to \infty.$$

*Proof.* Let $x$ be a nonoscillatory solution of (3.10), say $x(t) > 0$ for all $t \in \mathbb{N}_{t_1}$, where $t_1$ is a positive integer. Let $F(t) := f(t,x(t))$ and $t \in \mathbb{N}_{t_1}$. Then by Lemma 2.8, we have

$$\begin{aligned} y(t) \leq & c_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)}|F(s+\alpha)| \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}e(s+\alpha) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)}f(s+\alpha, x(s+\alpha)). \end{aligned}$$

By using (3.11), we get

$$y(t) \leq c_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} \left|F(s+\alpha)\right| + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} h(s+\alpha) x^{\lambda}(s+\alpha)$$

and

$$\begin{aligned} y(t) \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} \Big\{ (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} \\ &\qquad \cdot [h(s+\alpha)x^{\lambda}(s+\alpha) - m(s+\alpha)x(s+\alpha)] \Big\} \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x(s+\alpha) \cdot \end{aligned} \tag{3.16}$$

Taking $X = h^{1/\lambda}(s+\alpha)x(s+\alpha)$, $Y = \left(\frac{1}{\lambda} m(s+\alpha) h^{-1/\lambda}(s+\alpha)\right)^{1/\lambda - 1}$ and $k = \lambda$. Then using Lemma 2.5, we see that

$$h(s+\alpha)x^{\lambda}(s+\alpha) - m(s+\alpha)x(s+\alpha) \leq \lambda_1 m^{\lambda/(\lambda-1)}(s+\alpha) h^{1/1-\lambda}(s+\alpha),$$

where $\lambda_1 = (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}$. Thus (3.16) becomes,

$$\begin{aligned} \Delta x(t) \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} g_2(t) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x(s+\alpha), \end{aligned}$$

where $g_2$ is defined by (3.12).

It follows that

$$\Delta x(t) \leq w(t), \tag{3.17}$$

where

$$\begin{aligned} w(t) := c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} g_2(t) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x(s+\alpha). \end{aligned}$$

Summing (3.17) from $t_1$ to $t-1$ and noting that $w(t)$ is an increasing function of $t$, we have

$$\begin{aligned} x(t) \leq & d_1 + \sum_{s=t_1}^{t-1} w(s), \quad d_1 = x(t_1) \\ \leq & (d_1 + w(t))t. \end{aligned}$$

Proceeding as in the proof of Theorem 3.1, we have

$$x(t) \leq At + \frac{M_1 t}{\Gamma(\alpha)} \left[ \sum_{s=t_1-\alpha}^{t-\alpha-1} m^q(s+\alpha) x^q(s+\alpha) \right]^{1/q},$$

where

$$\begin{aligned} M_1 = & \left[ \Big( \frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)} \Big) \Big( \frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)} \Big) \right]^{1/p} \\ & \cdot \left[ \frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]} \right]^{1/p}. \end{aligned}$$

In view of (3.14) and (3.15), $A$ is an upper bound for

$$\begin{aligned} c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} g_2(t). \end{aligned}$$

Therefore, we have

$$\left(\frac{x(t)}{t}\right)^q \leq \left(A + \frac{M_1}{\Gamma(\alpha)}\left(\sum_{s=t_1-\alpha}^{t-\alpha-1} m^q(s+\alpha)x^q(s+\alpha)\right)^{1/q}\right)^q$$
$$\leq 2^{q-1}A^q + 2^{q-1}\frac{M_1^q}{\Gamma^q(\alpha)}\sum_{s=t_1-\alpha}^{t-\alpha-1}(s+\alpha)^q m^q(s+\alpha)\left[\frac{x(s+\alpha)}{s+\alpha}\right]^q.$$

Applying Lemma 2.10, we have

$$\left(\frac{x(t)}{t}\right)^q \leq A^q 2^{2(q-1)}\frac{M_1^q}{\Gamma^q(\alpha)}\exp\left(\sum_{s=t_1-\alpha}^{t-\alpha-1}(s+\alpha)^q m^q(s+\alpha)\right).$$

By using (3.13), we have

$$\limsup_{t\to\infty}\frac{x(t)}{t} < \infty,$$

which ends the proof.

□

**Remark 3.4.** If $x(t)$ is eventually negative, then we can set $y = -x$ to see that $y$ satisfies (1.1) with $e(t)$ replaced by $-e(t)$ and $f(t,x)$ by $-f(t,-y)$. It follows in the similiar manner that

$$\limsup_{t\to\infty}\frac{-x(t)}{t} < \infty.$$

## 3.3. NONOSCILLATION SOLUTIONS OF EQ. (1.1) WITH (1.4)

Consider the equation

$$\begin{cases} {}_C\Delta^\alpha x(t) = e(t+\alpha) + f(t+\alpha, x(t+\alpha)), \quad t \in \mathbb{N}_{1-\alpha}, \\ x(t)|_{t=0} = c_0. \end{cases} \tag{3.18}$$

It is assumed that the conditions (3.11) and (3.12) hold.

**Theorem 3.5.** *Let $q$ be a conjugate number of $p > 1$, (that is, $q = p/(p-1)$), $p(\alpha - 1) + 1 > 0$, and $\gamma = 2 - \alpha - \frac{1}{p}$. Suppose that for any positive integer $t_1$,* (3.14) *and* (3.15) *hold and*

$$\sum_{s=t_1-\alpha}^{\infty} m^q(s+\alpha) < \infty. \tag{3.19}$$

*Then every nonoscillatory solution $x$ of* (3.18) *is bounded.*

*Proof.* Let $x$ be a nonoscillatory solution of (3.18), say $x(t) > 0$ for all $t \in \mathbb{N}_{t_1}$, where $t_1$ is a positive integer. Let $F(t) := f(t, x(t))$ and $t \in \mathbb{N}_{t_1}$. Then by Lemma 2.8, we have

$$\begin{aligned}x(t) \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)).\end{aligned}$$

By using (3.11), we get

$$\begin{aligned}x(t) \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} s^{(\gamma-1)} h(s+\alpha) x^{\lambda}(s+\alpha)\end{aligned}$$

and

$$\begin{aligned}x(t) \leq c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} \Big\{ (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} \\ &\qquad \cdot [h(s+\alpha)x^{\lambda}(s+\alpha) - m(s+\alpha)x(s+\alpha)] \Big\} \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x(s+\alpha).\end{aligned} \tag{3.20}$$

Putting $X = h^{1/\lambda}(s+\alpha)x(s+\alpha)$, $Y = \left(\frac{1}{\lambda} m(s+\alpha) h^{-1/\lambda}(s+\alpha)\right)^{1/\lambda-1}$ and $k = \lambda$, and then using Lemma 2.5, we get

$$h(s+\alpha)x^{\lambda}(s+\alpha) - m(s+\alpha)x(s+\alpha) \leq \lambda_1 m^{\lambda/(\lambda-1)}(s+\alpha) h^{1/1-\lambda}(s+\alpha),$$

where $\lambda_1 = (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}}$. Thus, (3.20) becomes

$$\begin{aligned} x(t) \le c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) \\ &+ \frac{\lambda_1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m^{\lambda/(\lambda-1)}(s+\alpha) h^{1/1-\lambda}(s+\alpha) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(\gamma-1)} m(s+\alpha) x(s+\alpha). \end{aligned}$$

By using (3.12), we obtain

$$x(t) \le w(t), \tag{3.21}$$

where

$$\begin{aligned} w(t) := c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} g_2(t) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t_1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} s^{(\gamma-1)} m(s+\alpha) x(s+\alpha). \end{aligned}$$

Proceeding as in the proof of Theorem 3.1, we have

$$x(t) \le A + \frac{M_1}{\Gamma(\alpha)} \left[ \sum_{s=t_1-\alpha}^{t-\alpha} m^q(s+\alpha) x^q(s+\alpha) \right]^{1/q},$$

where

$$\begin{aligned} M_1 = &\left[ \Big( \frac{\Gamma(1-p\alpha+p)}{\Gamma^p(2-\alpha)} \Big) \Big( \frac{\Gamma(1-p\gamma+p)}{\Gamma^p(2-\gamma)} \Big) \right]^{1/p} \\ &\cdot \left[ \frac{\Gamma[p(\gamma-1)+1]}{\Gamma[p(\gamma-1)+p(\alpha-1)+2]} \right]^{1/p}. \end{aligned}$$

In view of (3.14) and (3.15), $A$ is an upper bound for

$$\begin{aligned} c_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t_1-1-\alpha} (t-s-1)^{(\alpha-1)} |F(s+\alpha)| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} e(s+\alpha) + \frac{\lambda_1}{\Gamma(\alpha)} g_2(t). \end{aligned}$$

Therefore, we have

$$x^q(t) \le \left(A + \frac{M_1}{\Gamma(\alpha)}\left(\sum_{s=t_1-\alpha}^{t-\alpha} m^q(s+\alpha)x^q(s+\alpha)\right)^{1/q}\right)^q$$
$$\le 2^{q-1}A^q + 2^{q-1}\frac{M_1^q}{\Gamma^q(\alpha)}\sum_{s=t_1-\alpha}^{t-\alpha} m^q(s+\alpha)x^q(s+\alpha).$$

Finally, if we apply Lemma 2.10, we have

$$x^q(t) \le A^q 2^{2(q-1)}\frac{M_1^q}{\Gamma^q(\alpha)}\exp\left(\sum_{s=t_1-\alpha}^{t-\alpha} m^q(s+\alpha)\right).$$

By using (3.19), we end up with

$$\limsup_{t\to\infty} x(t) < \infty,$$

which completes the proof. □

**Remark 3.6.** If $x$ is eventually negative, then we can set $y = -x$ to see that $y$ satisfies (1.1) with $e(t)$ replaced by $-e(t)$ and $f(t,x)$ by $-f(t,-y)$. It follows in similar manner that

$$\limsup_{t\to\infty} -x(t) < \infty.$$

## 4. APPLICATIONS

In this section, we present some examples for the illustrate the obtained results.

**Example 4.1.** Consider the following fractional difference equation

$$_C\Delta^{7/4}x\,(t) = e^{-3(t+3/4)} + \frac{\Gamma(t+\frac{7}{4})}{\Gamma(t+2)}e^{-(t+3/4)}x^{1/3}(t+3/4), \quad t\in\mathbb{N}_{1/4}, \tag{4.1}$$

where $\alpha = 3/4$, $e\,(t)$=$e^{-3t}$ and $f\,(t,x) = \frac{\Gamma(t+1)}{\Gamma(t+\frac{5}{4})}e^{-t}x^{1/3}$. If we consider $p = q = 2$,

$\gamma = 3/4$ and $h\,(t) = e^{-t} = m\,(t)$, then it is straightforward to check that all conditions of Theorem 3.3 are satisfied and then consequently every nonoscillatory solution $x$ of Eq. 4.1 satisfies

$$|x(t)| = O(t), \quad t\to\infty.$$

**Example 4.2.** Consider the following fractional difference equation

$$_C\Delta x(t) = e^{-2(t+1)} + \frac{\Gamma(t+2)}{\Gamma(t+\frac{8}{3})}e^{-(t+1)}x^{1/2}(t+1), \quad t\in\mathbb{N}_0. \tag{4.2}$$

In view of equation (3.18), the above equation (4.2) is a particular case with

$$e(t) = e^{-2t}, \quad f(t,x) = \frac{\Gamma(t+1)}{\Gamma(t+\frac{5}{3})} e^{-t} x^{1/2}, \quad \alpha = 1, \quad \text{and} \quad h(t) = e^{-t}.$$

Putting $p = \frac{3}{2} > 1$. Then $q = \frac{p}{p-1} = 3$, $\gamma = \frac{1}{3}$, and $p(\alpha - 1) + 1 = 1 > 0$. If we let $m(t) = h(t)$, then

$$g(t) = \sum_{s=0}^{t-1} (t-s-1)^{(1-1)} (s+1)^{(\frac{1}{3}-1)} e^{-(s+1)} = \sum_{s=0}^{t-1} \frac{\Gamma(s+1)}{\Gamma(s+1+\frac{2}{3})} e^{-(s+1)},$$

which is convergent as $t \to \infty$. Moreover, we have

$$\sum_{s=t_1-1}^{\infty} m^q(s+1) = \sum_{s=t_1-1}^{\infty} e^{-3(s+1)} < \infty$$

It follows that

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} (t-s-1)^{(\alpha-1)} |e(s+1)| = \lim_{t\to\infty} \sum_{s=0}^{t-1} (t-s-1)^{(1-1)} e^{-2(s+1)} < \infty.$$

Therefore, by Theorem 3.5 every nonoscillatory solution of (4.2) is bounded.

## 5. CONCLUSION

Following the trend in studying qualitative properties of solutions of fractional difference equations, we investigated the nonoscillatory behavior of three different classes of fractional difference equations. The main results are obtained by the use of the fractional Taylor's difference formula, some features of discrete fractional calculus and certain mathematical inequalities. To ensure consistency with the theoretical findings, numerical examples are provided. We claim that our results are new and have not been considered earlier.

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