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# On Uniqueness of Meromorphic Functions Sharing Three Sets with Finite Weights

by

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**Summary.** We prove the uniqueness of meromorphic functions sharing some three sets with finite weights.

**1. Introduction, definitions and results.** In the paper we will denote by $\mathbb{C}$ the set of all complex numbers, by $\mathbb{N}$ the set of all positive integers and write $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{0, \infty\}$. Throughout the paper the letters $n$, $m$ are reserved for elements of $\mathbb{N}$, while $k, l, p \in \overline{\mathbb{N}}$, $z, w \in \mathbb{C}$. Also it is tacitly assumed that all meromorphic functions considered are defined on $\mathbb{C}$ and that they are non-constant.

For such a function $f$ and $a \in \overline{\mathbb{C}}$, each $z$ with $f(z) = a$ will be called an *$a$-point* of $f$. For a meromorphic function $f$ and a set $S \subset \overline{\mathbb{C}}$ we define $E_f(S)$ (resp. $\overline{E}_f(S)$) as the set of all $a$-points of $f$, when $a \in S$, together with their multiplicity (resp. without their multiplicity). If $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$) then we simply say $f$, $g$ *share* $S$ Counting Multiplicities or $CM$ (resp. Ignoring Multiplicities or IM).

More formally we define

Definition 1.1. If $f$ is a meromorphic function and $S \subset \overline{\mathbb{C}}$ then if $z_0 \in f^{-1}(S)$, the value of $E_f(S)$ at the point $z_0$ is denoted by $E_f(S)(z_0) : f^{-1}(S) \to \mathbb{N}$ and is equal to the multiplicity of zero of the function $f(z) - f(z_0)$ at $z_0$, i.e. the order of the pole of the function $(f(z) - f(z_0))^{-1}$ at $z_0$ if $f(z_0) \in \mathbb{C}$ (resp. of the function $f(z)$ if $z_0$ is a pole for $f$).

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The following notion of weighted sharing of values and sets was introduced by Lahiri [8, 9]. It expedited new directions of research in value distribution theory.

DEFINITION 1.2. For $k \in \overline{\mathbb{N}}$ and $z_0 \in f^{-1}(S)$ we put $E_f(S,k)(z_0) = \min\{E_f(S)(z_0), k+1\}$. Given $S \subset \overline{\mathbb{C}}$, we say that meromorphic functions $f$ and $g$ *share* the set $S$ *up to multiplicity* $k$ (or *share* $S$ *with weight* $k$, or simply *share* $(S,k)$) if $f^{-1}(S) = g^{-1}(S)$ and for each $z_0 \in f^{-1}(S)$ we have $E_f(S,k)(z_0) = E_g(S,k)(z_0)$, which is represented by the notation $E_f(S,k) = E_g(S,k)$.

The subject of the paper is closely related to a problem posed by H. X. Yi [13]. The problem was to find three, possibly small, finite subsets $S_1$, $S_2$, $S_3$ of $\overline{\mathbb{C}}$ such that for any two meromorphic functions $f$, $g$ which share each of the three sets $S_i$, $i = 1, 2, 3$ CM, we have $f \equiv g$. The problem has drawn attention of many mathematicians. It was solved by W. C. Lin and H. X. Yi [10] who proved that the sets $S_1 = \{0\}$, $S_2 = \{z \in \mathbb{C} : az^n - n(n-1)z^2 + 2n(n-2)bw = (n-1)(n-2)b^2\}$ and $S_3 = \{\infty\}$ have the above property, for $n \geq 5$, where $a$ and $b$ are complex numbers satisfying $ab^{n-2} \neq 2{,}0$. Later the result was strengthened by H. Y. Xu, H. X. Zhang and C. F. Yi [11] and the first author of the present paper [2]–[3].

In this paper we modify the sets $S_1$, $S_2$ so that $S_1 = \{0, 1\}$, and the number of elements in the new set $S_2$ is decreased by 1 in the optimal case. Moreover the conditions on the sharing sets $S_i$, $i = 1, 2, 3$, are relaxed to the conditions of sharing $(S_i, k_i)$, $i = 1, 2, 3$, where $k_1$, $k_2$, $k_3$ are relatively small.

The main result of the paper is the following.

THEOREM 1.1. *Let* $S_1 = \{0, 1\}$,

$$S_2 = \left\{ z : \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c = 0 \right\},$$

*where* $n \geq 4$, $c \in \mathbb{C}$, $c \neq 0, 1, 1/2$, *and* $S_3 = \{\infty\}$. *If two meromorphic functions* $f$ *and* $g$ *share* $(S_1, p)$, $(S_2, m)$ *and* $(S_3, k)$, *where* $p \leq 1$, $2 \leq m < \infty$ *and*

$$0 < \frac{9 - 4p/3 - 2m}{m+1} < 2 - \frac{4 - 2p/3}{k+2},$$

*then* $f \equiv g$.

COROLLARY 1.1. *If* $(p, m, k)$ *is one of the triplets* $(0, 2, 11)$, $(0, 3, 2)$, $(0, 4, 1)$, $(1, 2, 3)$, $(1, 3, 1)$ *then the conclusion of Theorem* 1.1 *holds.*

**2. Auxiliary definitions and lemmas.** The proofs of the main theorems depend heavily on the value distribution of meromorphic functions, as in [6]. We will use standard definitions and notations from this theory. In particular $N(r, a; f)$ (resp. $\overline{N}(r, a; f)$) denotes the counting function (resp.

reduced counting function) of $a$-points of a meromorphic function $f$, $T(r,f)$ is the Nevanlinna characteristic function of $f$, and $S(r,f)$ is used to denote each function which is of smaller order than $T(r,f)$ when $r\to\infty$. Moreover we will need the following notation.

Definition 2.1 ([7]). For $a\in\overline{\mathbb{C}}$ we denote by $N(r,a;f\mid=1)$ the counting function of simple $a$-points of $f$. For a positive integer $m$ we denote by $N(r,a;f\mid\geq m)$ the counting function of those $a$-points of $f$ whose multiplicities are not less than $m$, where each $a$-point is counted according to its multiplicity. We denote by $\overline{N}(r,a;f\mid\geq m)$ the reduced form of $N(r,a;f\mid\geq m)$.

Definition 2.2 ([14]). Let $f$ and $g$ be meromorphic functions sharing $(a,0)$ where $a\in\mathbb{C}\cup\{\infty\}$. We denote by $\overline{N}_L(r,a;f>g)$ the reduced counting function of those $a$-points of $f$ whose multiplicity corresponding to $f$ is greater than that corresponding to $g$.

Definition 2.3 ([8, 9]). Let $f$, $g$ share $(a,0)$. We denote

$$\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)=\overline{N}_L(r,a;f>g)+\overline{N}_L(r,a;g>f).$$

For fixed $n\geq 3$ and $c\in\mathbb{C}\setminus\{0,1,1/2\}$ we set

$$Q(z):=\frac{(n-1)(n-2)}{2}z^2-n(n-2)z+\frac{n(n-1)}{2}\quad\text{and}\quad P(z):=z^{n-2}Q(z).$$

To meromorphic functions $f$, $g$ we associate $F$, $G$ by

$$F=\frac{P(f)}{c},\qquad G=\frac{P(g)}{c},\tag{2.1}$$

and to $F$, $G$ we associate $H$ by the formula

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).\tag{2.2}$$

Lemma 2.1 ([9, Lemma 1]). *Let $F$, $G$ be meromorphic functions sharing $(1,1)$ and let $H$ be given by* (2.2). *If $H\not\equiv 0$, then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. *Let $F$, $G$, $H$ be as in* (2.1), (2.2) *and let $S_i$ $i=1,2,3$, be as defined in Theorem* 1.1. *If $H\not\equiv 0$ and $f$, $g$ share $(S_1,p)$, $(S_2,0)$ and $(S_3,0)$, where $p<\infty$, then*

$$\begin{aligned}N(r,H)\leq\ &\overline{N}(r,0;f\mid\geq p+1)+\overline{N}(r,1;f\mid\geq p+1)+\overline{N}_*(r,1;F,G)\\&+\overline{N}_*(r,\infty;f,g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),\end{aligned}$$

*where $\overline{N}_0(r,0;f')$ is the reduced counting function for the points $\{z\in\mathbb{C}: f'(z)=0,\ f(z)\neq 0,1;\ F(z)\neq 1\}$, and $\overline{N}_0(r,0;g')$ is defined similarly.*

*Proof.* Since

$$F-1=\frac{P(f)-c}{c},\qquad G-1=\frac{P(g)-c}{c}$$

and $E_f(S_2,0)=E_g(S_2,0)$ we see that $F$ and $G$ share $(1,0)$. It is easy to check that

$$H=\frac{2f'}{f-1}-\frac{2g'}{g-1}+\frac{(n-3)f'}{f}-\frac{(n-3)g'}{g}+\frac{f''}{f'}-\frac{g''}{g'}-\left(\frac{2F'}{F-1}-\frac{2G'}{G-1}\right).$$

Since $E_f(S_1,p)=E_g(S_1,p)$ we deduce that $z\in f^{-1}(\{0,1\})$ if and only if $z\in g^{-1}(\{0,1\})$. Hence

$$\begin{aligned}\overline{N}(r,0;f\mid\geq p+1)&+\overline{N}(r,1;f\mid\geq p+1)\\&=\overline{N}(r,0;g\mid\geq p+1)+\overline{N}(r,1;g\mid\geq p+1).\end{aligned}$$

It can also be easily verified that possible poles of $H$ occur at (i) zeros (or 1-points) of $f$ and $g$ with multiplicity greater than $p$, (ii) poles of $f$ and $g$ with different multiplicities, (iii) 1-points of $F$ and $G$ with different multiplicities, (iv) zeros of $f'$ which are not zeros of $f(f-1)$ and $F-1$, (v) zeros of $g$ which are not zeros of $g(g-1)$ and $G-1$.

Since $H$ has only simple poles, clearly the lemma follows from the above explanations. ■

LEMMA 2.3 ([12]). *If $f$ is a meromorphic function and $R$ a polynomial of degree $n$ then*

$$T(r,R(f))=nT(r,f)+O(1).$$

LEMMA 2.4 ([4, Lemma 2.10]). *If meromorphic functions $f$, $g$ share $(1,m)$, then*

$$\begin{aligned}\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)&\\ \leq\frac{1}{2}[N(r,1;f)+N(r,1;g)]&.\end{aligned}$$

LEMMA 2.5. *If meromorphic functions $f$, $g$ share $(\{0,1\},0)$ and $(\infty,0)$ then $P(f)P(g)$ is not a constant.*

*Proof.* On the contrary, assume that

$$(n-1)^2(n-2)^2f^{n-2}(f-\gamma)(f-\delta)g^{n-2}(g-\gamma)(g-\delta)\equiv 4c^2,\tag{2.3}$$

where $\gamma$ and $\delta$ are the roots of the equation $Q(z)=0$.

If $f$ has a pole then $g$ will also have a pole, which is impossible by (2.3). So $f$ and $g$ have no poles. Similarly $f$ (resp. $g$) cannot have any zero, $\gamma$-points or $\delta$-points as they can only be neutralized by poles of $g$ (resp. $f$). So $f$ and $g$ omit 0, $\infty$ as well as $\gamma$, $\delta$, which is impossible. ■

LEMMA 2.6 ([5, p. 192]). *Let*

$$R(z) = (n-1)^2(z^n-1)(z^{n-2}-1) - n(n-2)(z^{n-1}-1)^2.$$

*Then* $R(z) = (z-1)^4 W(z)$ *and all the* $2n-6$ *roots of the polynomial* $W$ *are distinct and different from* 0, 1.

LEMMA 2.7. *If* $n \geq 4$ *and meromorphic functions* $f$, $g$ *share* $(\{0,1\},0)$ *and* $P(f) \equiv P(g)$ *then* $f \equiv g$.

*Proof.* From the assumption we can write

$$f^{n-2}(f-\gamma)(f-\delta) \equiv g^{n-2}(g-\gamma)(g-\delta). \tag{2.4}$$

Clearly (2.4) implies that $f$ and $g$ share $(\infty,\infty)$. Since $E_f(\{0,1\},0) = E_g(\{0,1\},0)$ it follows that if $z_0$ is a zero of $f$ (resp. $g$) then it cannot be a 1-point of $g$ (resp. $f$) as none of $\gamma$ and $\delta$ is zero. So $f$ and $g$ share $(0,\infty)$ and $(1,\infty)$. Suppose $h = f/g$. Clearly $h$ has no zero and no pole. Substituting $f = hg$ in (2.4) we get

$$\frac{(n-1)(n-2)}{2}(h^n-1)g^2 - n(n-2)(h^{n-1}-1)g + \frac{n(n-1)}{2}(h^{n-2}-1) \equiv 0. \tag{2.5}$$

Suppose $h$ is not a constant. Then by a simple calculation we deduce from (2.5) that

$$\{(n-1)(n-2)(h^n-1)g - n(n-2)(h^{n-1}-1)\}^2 \equiv -n(n-2)R(h), \tag{2.6}$$

where $R(z)$ is as in Lemma 2.6. So using Lemma 2.6 we have

$$\{(n-1)(n-2)(h^n-1)g - n(n-2)(h^{n-1}-1)\}^2 \equiv -n(n-2)(h-1)^4(h-\beta_1)\dots(h-\beta_{2n-6}), \tag{2.7}$$

where $\beta_j \in \mathbb{C} - \{0,1\}$ $(j = 1,\dots,2n-6)$ are distinct. From (2.7) we see that $h - \beta_j$ $(j = 1,\dots,2n-6)$ each have multiplicity at least 2. So by the Second Fundamental Theorem we get

$$\begin{aligned}(2n-6)T(r,h) &\leq \overline{N}(r,\infty;h) + \overline{N}(r,0;h) + \sum_{j=1}^{2n-6} \overline{N}(r,\beta_j;h) + S(r,h)\\ &\leq \frac{1}{2}\sum_{j=1}^{2n-6} N(r,\beta_j;h) + S(r,h)\\ &\leq (n-3)T(r,h) + S(r,h),\end{aligned}$$

which is a contradiction for $n \geq 4$. So $h$ is a constant. From (2.5) we have $h^n - 1 = 0$, $h^{n-1} - 1 = 0$. It follows that $h \equiv 1$ and so $f \equiv g$. ∎

Lemma 2.8. *Let $n \geq 3$ and $S_i$, $i = 1, 2, 3$, be as in Theorem 1.1. Also let meromorphic functions $f$ and $g$ share $(S_1, p)$, $(S_2, m)$, $(S_3, k)$, where $p < \infty$. If $F$, $G$ are given by (2.1) and*

$$\Phi := \frac{F'}{F-1} - \frac{G'}{G-1} \not\equiv 0,$$

*then*

$$\min\{(n-2)p + (n-3),\ 3p+2\}\{\overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1)\} \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g).$$

*Proof.* By the assumptions, $F$ and $G$ share $(1, m)$. Also we see that

$$\Phi = \frac{n(n-1)(n-2)f^{n-3}(f-1)^2 f'}{2c(F-1)} - \frac{n(n-1)(n-2)g^{n-3}(g-1)^2 g'}{2c(G-1)}.$$

Let $z_0$ be a zero or a 1-point of $f$ with multiplicity $r$. Since $E_f(S_1, p) = E_g(S_1, p)$, $z_0$ is a zero of $\Phi$ of multiplicity

$$\min\{(n-3)r + r - 1, 2r + r - 1\} = \min\{(n-2)r - 1,\ 3r - 1\},$$

if $r \leq p$, and of multiplicity at least

$$\min\{(n-3)(p+1) + p, 2(p+1) + p\} = \min\{(n-2)p + (n-3), 3p+2\}$$

if $r > p$. So by a simple calculation we can write

$$\begin{aligned}
\min\{(n-2)p + (n-3),\ 3p+2\}&\{\overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1)\} \\
&\leq N(r, 0; \Phi) \leq T(r, \Phi) \\
&\leq N(r, \infty; \Phi) + S(r, F) + S(r, G) \\
&\leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g). \quad \blacksquare
\end{aligned}$$

Lemma 2.9. *Let $S_i$, $i = 1, 2, 3$, be as in Theorem 1.1 and $F$, $G$, $H$ be given by (2.1) and (2.2). If meromorphic functions $f$ and $g$ share $(S_1, p)$, $(S_2, m)$ and $(S_3, k)$, where $p < \infty$, $2 \leq m < \infty$ and $H \not\equiv 0$, then*

$$\begin{aligned}
(n+1)&\{T(r, f) + T(r, g)\} \\
&\leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 1; f)\} + \overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1) \\
&\quad + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) \\
&\quad + \frac{1}{2}[N(r, 1; F) + N(r, 1; G)] \\
&\quad - \left(m - \frac{3}{2}\right)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).
\end{aligned}$$

*Proof.* By the Second Fundamental Theorem we get

$$(2.8)\qquad (n+1)\{T(r,f)+T(r,g)\}$$
$$\le \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)+\overline{N}(r,0;g)$$
$$+\overline{N}(r,1;g)+\overline{N}(r,\infty;g)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g).$$

Using Lemmas 2.1–2.4 we see that

$$(2.9)\qquad \overline{N}(r,1;F)+\overline{N}(r,1;G)$$
$$\le \frac{1}{2}[N(r,1;F)+N(r,1;G)]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)$$
$$\le \frac{1}{2}[N(r,1;F)+N(r,1;G)]+\overline{N}(r,0;f\mid\ge p+1)+\overline{N}(r,1;f\mid\ge p+1)$$
$$+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')$$
$$+S(r,f)+S(r,g).$$

Applying (2.9) in (2.8) and noting that

$$\overline{N}(r,0;f)+\overline{N}(r,1;f)=\overline{N}(r,0;g)+\overline{N}(r,1;g),$$

the lemma follows. ∎

Lemma 2.10 ([14, Lemma 6]). *If $H\equiv 0$, then $F$, $G$ share $(1,\infty)$. If further $F$, $G$ share $(\infty,0)$ then they share $(\infty,\infty)$.*

Lemma 2.11. *Let $F$, $G$ be given by (2.1) and suppose they share $(1,m)$. Also let $\alpha_1,\dots,\alpha_n$ be the distinct elements of the set*

$$\left\{z:\frac{(n-1)(n-2)}{2}z^n-n(n-2)z^{n-1}+\frac{n(n-1)}{2}z^{n-2}-c=0\right\},$$

*where $c\ne 0,1,1/2$ is a complex number and $n\ge 3$. Then*

$$\overline{N}_L(r,1;F>G)\le\frac{1}{m+1}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_\otimes(r,0;f')]+S(r,f),$$

*where $N_\otimes(r,0;f')$ is the counting function of those 0-points of $f'$ which are not in $f^{-1}(\{0,\alpha_1,\dots,\alpha_n\})$.*

*Proof.* The proof can be carried out along the lines of the proof of [1, Lemma 2.14]. ∎

## 3. Proof of the theorem

*Proof of Theorem 1.1.* Let $F$, $G$ be given by (2.1) and (2.2). Then $F$, $G$ share $(1,m)$ and $f$, $g$ share $(\infty,k)$ . We consider two cases, each of them split into several subcases.

Case 1. Suppose that $\Phi\not\equiv 0$.

SUBCASE 1.1. Let $H \not\equiv 0$. First suppose $p = 0$.

In view of Definition 2.3 we observe that

$$\begin{aligned}\overline{N}_*(r,\infty;f,g) &= \overline{N}_L(r,\infty;f) + \overline{N}_L(r,\infty;g)\\ &\le \overline{N}(r,\infty;f\mid \ge k+2) + \overline{N}(r,\infty;g\mid \ge k+2)\\ &\le \frac{1}{k+2}\{N(r,\infty;f) + N(r,\infty;g)\}.\end{aligned}$$

Then using Lemma 2.3, Lemma 2.8 with $p = 0$ and Lemma 2.11 we deduce that

$$\begin{aligned}(3.1)\qquad &(n+1)\,\{T(r,f)+T(r,g)\}\\ &\le 3\{\overline{N}(r,0;f)+\overline{N}(r,1;f)\} + \left\{1+\frac{1}{k+2}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}\\ &\quad + \frac{1}{2}[N(r,1;F)+N(r,1;G)] - \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\ &\quad + S(r,f)+S(r,g)\\ &\le 3\overline{N}_*(r,\infty;f,g) + \left\{1+\frac{1}{k+2}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}\\ &\quad + \frac{n}{2}\{T(r,f)+T(r,g)\}\\ &\quad - \left(m-\frac{9}{2}\right)\overline{N}_*(r,1;F,G) + S(r,f)+S(r,g)\\ &\le \left\{\frac{n}{2}+1+\frac{4}{k+2}\right\}\{T(r,f)+T(r,g)\}\\ &\quad - \frac{2m-9}{2(m+1)}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\}\\ &\quad + S(r,f)+S(r,g)\\ &\le \left\{\frac{n}{2}+1+\frac{4}{k+2}+\frac{9-2m}{m+1}\right\}\{T(r,f)+T(r,g)\} + S(r,f)+S(r,g).\end{aligned}$$

Since $2-\frac{4}{k+2} > \frac{9-2m}{m+1} > 0$, (3.1) gives a contradiction for $n \ge 4$.

Next suppose $p = 1$.

Using Lemma 2.3, Lemma 2.8 for $p = 0$ and again for $p = 1$, and Lemma 2.11, we get

$$\begin{aligned}(3.2)\qquad &(n+1)\{T(r,f)+T(r,g)\}\\ &\le \frac{7}{3}\{\overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)\}\\ &\quad + \left\{1+\frac{1}{k+2}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}[N(r,1;F)+N(r,1;G)]\\
&-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\le{}&\left\{1+\frac{10}{3(k+2)}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}+\frac{n}{2}\{T(r,f)+T(r,g)\}\\
&-\left(m-\frac{23}{6}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\le{}&\left\{\frac{n}{2}+1+\frac{10}{3(k+2)}\right\}\{T(r,f)+T(r,g)\}-\frac{6m-23}{6(m+1)}\{2T(r,f)+2T(r,g)\}\\
&+S(r,f)+S(r,g)\\
\le{}&\left\{\frac{n}{2}+1+\frac{10}{3(k+2)}+\frac{23-6m}{3(m+1)}\right\}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}
$$

Since the assumption for $p=1$ implies $2-\frac{10}{3(k+2)}>\frac{23-6m}{3(m+1)}>0$, (3.8) gives a contradiction for $n\ge 4$.

Subcase 1.2. Suppose $H\equiv 0$. Then

$$F\equiv\frac{AG+B}{CG+D}, \tag{3.3}$$

where $A$, $B$, $C$, $D$ are constants such that $AD-BC\ne 0$. Also $T(r,F)=T(r,G)+O(1)$, i.e.,

$$T(r,f)=T(r,g)+O(1). \tag{3.4}$$

In view of Lemma 2.10 it follows that $F$ and $G$ share $(1,\infty)$ and $(\infty,\infty)$, that is, $f$ and $g$ share $(\infty,\infty)$. So in view of Lemma 2.8, $\overline{N}(r,0;f)+\overline{N}(r,1;f)=S(r,f)+S(r,g)$. Since $P(1)=1$, by a simple computation it can be easily seen that 1 is a zero with multiplicity 3 of $F-\frac{1}{c}=\frac{P(f)-1}{c}$ and hence

$$F-\frac{1}{c}=(f-1)^3Q_{n-3}(f),$$

where $Q_{n-3}(f)$ is a polynomial in $f$ of degree $n-3$ and thus

$$
\begin{aligned}
\overline{N}\left(r,\frac{1}{c};F\right)&\le\overline{N}(r,1;f)+\overline{N}(r,0;Q_{n-3}(f))\\
&\le\overline{N}(r,1;f)+(n-3)T(r,f)+S(r,f).
\end{aligned}
$$

We now consider the following cases.

Subcase 1.2.1. Let $AC\ne 0$. From (3.3) we get

$$\overline{N}(r,\infty;G)=\overline{N}\left(r,\frac{A}{C};F\right). \tag{3.5}$$

Since $F$ and $G$ share $(1,\infty)$, it follows that $A/C \neq 1$. Suppose $A/C \neq 1/c$. Then in view of Lemma 2.3 and (3.4), by the Second Fundamental Theorem we get

$$\begin{aligned}(n+1)T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)\\ &\quad+\overline{N}\left(r,\frac{A}{C};F\right)+S(r,f)+S(r,g)\\ &=\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)\\ &\leq 2T(r,f)+S(r,f),\end{aligned}$$

which gives a contradiction for $n \geq 4$.

Next suppose $A/C = 1/c$. Then

$$F-\frac{A}{C} \equiv \frac{BC-AD}{C(CG+D)} \quad \text{i.e.,} \quad (f-1)^3 Q_{n-3}(f) \equiv \frac{BC-AD}{C(CG+D)}.$$

Suppose

$$Q_{n-3}(f) = (f-\alpha_1')\dots(f-\alpha_{n-3}'),$$

where $\alpha_i'$'s, $i=1,\dots,n-3$ are distinct. Then the above expression implies that any $\alpha_i'$-point of $f$ of order $p$ (say) will be a pole of order $q$ (say) of $g$. Consequently, we have

$$p = nq \geq n.$$

Noting that $\overline{N}(r,0;f)+\overline{N}(r,1;f) = S(r,f)+S(r,g)$, in view of (3.4) the Second Fundamental Theorem yields

$$\begin{aligned}&(n-2)T(r,f)\\ &\quad\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\sum_{i=1}^{n-3}\overline{N}(r,\alpha_i';f)+S(r,f)\\ &\quad\leq \overline{N}(r,\infty;f)+\frac{n-3}{n}T(r,f)+S(r,f)\\ &\quad\leq \left(1+\frac{n-3}{n}\right)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction for $n \geq 4$.

Subcase 1.2.2. Let $A \neq 0$ and $C = 0$. Then $F \equiv \alpha_0 G + \beta_0$, where $\alpha_0 = A/D$ and $\beta_0 = B/D$.

We note that 1 cannot be a Picard exceptional value (P.e.v.) of $F$ (or $G$). For, if it happens, then $f$ (resp. $g$) omits $n \geq 4$ values, which is a contradiction.

So $F$ and $G$ have some 1-points. Then $\alpha_0+\beta_0 = 1$ and so

$$F \equiv \alpha_0 G + 1 - \alpha_0. \tag{3.6}$$

Suppose $\alpha_0 \neq 1$. If $1-\alpha_0 \neq 1/c$ then using Lemma 2.3, (3.4) and the Second Fundamental Theorem we get

$$\begin{aligned}
&2nT(r,f)\\
&\quad\le \overline{N}(r,0;F)+\overline{N}(r,1-\alpha_0;F)+\overline{N}\left(r,\frac{1}{c};F\right)+\overline{N}(r,\infty;F)+S(r,F)\\
&\quad\le \overline{N}(r,0;f)+2T(r,f)+\overline{N}(r,0;G)+\overline{N}(r,1;f)\\
&\qquad+(n-3)T(r,f)+\overline{N}(r,\infty;f)+S(r,f)\\
&\quad\le (n-1)T(r,f)+3T(r,g)+\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\quad\le (n+3)T(r,f)+S(r,f),
\end{aligned}$$

which implies a contradiction since $n \ge 4$.

If $1-\alpha_0 = 1/c$, then from (3.6) we have $cF \equiv (c-1)G+1$.

Noting that $c \neq 1/2$ and $\overline{N}(r,0;f)+\overline{N}(r,1;f)=\overline{N}(r,0;g)+\overline{N}(r,1;g)$, using Lemma 2.3, (3.4) and (3.6) we obtain, by the Second Fundamental Theorem,

$$\begin{aligned}
&2nT(r,g)\\
&\le \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{c};G\right)+\overline{N}\left(r,\frac{1}{1-c};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\
&\le 2T(r,g)+\overline{N}(r,0;g)+(n-3)T(r,g)+\overline{N}(r,1;g)+2T(r,f)+\overline{N}(r,0;f)\\
&\quad+\overline{N}(r,\infty;g)+S(r,g)\\
&\le 3T(r,f)+nT(r,g)+S(r,f)+S(r,g)\\
&\le (n+3)T(r,g)+S(r,g),
\end{aligned}$$

which implies a contradiction as $n \ge 4$. Therefore $\alpha_0 = 1$ and hence $F \equiv G$. This implies $\Phi \equiv 0$, a contradiction to the initial assumption.

Subcase 1.2.3. Let $A = 0$ and $C \neq 0$. Then

$$F \equiv \frac{1}{\gamma_0 G+\delta_0},$$

where $\gamma_0 = C/B$ and $\delta_0 = D/B$.

Clearly 1 cannot be a P.e.v. of $F$ and so of $G$. Since $F$ and $G$ have some 1-points we have $\gamma_0+\delta_0 = 1$ and so

$$F \equiv \frac{1}{\gamma_0 G+1-\gamma_0}. \tag{3.7}$$

Suppose $\gamma_0 \neq 1$. If $\gamma_0 \neq 1-c$, then noting that

$$\overline{N}(r,0;G) = \overline{N}\left(r,\frac{1}{1-\gamma_0};F\right) \neq \overline{N}\left(r,\frac{1}{c};F\right),$$

by the Second Fundamental Theorem, using Lemma 2.3 we can again deduce a contradiction as above when $n \ge 4$.

If $\gamma_0 = 1 - c$, from (3.7) we have

$$F \equiv \frac{1}{(1-c)G + c}.$$

If possible suppose that $\frac{1}{c} \neq \frac{c}{c-1}$. Now in the same way as above using (3.4), Lemma 2.3, and the Second Fundamental Theorem yields

$$\begin{aligned}
&2nT(r,g)\\
&\quad\le \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{c};G\right) + \overline{N}\left(r,\frac{c}{c-1};G\right) + \overline{N}(r,\infty;G) + S(r,G)\\
&\quad\le \overline{N}(r,0;g) + \overline{N}(r,1;g) + 2T(r,g) + (n-3)T(r,g) + \overline{N}(r,\infty;F)\\
&\qquad + \overline{N}(r,\infty;G) + S(r,f) + S(r,g)\\
&\quad\le nT(r,g) + N(r,\infty;f) + S(r,f) + S(r,g),
\end{aligned}$$

which implies a contradiction for $n \geq 4$.

Next suppose $\frac{1}{c} = \frac{c}{c-1}$. Then

$$F \equiv \frac{1}{-c^2(G - \frac{1}{c})}, \quad \text{i.e.,} \quad F\left(G - \frac{1}{c}\right) \equiv \frac{1}{-c^2}.$$

Since $F$, $G$ share $(\infty,\infty)$, it follows that 0 is a P.e.v. of $F$, which implies $f$ omits three distinct complex numbers, which is impossible. So we must have $\gamma_0 = 1$, i.e., $FG \equiv 1$, which is impossible by Lemma 2.5.

Case 2. Suppose that $\Phi \equiv 0$. On integration we get $F - 1 \equiv A(G-1)$ for some non-zero constant $A$. So in view of Lemma 2.3, (3.4) is satisfied. Since by the assumption of the theorem $E_f(S_1,0) = E_g(S_1,0)$, we consider the following cases.

Subcase 2.1. First assume $f$ and $g$ share $(0,0)$ and $(1,0)$. If none of 0 and 1 is a P.e.v. of $f$ and $g$, then we have $A = 1$. Similarly if one of 0 or 1 is a P.e.v. of $f$ and $g$, then we get $A = 1$ and so in both cases we have $F \equiv G$, which in view of Lemma 2.7 implies $f \equiv g$. If both 0 and 1 are P.e.v. of $f$ as well as of $g$ then noting that here $F \equiv AG + (1-A)$ which is similar to (3.6), we can handle the situation as in Subcase 1.2.2. So we omit the details.

Subcase 2.2. Next suppose that $f$, $g$ do not share $(0,0)$, $(1,0)$. Here we have to consider the following subcases.

Subcase 2.2.1. Suppose there exist $z_0$, $z_1$ such that

$$f(z_0) = 0, \quad g(z_0) = 1, \quad f(z_1) = 1, \quad g(z_1) = 0.$$

i.e., none of 0 and 1 is a P.e.v. of $f$ and $g$. We note that from $F - 1 \equiv A(G-1)$ we get $P(f) - c(1-A) \equiv AP(g)$. If $A \neq 1$, then $c(1-A) \neq 0$. If $c(1-A) = 1$, then $A = \frac{c-1}{c}$. So $F - \frac{1}{c} \equiv \frac{c-1}{c}G$. We have $F(z_0) = 0$ and $G(z_0) = 1/c$. Putting these values we obtain $\frac{-1}{c} = \frac{c-1}{c^2}$, which implies $c = \frac{1}{2}$, a contradiction. So $c(1-A) \neq 0, 1$. Hence $P(f) - c(1-A)$ has simple zeros and so

$$(f-\omega_1)\dots(f-\omega_n)\equiv A\frac{(n-1)(n-2)}{2}g^{n-2}(g-\gamma)(g-\delta),$$

where $\omega_i$ $(i=1,\dots,n)$ are the distinct zeros of $P(f)-c(1-A)$. Since $f$, $g$ share the set $S_1$, from the above we see that 0 is a P.e.v. of $g$, a contradiction.

Subcase 2.2.2. If no such $z_0$ exists, i.e., if 0 is a P.e.v. of $f$ and 1 is a P.e.v. of $g$, then again as above from $\Phi\equiv 0$ we get

$$F\equiv AG+1-A, \tag{3.8}$$

i.e.,

$$\frac{P(f)}{A}\equiv P(g)-\frac{c(A-1)}{A}. \tag{3.9}$$

Clearly, $\frac{c(A-1)}{A}\neq 0$ as $c\neq 0$ and $A\neq 1$. Now if $\frac{c(A-1)}{A}=1$ then $A=\frac{c}{c-1}$. Since any 1-point of $f$ is 0-point of $g$, from (3.8) we have $\frac{1}{c}=1-A$, i.e., $A=\frac{c-1}{c}$. Therefore

$$\frac{c-1}{c}=\frac{c}{c-1},$$

which implies $c=\frac{1}{2}$, a contradiction. This implies $\frac{c(A-1)}{A}\neq 1$ and so $P(g)-\frac{c(A-1)}{A}$ has $n$ distinct zeros $\beta'_j$, say $(j=1,\dots,n)$. Hence from (3.9) we have

$$\frac{(n-1)(n-2)}{2A}f^{n-2}(f-\gamma)(f-\delta)\equiv(g-\beta'_1)\dots(g-\beta'_n).$$

Now by the Second Fundamental Theorem and (3.4) we get

$$\begin{aligned}nT(r,g)&\le\overline{N}(r,0;g)+\overline{N}(r,1;g)+\sum_{j=1}^{n}\overline{N}(r,\beta'_j;g)+S(r,g)\\&\le\overline{N}(r,0;g)+\overline{N}(r,\gamma;f)+\overline{N}(r,\delta;f)+S(r,g)\\&\le 3T(r,g)+S(r,g),\end{aligned}$$

which is a contradiction for $n\ge 4$.

Subcase 2.2.3. If no such $z_0$, $z_1$ exist at all, i.e., 0 and 1 are both Picard exceptional values of $f$ and $g$ then again we can obtain either (3.9) or

$$P(f)-c(1-A)\equiv AP(g). \tag{3.10}$$

We prove that either the right hand side of (3.9) or the left hand side of (3.10) will have $n$ distinct factors. Now if $\frac{c(A-1)}{A}=1$, i.e., the right hand side of (3.9) does not have $n$ distinct factors, then $A=\frac{c}{c-1}$ and hence $c(1-A)=-A=\frac{c}{1-c}\neq 1$ as $c\neq\frac{1}{2}$. So $P(f)-c(1-A)$ has simple zeros and consequently we have $(f-\omega_1)\dots(f-\omega_n)\equiv A\frac{(n-1)(n-2)}{2}g^{n-2}(g-\gamma)(g-\delta)$. Therefore by the Second Fundamental Theorem and (3.4),

$$\begin{aligned} nT(r,f) &\leq \sum_{i=1}^{n} \overline{N}(r,\omega_i;f) + \overline{N}(r,0;f) + \overline{N}(r,1;f) + S(r,f) \\ &\leq \overline{N}(r,\gamma;g) + \overline{N}(r,\delta;g) + S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 3$. ■

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### References


[1] A. Banerjee, *Some uniqueness results on meromorphic functions sharing three sets*, Ann. Polon. Math. 92 (2007), 261–274.

[2] A. Banerjee, *On uniqueness of meromorphic functions sharing three sets*, Analysis 28 (2008), 299–312.

[3] A. Banerjee, *Some further results on the uniquenss of meromorphic functions sharing three sets*, Southeast Asian Bull. Math. 34 (2010), 835–846.

[4] A. Banerjee and P. Bhattacharajee, *Uniqueness and set sharing of derivatives of meromorphic functions*, Math. Slovaca 61 (2011), 197–214.

[5] G. Frank and M. Reinders, *A unique range set for meromorphic functions with 11 elements*, Complex Var. Theory Appl. 37 (1998), 185–193.

[6] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[7] I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sci. 28 (2001), 83–91.

[8] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. 161 (2001), 193–206.

[9] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. 46 (2001), 241–253.

[10] W. C. Lin and H. X. Yi, *Uniqueness theorems for meromorphic functions that share three sets*, Complex Var. Theory Appl. 48 (2003), 315–327.

[11] H. Y. Xu, H. X. Zhang and C. F. Yi, *Uniqueness theorems for meromorphic functions that share three sets with weights*, Novi Sad J. Math. 38 (2008), 71–81.

[12] C. C. Yang, *On deficiencies of differential polynomials II*, Math. Z. 125 (1972), 107–112.

[13] H. X. Yi, *Uniqueness of meromorphic functions and question of Gross*, Sci. China Ser. A 37 (1994), 802–813.

[14] H. X. Yi, *Meromorphic functions that share one or two values II*, Kodai Math. J. 22 (1999), 264–272.


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