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Hybrid reliability modelling under general uncertainty

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uncertain measure, hybrid lifetime, uncertain parameter, average chance reliability

Abstract

The real world phenomena are often facing the co-existence reality of different formality of uncertainty and thus the probabilistic reliability modeling practices are very doubtful. Under complicated uncertainty environments, hybrid variable modeling is important in reliability and risk analysis, which includes Bayesian distributional theory, random fuzzy distributional theory, as well as fuzzy random distributional theory as special distribution families. In this paper, we define a new hybrid lifetime which is specified by a random lifetime distribution with an uncertain distributed parameter, which is called as random uncertain hybrid lifetime. We furthermore define the average chance distribution as a quality index for quantifying the hybrid lifetime and accordingly the average chance reliability is derived.

1. Introduction

System reliability, as a quality index, is the capability to complete the specified functions accurately in mutually harmonious manner under the specified conditions within specified period. The quality and reliability engineering facilitates the specification of the system reliability function on the ground of probability and statistics theory. The Toyota crisis does not only tear off the brand image of quality but also shake the belief of existing quality and reliability engineering practices and the underlying probability and statistics theory, which treat the random uncertainty. Uncertainty in real world is intrinsic and diversified in formality. For example, the vagueness is another form of uncertainty, which is more and more aware of in today's industrial environments, just as Carvalho and Machado [1] commented, "In a global market, companies must deal with a high rate of changes in business environment. The parameters, variables and restrictions of the production system are inherently vagueness." Therefore quality and reliability engineering is no longer a blind exercise of applying the traditional techniques from existing probabilistic reliability engineering literature.

The coexistence of randomness and other forms of uncertainty in reliability concept is intrinsic and inherent and therefore modern reliability analysis inevitably engages hybrid lifetime modeling.

Accordingly, the methodology to solve the reliability of hybrid lifetime should be developed in terms of the basic concept of general uncertain measure theory.

2. A review of uncertain measure theory

Uncertain measure [6], [7] is an axiomatically defined set function mapping from a σ-algebra of a given space (set) to the unit interval [0,1], which provides a measuring grade system of an uncertain phenomenon and facilitates the formal definition of an uncertain variable.

Let Ξ be a nonempty set (space), and $\mathfrak{A}(\Xi)$ the σ-algebra on Ξ. Each element, let us say, $A\subset\Xi$, $A\in\mathfrak{A}(\Xi)$ is called an uncertain event. A number denoted as $\lambda\{A\}$, $0\le\lambda\{A\}\le 1$, is assigned to event $A\in\mathfrak{A}(\Xi)$, which indicates the uncertain measuring grade with which event $A \in \mathfrak{A}(\Xi)$ occurs. The normal set function $\lambda\{A\}$ satisfies following axioms given by [6], [7]:

Axiom 1. (Normality) $\lambda\{\Xi\}=1$.

Axiom 2. (Monotonicity) $\lambda\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B, \lambda\{A\} \le \lambda\{B\}$.

Axiom 3. (Self-Duality) $\lambda\{\}$ is self-dual, i.e., for any $A \in \mathfrak{A}(\Xi), \lambda\{A\}+\lambda\{A^c\}=1$.

Axiom 4. (σ- Subadditivity) $\lambda\left\{\bigcup_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} \lambda\{A_i\}$ for any countable event sequence $\{A_i\}$.

Definition 2.1. [6], [7] Any set function $\lambda : \mathfrak{A}(\Xi) \to [0,1]$ satisfies *Axioms 1-4* is called an uncertain measure. The triple $(\Xi, \mathfrak{A}(\Xi), \lambda)$ is called the uncertain measure space.

Definition 2.2. [6], [7] An uncertain variable ξ is a measurable mapping, i.e., $\xi : (\Xi, \mathfrak{A}(\Xi)) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, where $\mathfrak{B}(\mathbb{R})$ denotes the Borel σ -algebra on $\mathbb{R}=(-\infty,+\infty)$.

Remark 2.3. The fundamental difference between a random variable and an uncertain variable is the measure space on which they are defined. In the triples, the first two factors are similar in formation: the set and the σ- algebra on the set. However, the third factor in the triples: the measures defined on the σ- algebras are not similar. The former (i.e. the probability measure) obeys σ- additivity and the later (i.e. the uncertain measure) obeys σ- subadditivity. The way for specifying measure inevitably has impacts on the behaviour of the measurable function on the triple.

Definition 2.4. [7] Let ξ be an uncertain variable on $(\Xi, \mathfrak{A}(\Xi), \lambda)$. A nonnegative, non-decreasing function $\Psi : \mathbb{R} \to [0,1]$ if

$$\Psi(x) = \lambda\{\tau \in \Xi : \xi(\tau) \le x\} \tag{1}$$

is called as uncertainty distribution for the uncertain variable ξ.

Definition 2.5. [7] (Identification function of the first kind) If function $\iota : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\sup_{x \ne y}\{\iota(x)+\iota(y)\}=1 \tag{2}$$

Then, $\iota(\cdot)$ is termed as the identification function of the first kind for an uncertain variable ξ.

Theorem 2.6. [7] If $\iota(\cdot)$ is an identification function of the first kind, then for $\forall B \subset \mathbb{R}$, an uncertain measure is defined by

$$\lambda\{B\} = \begin{cases} \sup_{x\in B} \iota(x) & \text{if } \sup_{x\in B} \iota(x) < 0.5 \\ 1-\sup_{x \in B^c} \iota(x) & \text{if } \sup_{x\in B} \iota(x) \ge 0.5 \end{cases}. \tag{3}$$

Proof: The set function $\lambda\{\cdot\}$ satisfies normality, monotonicity, self-duality and σ- sub-additivity with the support of equality : $\sup_{x \ne y}\{\iota(x)+\iota(y)\}=1$, thus it is an uncertain measure. An uncertain variable ξ mapping from the uncertain space $(\Xi, \mathfrak{A}(\Xi), \lambda)$ to $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \nu)$.

3. Hybrid variable concept

Since [9], [10] proposed fuzzy set theory, fuzzy random fuzzy set, a special case of hybrid variable, soon proposed by [4]. [5] defined that a random fuzzy variable, another special case of hybrid variable, is a mapping from the credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$ to a set of random variables. Let us start with a general hybrid variable definition.

Definition 3.1. A hybrid variable is a real-valued measurable mapping, i.e., $\eta : (\Xi, \mathfrak{A}) \to (\mathbb{R}, \mathfrak{B})$.

Remark 3.2. It is obvious that the order of the formation of a hybrid variable does matter. For example, Random fuzzy variable [6] and fuzzy random variable [4] are two types of hybrid variable, even with the same component uncertain variables. Therefore, it is necessary to define them separately when specifying the hybrid variable with different uncertain variables.

Definition 3.3. A random-uncertain hybrid variable is a measurable mapping η from product space $(\Xi, \mathfrak{A}(\Xi), \lambda) \times (\Omega, \mathfrak{F}(\Omega), \mathrm{Pr})$ into $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \nu)$, which is called as hybrid variable of Type I; An uncertain-

random hybrid variable is a measurable mapping η from product space $(\Omega, \mathfrak{F}(\Omega), \Pr) \times (\Xi, \mathfrak{A}(\Xi), \lambda)$ into $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \nu)$, which is called hybrid variable of Type II.
In the remaining of the paper, we only deal with hybrid variable of Type I, i.e., random-uncertain hybrid variable. Therefore, for convenience we simply use the term hybrid variable. For reliability engineers and managers armed with introductory probability and statistics, this definition will be difficult to understand. For a more intuitive understanding, we would like to present a definition similar to that of stochastic process in probability theory and expect readers who are familiar with the basic concept of stochastic processes can understand our comparative definition.

Definition 3.4. A hybrid variable (of Type I), denoted by $\xi = \left\{X_{\beta(\tau)}, \tau \in \Xi\right\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, \mathfrak{F}, \Pr)$ and indexed by an uncertain variable $\beta(\tau)$ defined on the uncertainty space $(\Xi, \mathfrak{A}(\Xi), \lambda)$.
Similar to the interpretation of a stochastic process $X = \left\{X_t, t \in \mathbb{R}^+\right\}$, a hybrid variable is also a bivariate mapping from $(\Omega \times \Xi, \mathfrak{F} \times \mathfrak{A})$ to the space $(\mathbb{R}, \mathfrak{B})$. As to the index set, in stochastic process theory, index set used is referred to as *time* typically, which is a positive (scalar variable), while in the random fuzzy variable theory, the "index" is an uncertain variable β. Using uncertain parameter as index is not starting in hybrid variable definition. In stochastic process theory we already know that the stochastic process $X = \left\{X_{\tau(\omega)}, \omega \in \Omega\right\}$ uses stopping time $\tau(\omega)$, $\omega \in \Omega$, which is an random variable as its index.

4. Average chance measure and average chance distribution for a hybrid variable

Hybrid variable can be quantified in terms of chance measure concept, see [5], [6], and [8].

Definition 4.1. Let ξ be a random-uncertain hybrid variable and B a Borel set of real numbers. Then the chance measure of random fuzzy event $\{\xi \in B\}$ is a function mapping from $(0,1]$ to $[0,1]$,

$$\mathrm{Ch}\{\xi \in B\}(\alpha) = \sup_{\lambda\{A\} \geq \alpha} \inf_{\theta \in A} \Pr\{\theta : \xi(\theta) \in B\} \tag{4}$$

However, we notice the potential mathematical complexity associated with the chance measure formulation. Therefore, it is necessary to explore a convenient way to deal with the chance measure specification. Recall that in probability theory, the distribution of a random variable ξ on probability space $(\Omega, \mathfrak{A}, \Pr)$, $F_\xi(\cdot)$ links to the probability measure of event $\forall\{\omega : \xi(\omega) \leq x\} \in \mathfrak{A}$

$$F_\xi(x) = \Pr\{\omega : \xi(\omega) \leq x\}. \tag{5}$$

In random-uncertain hybrid variable theory, we may say that that average chance measure plays an equivalent role similar to probability measure, denoted as $\Pr$, in probability theory.

Definition 4.2. Let ξ be a random-uncertain hybrid variable, then the *average* chance measure, denoted as $\mathrm{ch}\{\cdot\}$, of a random-uncertain event $\{\tau \in \Xi : \xi(\tau) \leq x\}$, is

$$\mathrm{ch}\{\xi \leq x\} = \int_0^1 \lambda\{\tau \in \Xi | \Pr\{\xi(\tau) \leq x\} \geq \alpha\} \,\mathrm{d}\alpha \tag{6}$$

Then function $\Psi(\cdot)$ is called as *average* chance distribution if and only if

$$\Psi(x) = \mathrm{ch}\{\xi \leq x\} \tag{7}$$

Now, we are required to establish a theoretical framework in terms of average chance measure concepts. Once the average chance measure for the basic event form $\{\xi \leq x\}$ is given, then the average chance measure for any event A should be established in terms of the basic event $\{\xi \leq x\}$. In this way, we may define average chance measure for an arbitrary event A. The triple space $(\Omega \times \Xi, \mathfrak{F} \times \mathfrak{A}, \mathrm{ch})$ is called the average chance space.

Proposition 4.3. Let $\mathrm{ch}\{\cdot\}$ be an average chance measure on a product measure space $(\Omega \times \Xi, \mathfrak{F}(\Omega) \times \mathfrak{A}(\Xi))$. Then

(i) $\mathrm{ch}\{\varnothing\} = 0$ and $\mathrm{ch}\{\Omega \times \Xi\} = 1$;

(ii) (Normality) $\forall A \in \mathfrak{F} \times \mathfrak{A}$, $0 \leq \mathrm{ch}\{A\} \leq 1$;

(iii) (Self-Duality) For $\forall A \in \mathfrak{F} \times \mathfrak{A}$, then $\mathrm{ch}\{A^c\} = 1 - \mathrm{ch}\{A\}$

(iv) (Weak monotone increasing) For $\forall A \subset B,\ A, B \in \mathfrak{F} \times \mathfrak{A}$, $\text{ch}(A) \le \text{ch}(B)$;

(v) (Semi-Continuity) For $\forall A_n,\ A \in \mathfrak{F} \times \mathfrak{A}$, $n = 1, 2, \cdots$, if $A_n \to A$, then

$$\lim_{A_n \to A} \text{ch}(A_n) = \text{ch}(A) \tag{8}$$

if and only if one of the following conditions holds:
(a) $\lambda(A_n) \le 0.5\ \&\ A_n \uparrow A$,
(b) $\lim_{n\to\infty} \lambda(A_n) < 0.5\ \&\ A_n \uparrow A$,
(c) $\lambda(A_n) \ge 0.5\ \&\ A_n \downarrow A$, and
(d) $\lim_{n\to\infty} \lambda(A_n) > 0.5\ \&\ A_n \downarrow A$.
(vi) (Sub-Additivity) For $\forall A \subset B,\ A, B \in \mathfrak{F} \times \mathfrak{A}$,

$$\text{ch}(A \cup B) \le \text{ch}(A) + \text{ch}(B) \tag{9}$$

Proposition 4.4. Let $\Psi_\xi(\cdot)$ be average chance distribution of (random-uncertain) hybrid variable ξ on the chance measure space $(\Omega \times \Xi, \mathfrak{F} \times \mathfrak{A}, \text{ch})$. Then
(i) $\Psi_\xi(-\infty) = 0$ and $\Psi_\xi(+\infty) = 1$;
(ii) For $\forall x \in \mathbb{R} = (-\infty, +\infty)$, $0 \le \Phi_\xi(x) \le 1$;
(iii)Nonnegative real-valued function $\psi_\xi(\cdot)$ is called average chance density for a (random-uncertain) hybrid variable ξ if for $\psi_\xi(x) \ge 0,\ x \in \mathbb{R}$ and

$$\Psi_\xi(x) = \int_{-\infty}^{x} \psi_\xi(u)\, \text{d}u \tag{10}$$

5. Construction of random-uncertain hybrid variable

Liu [5] mentioned an exponentially distributed random fuzzy variable ξ has a density function

$$\phi(x) = \begin{cases} \dfrac{1}{\beta}\exp\left(-\dfrac{x}{\beta}\right) & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

if the value of β is assumed to be a fuzzy variable, then ξ is a random fuzzy variable. Similarly, let parameter β be an uncertain variable following a distribution function $\Lambda_\beta(\cdot)$, and the probability density is defined by Equation (11), then the random-uncertain hybrid variable ξ is said to be exponentially distributed. This example hints a constructive definition for specifying hybrid variable, i.e., random-uncertain variable or equivalently, the average chance distribution.

Definition 5.1. Let $\{F(x;\beta(\tau)), \tau \in \Xi\}$ be a family of probability distributions on the probability space $(\Omega, \mathfrak{A}, \Pr)$ with a common uncertain parameter β on the uncertain measure space $(\Xi, \mathfrak{A}(\Xi), \lambda)$, then the average distribution derived from $(F(x, \beta), \lambda)$ defines a (random-uncertain) hybrid variable ξ.

Theorem 5.2. Let ξ be a random-uncertain hybrid variable. If the expectation $E_P\left[\xi(\tau_0)\right]$ exists for any given $\tau_0 \in \Xi$, then $E_P\left[\xi(\cdot)\right]$ is an uncertain variable.

6. Random uncertain hybrid lifetimes

Analyzing hybrid lifetimes, or survival times, or failure times, is the focus of lifetime modeling and analysis under randomness and general uncertainty co-existence environments. Different from the statistical lifetime modeling and analysis, where the random lifetimes are concerned, also different from the uncertainty lifetime modeling and analysis, where the uncertainty lifetimes are concerned, hybrid lifetime modeling analysis provides a general guideline with a rigorous theoretical foundation.
A (random-uncertain) hybrid lifetime, denoted by ξ, which is a special case of hybrid (of Type I), takes only a positive real values. In other words, hybrid lifetime is a bivariate mapping from $(\Omega \times \Xi, \mathfrak{F} \times \Xi)$ to the space $\left(\mathbb{R}^+, \mathfrak{B}\left(\mathbb{R}^+\right)\right)$.

6.1. Basic construction of continuous hybrid lifetimes

In statistical lifetime modeling and analysis, the probability distribution contains the full information on system lifetime and there are many related concepts, particularly, hazard function reveals an aspect of lifetime distribution, which links to the physical structure of a system.

Theorem 4.1. Let ξ be a continuous hybrid lifetime having probability distribution function $F(t;\beta(\tau))$, where the uncertain parameter β is defined on the uncertain measure space $(\Xi, \mathfrak{A}(\Xi), \lambda)$. Then function $\Pi(t;\beta) = \Lambda(F(t;\beta))$ can uniquely define the hybrid lifetime ξ if the operator or function Λ is invertible. *Table 1* lists four commonly used operators or functions.

Table 1. Examples of operators or functions

Name	Form of $\Pi(t;\beta)$	$\Lambda(\cdot)$
Survival function	$\bar{F}(t;\beta) = 1 - F(t;\beta)$	$F(t;\beta) = 1 - \bar{F}(t;\beta)$
Density function	$f(t;\beta) = \mathrm{d}F(t;\beta)/\mathrm{d}t$	$F(t;\beta) = \int_0^t f(u;\beta)\,\mathrm{d}u$
Hazard function	$h(t;\beta) = f(t) / (1 - F(t;\beta))$	$F(t;\beta) = 1 - \exp\left(-\int_0^t h(u;\beta)\,\mathrm{d}u\right)$
Moment generating function	$m(\theta;\beta) = \int_0^{+\infty} e^{\theta t}\mathrm{d}F(t;\beta)$	$F(t;\beta) = \int_0^t \left(\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} m(s;\beta)\, e^{su}\mathrm{d}s\right)\mathrm{d}u$

6.2. Continuous hybrid lifetime models

In statistical lifetime modeling and analysis, the elementary lifetime models are exponential, Weibull, Log-normal, gamma, Cox-Lewis, bathtub, and etc. These are essential for the construction of hybrid lifetimes. *Table 2* lists these models.

Table 2. Commonly used distributional lifetime models

Name	Probability density & hazard function	
Exponential	density	$\beta\exp(-\beta t)$
	hazard	β
Weibull	density	$(\beta/\eta)(t/\eta)^{\beta-1}\exp\left(-(t/\eta)^{\beta}\right)$
	hazard	$(\beta/\eta)(t/\eta)^{\beta-1}$
Extreme - value	density	$(1/u)\exp((t-b)/u)\exp(-\exp((t-b)/u))$
	hazard	$(1/u)\exp((t-b)/u)$
Log-Normal	density	$1/(\sqrt{2\pi}\sigma t)\exp\left(-(\ln t-\mu)^2/2\sigma^2\right)$
	hazard	$\left(1/(\sqrt{2\pi}\sigma t)\exp\left(-(\ln t-\mu)^2/2\sigma^2\right)\right)/\left(1-\Phi((\ln t-\mu)/\sigma)\right)$
Gamma	density	$\lambda(\lambda t)^{\beta-1}/\Gamma(\beta)\, e^{-\lambda t}$
	hazard	$\left(\lambda(\lambda t)^{\beta-1}/\Gamma(\beta)\, e^{-\lambda t}\right)/\left(1-I(\beta,\lambda t)\right)$
Bathtub	density	$(\beta/\eta)(t/\eta)^{\beta-1}\exp\left((t/\eta)^{\beta-1}\right)\exp\left(-\exp\left((t/\eta)^{\beta-1}\right)\right)$
	hazard	$(\beta/\eta)(t/\eta)^{\beta-1}\exp\left((t/\eta)^{\beta-1}\right)$

In *Table 2*, $I(\beta, \lambda t)$ denotes the incomplete gamma function of the first-type and $\Phi(\cdot)$ represents the cumulative distribution of a standard normal variable.

6.3. Proportional hazard models

Covariate models play very important roles in lifetime analysis. Cox [2] initiated proportional hazards (abbreviated as PH) model as following:

$$h(t;\beta,\gamma) = h_0(t;\beta)\,\varsigma(\gamma^T y) \tag{12}$$

where $h_0(t;\beta)$ is called the baseline hazard function having a fuzzy parameter β defined on the credibility measure space $(\Xi, \mathfrak{A}, \lambda)$, while $\varsigma : \mathbb{R} \to \mathbb{R}^+$ with

$$\gamma^T y = \gamma_0 + \gamma_1 y_1 + \cdots + \gamma_p y_p \tag{13}$$

where $y = (1, y_1, \cdots, y_p)^T$ is covariate vector and $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_p)^T$ is covariate effect parameter vector. A typically function of $\varsigma : \mathbb{R} \to \mathbb{R}^+$ used is the exponential function $\varsigma(x) = e^x$. It is easy to show that the accumulated hazard if covariate *y* is not time-dependent is

$$H(t;\beta,\gamma) = H_0(t;\beta)\, e^{\gamma^T y}. \tag{14}$$

And therefore the average chance distribution with covariate *y* is

$$\Phi(t,y) = \int_0^1 \lambda\left\{(\tau_1,\tau_2) : H_0(t;\beta(\tau_1))\, e^{\gamma^T y(\tau_2)} \geq -\ln(1-\alpha)\right\}\mathrm{d}\alpha \tag{15}$$

where covariate *y* is assumed to be uncertain distributed but parameter γ is assumed to be determined. Other options are also possible to be formulated.

7. Exponentially distributed hybrid lifetimes

The purpose to have this section is double-folded: (a) exponential hybrid lifetime is an important member for system lifetime analysis; (b) the arguments for deriving the average chance distribution are demonstration in line with hybrid variable reliability analysis. Bearing this agenda in mind, the following step-by-step developments will be very beneficial.

Let us use exponentially distributed hybrid lifetime which has probability density

$$f(t;\beta) = \begin{cases} 0 & t \leq 0 \\ \beta e^{-\beta t} & t > 0 \end{cases} \tag{16}$$

where the uncertain parameter has a trapezoidal identification function [7]

$$\iota_\beta(x)=\begin{cases}\dfrac{x-a}{2(b-a)} & a<x\le b\\ 0.5 & b<x\le c\\ \dfrac{d-x}{2(d-c)} & c<x\le d\\ 0 & \text{otherwise}\end{cases} \tag{17}$$

Note that

$$\Pr\{\xi(\theta)\le t\}=1-e^{\beta(\theta)t} \tag{18}$$

Therefore event $\{\tau:\Pr\{\xi(\tau)\le t\}\ge\alpha\}$ is an uncertain event and is equivalent to the uncertain event $\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$. As a critical toward the derivation of the average chance distribution, it is necessary to calculate the uncertain measure for the uncertain event $\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$, i.e., obtain the expression for

$$\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\} \tag{19}$$

For the trapezoidal uncertain variable (parameter), β, the uncertain measure takes the form

$$\lambda\{\tau\in B\}=\begin{cases}\sup_{x\in B}\iota(x) & \text{if } \sup_{x\in B}\iota(x)<0.5\\ 1-\sup_{x\in B^c}\iota(x) & \text{if } \sup_{x\in B}\iota(x)\ge 0.5\end{cases}$$

i.e., the uncertain parameter β follows the uncertainty distribution:

$$\Lambda(x)=\lambda\{\xi\le x\}=\begin{cases}0 & \text{if } x\le a\\ \dfrac{x-a}{2(b-a)} & \text{if } a<x\le b\\ 0.5 & \text{if } b<x\le c\\ \dfrac{x+d-2c}{2(d-c)} & \text{if } c<x\le d\\ 1 & \text{if } x>d\end{cases}. \tag{20}$$

Accordingly, the range for integration with respect to α can be determined as shown in *Table 3*. Recall that the expression of $x=-\ln(1-\alpha)/t$ appears in Equations (19), which facilitates the link between intermediate variable α and average chance measure.

Table 3. Range analysis for α

Range for x	α and credibility measure expression	
$-\infty<x\le a$	Range for α	$0\le\alpha\le 1-e^{-at}$
	$\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$	1
$a<x\le b$	Range for α	$1-e^{-at}<\alpha\le 1-e^{-bt}$
	$\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$	$1-(x-a)/2(b-a)$
$b<x\le c$	Range for α	$1-e^{-bt}<\alpha\le 1-e^{-ct}$
	$\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$	0.5
$c<x\le d$	Range for α	$1-e^{-ct}<\alpha\le 1-e^{-dt}$
	$\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$	$(d-x)/2(d-c)$
$d<x<+\infty$	Range for α	$1-e^{-dt}<\alpha\le 1$
	$\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$	0

The average chance distribution for the exponentially distributed hybrid lifetime is then derived by splitting the integration into five terms according to the range of α and the corresponding mathematical expression for the uncertain measure $\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}$, which is detailed in Table 3. Then the exponential random fuzzy lifetime has an average chance distribution function:

$$\begin{aligned}\Psi_\xi(t)&=\int_0^1\lambda\{\theta:\beta(\theta)\ge-\ln(1-\alpha)/t\}\,d\alpha\\ &=1+\frac{e^{-bt}-e^{-at}}{2(b-a)t}+\frac{e^{-dt}-e^{-ct}}{2(d-c)t}\end{aligned} \tag{21}$$

and the average chance density is

$$\begin{aligned}\psi_\xi(t)&=\frac{e^{-at}-e^{-bt}}{2(b-a)t^2}+\frac{be^{-bt}-ae^{-at}}{2(b-a)t}\\ &+\frac{e^{-ct}-e^{-dt}}{2(d-c)t^2}+\frac{ce^{-ct}-de^{-dt}}{2(d-c)t}\end{aligned} \tag{22}$$

Similar to the probabilistic reliability theory, we define a reliability function or survival function for a random fuzzy lifetime and accordingly name it as the average chance reliability function, which is defined accordingly as

$$R_\xi(t)=1-\Psi_\xi(t) \tag{23}$$

Then, for exponential random fuzzy lifetime, its average chance reliability function is

$$R_\xi(t)=\frac{e^{-at}-e^{-bt}}{2(b-a)t}+\frac{e^{-ct}-e^{-dt}}{2(d-c)t} \tag{24}$$

In standard statistical lifetime modelling and analysis reliability function reveals the system functioning behaviour. The average chance reliability function should play similar roles in hybrid lifetime modelling and analysis. In order to gain an intuitive perceptions on the average chance reliability function, let us assume that the trapezoidal identification function defined by (0.1, 0.15, 0.25, 0.30), i.e., the parameters for specifying the identification function are $a = 0.1,\ b = 0.15$, $c = 0.25, d = 0.30$. For comparison purpose, we define an exponentially distributed random lifetime with fixed valued parameter, 0.20, which is obtained by

$$m_\beta = E(\beta) = 0.20 \qquad (25)$$

Then the reliability function for the exponentially distributed random lifetime with parameter $m_\beta = 0.20$ is

$$R(t;0.20) = \exp(-0.2t) \qquad (26)$$

The corresponding average chance reliability function, $R_\xi(t;\beta)$:

$$R_\xi(t;\beta) = \frac{10(e^{-at} - e^{-bt})}{t} + \frac{10(e^{-ct} - e^{-dt})}{t} \qquad (27)$$

Figure 1 gives a comparison between $R_\xi(t;\beta)$ and $R(t;0.20)$.

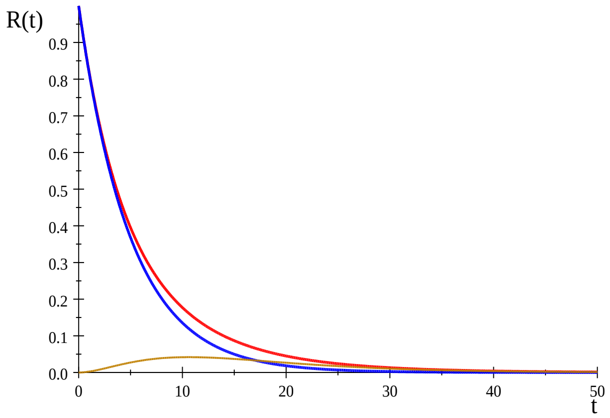

Figure 1. Exponential hybrid lifetime average chance reliability $R_\xi(t;\beta)$ (Red), corresponding exponential lifetime reliability $R(t;0.20)$ (Blue), and the difference function $d(R_\xi(t;\beta), R(t;0.20))$ (Sienna)

Intuitively, we can see that given two systems: the first one is an exponentially distributed hybrid system with trapezoidal uncertain distributed parameter $\beta = (0.10, 0.15, 0.25, 0.30)$ and the second one is an exponentially distributed random system with parameter $m_\beta = 0.20$, the first one enjoys a higher reliability than that of the second one. Definitely, a rigorous mathematical proof should be pursued before stating this impression as a general statement.

However, the purpose for us to develop hybrid lifetime analysis theory is a serious effort to facilitate a foundation for analyzing reliability data collected from system performance.

7. Concluding Remarks

In this paper, we develop a framework for modeling hybrid lifetimes (of Type I) and the average chance distribution as well as the average chance reliability. The models are constructive. We use exponentially distributed hybrid lifetime with a trapezoidal identification function as an example to illustrate the model developments on hybrid lifetimes. [3] demonstrated hybrid variable theory in repairable modeling, although in random fuzzy context. However, many research work need to be done, for example, the parameter estimation, the asymptotic distribution for the estimated parameters, the small sample asymptotic theory, etc.

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