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A coding theoretical interpretation of Gaussian-Pell polynomials

Abstract In this paper, we establish a new result followed from Gaussian Pell polynomials matrix, $Q^n(x)P(x)$ (cf. Serpil and Sinan (2018)) whose elements are Gaussian Pell polynomials and we develop a new coding and decoding method follow from Gaussian Pell polynomials matrix, $Q^n(x)P(x)$. The correction ability of this method is 93:33% .

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1. Introduction. The Pell numbers are defined by the recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2} \quad \text{for} \quad n \geqslant 2$$

with initial seeds

$$P_0 = 0, P_1 = 1.$$

The Pell numbers, P_n and silver ratio,

$$\mu = \lim_{n \longrightarrow \infty} \frac{P_n}{P_{n-1}} = 1 + \sqrt{2}$$

have appeared in sciences and information theory Stakhov (2006), Stakhov (2007), Horadam (1971), Basu and Prasad (2010), Halici (2011), El Naschie (2009). The Pell polynomials Serpil and Sinan (2018) are defined by the recurrence relation:

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x) \quad \text{for} \quad n \geqslant 2$$

with initial seeds

$$P_0(x) = 0, P_1(x) = 1.$$

The Gaussian Pell polynomials Serpil and Sinan (2018) are defined by the relation:

$$GP_n(x) = P_n(x) + iP_{n-1}(x)$$

where i is the imaginary unit which satisfies $i^2 = -1$.
The Gaussian Pell polynomials Serpil and Sinan (2018) are defined by the recurrence relation:

$$GP_n(x) = 2xGP_{n-1}(x) + GP_{n-2}(x) \text{ for } n \geqslant 2$$

with initial seeds

$$GP_0(x) = i, GP_1(x) = 1.$$

2. Gaussian Pell polynomials matrix, $Q^n(x)P(x)$. In this section, we defined a Gaussian Pell polynomials matrix, $Q^n(x)P(x)$,

$$Q^n(x)P(x) = \begin{pmatrix} GP_{n+2}(x) & GP_{n+1}(x) \\ GP_{n+1}(x) & GP_n(x) \end{pmatrix}$$

where

$$Q(x) = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$P(x) = \begin{pmatrix} 2x+i & 1 \\ 1 & i \end{pmatrix}$$

$DetQ(x) = -1$, $DetP(x) = 2(-1+ix)$ and $Det(Q^n(x)P(x)) = 2(-1)^n(-1+ix)$, which is known as Cassini formula for the Gaussian Pell polynomials.

3. Gaussian Pell polynomials coding and decoding method. In this paper, we introduce Gaussian Pell polynomials the coding and decoding method, which is applicable for a complex plane. In this method, we represent the message in the form of nonsingular square matrix, M of order 2 and we represent the Gaussian Pell polynomials matrix, $Q^n(x)P(x)$ of order 2 as coding matrix and its inverse matrix $(Q^n(x)P(x))^{-1}$ as a decoding matrix. We represent a transformation $M \times (Q^n(x)P(x)) = E$ as Gaussian Pell polynomials coding and a transformation $E \times (Q^n(x)P(x))^{-1} = M$ as Gaussian Pell polynomials decoding. We represent the matrix, E as code matrix.

3.1. Example of Gaussian Pell polynomials coding and decoding method. Let us represent the initial message in the form of the nonsingular square matrix, M of order 2

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}. \tag{1}$$

Let us assume that all elements of the matrix are positive integer i.e., m_1, m_2, m_3, $m_4 > 0$. We can select for any value of n from Gaussian Pell polynomials matrix, $Q^n(x)P(x)$ as a coding matrix. We simply write for $n = 1$

$$Q(x)P(x) = \begin{pmatrix} GP_3(x) & GP_2(x) \\ GP_2(x) & GP_1(x) \end{pmatrix} = \begin{pmatrix} 4x^2+1+i2x & 2x+i \\ 2x+i & 1 \end{pmatrix}. \tag{2}$$

Then the inverse of $Q(x)P(x)$ is given by

$$\begin{aligned}(Q(x)P(x))^{-1} &= \frac{1}{(-2)(-1+ix)}\begin{pmatrix} GP_1(x) & -GP_2(x) \\ -GP_2(x) & GP_3(x)\end{pmatrix} \\ &= \frac{1}{(-2)(-1+ix)}\begin{pmatrix} 1 & -2x-i \\ -2x-i & 4x^2+1+i2x\end{pmatrix}. \end{aligned} \quad (3)$$

Then the Gaussian Pell polynomials coding of the message (1) consists of the multiplication of the initial matrix (2) that is

$$\begin{aligned} M \times (Q(x)P(x)) &= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} 4x^2+1+i2x & 2x+i \\ 2x+i & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4m_1x^2+m_1+2m_2x+i(2m_1x+m_2) & 2m_1x+m_2+im_1 \\ 4m_3x^2+m_3+2m_4x+i(2m_3x+m_4) & 2m_3x+m_4+im_3 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E \end{aligned}$$

where

$$\begin{aligned} e_1 &= 4m_1x^2+m_1+2m_2x+i(2m_1x+m_2), \\ e_2 &= 2m_1x+m_2+im_1, \\ e_3 &= 4m_3x^2+m_3+2m_4x+i(2m_3x+m_4), \\ e_4 &= 2m_3x+m_4+im_3. \end{aligned}$$

Then the code message, $E = e_1,\ e_2,\ e_3,\ e_4$ is sent to a channel. The decoding of the code message, E performed by following way,

$$\begin{aligned} &\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}\frac{1}{(-2)(-1+ix)}\begin{pmatrix} 1 & -2x-i \\ -2x-i & 4x^2+1+i2x \end{pmatrix} \\ &= \frac{1}{(-2)(-1+ix)}\begin{pmatrix} e_1-2e_2x-ie_2 & -2e_1x+4e_2x^2+e_2+i(-e_1+2e_2x) \\ e_3-2e_4x-ie_4 & -2e_3x+4e_4x^2+e_4+i(-e_3+2e_4x) \end{pmatrix} \\ &= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = M. \end{aligned}$$

4. Determinant of the code matrix, E. The code matrix, E is defined by the following formula $E = M \times (Q^n(x)P(x))$. According to the matrix theory Hohn (2002) we have

$$\begin{aligned} Det\ E = Det(M \times (Q^n(x)P(x))) &= Det\ M \times Det\ (Q^n(x)P(x)) \\ &= Det\ M \times Det\ (Q^n(x))Det\ (P(x)). \end{aligned} \quad (4)$$

5. Relations between the code matrix elements Case I
When $n = 2k+1$ i.e. when n is an odd.

We can write the code matrix, E and the initial message, M in the following way

$$E = M \times (Q^{2k+1}(x)P(x)) = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} GP_{2k+3}(x) & GP_{2k+2}(x) \\ GP_{2k+2}(x) & GP_{2k+1}(x) \end{pmatrix}$$
$$= \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$$

and

$$M = E \times (Q^{2k+1}(x)P(x))^{-1}$$
$$= \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \frac{1}{2(-1+ix)} \begin{pmatrix} -GP_{2k+1}(x) & GP_{2k+2}(x) \\ GP_{2k+2}(x) & -GP_{2k+3}(x) \end{pmatrix} = \frac{1}{2(-1+ix)}$$
$$\times \begin{pmatrix} -e_1GP_{2k+1}(x) + e_2GP_{2k+2}(x) & e_1GP_{2k+2}(x) - e_2GP_{2k+3}(x) \\ -e_3GP_{2k+1}(x) + e_4GP_{2k+2}(x) & e_3GP_{2k+2}(x) - e_4GP_{2k+3}(x) \end{pmatrix}.$$

Since $m_1,\ m_2,\ m_3,\ m_4$ are positive integers, we have

$$m_1 = \frac{-e_1GP_{2k+1}(x) + e_2GP_{2k+2}(x)}{2(-1+ix)} > 0, \tag{5}$$

$$m_2 = \frac{e_1GP_{2k+2}(x) - e_2GP_{2k+3}(x)}{2(-1+ix)} > 0, \tag{6}$$

$$m_3 = \frac{-e_3GP_{2k+1}(x) + e_4GP_{2k+2}(x)}{2(-1+ix)} > 0, \tag{7}$$

$$m_4 = \frac{e_3GP_{2k+2}(x) - e_4GP_{2k+3}(x)}{2(-1+ix)} > 0. \tag{8}$$

From (5) and (6) we get

$$\frac{GP_{2k+3}}{GP_{2k+2}} < \frac{e_1}{e_2} < \frac{GP_{2k+2}}{GP_{2k+1}}. \tag{9}$$

From (7) and (8) we get

$$\frac{GP_{2k+3}}{GP_{2k+2}} < \frac{e_3}{e_4} < \frac{GP_{2k+2}}{GP_{2k+1}}. \tag{10}$$

Therefore, for large value of k we get

$$\frac{e_1}{e_2} \approx \mu,\ \frac{e_3}{e_4} \approx \mu \text{ where } \mu = 1+\sqrt{2}. \tag{11}$$

Case II
When $n = 2k$ i.e. when n is even.
We will get the same result as above in (11).

6. Error detection and correction.

6.1. Error detection. The main aim of the coding theory are the detection and correction of errors arising in the code message, E under the influence of noise in the communication channel. The most important idea is using the property of the determinant of the matrix as the check criterion of the transmitted message, E. Let the initial message, M be given by

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \tag{12}$$

where all elements m_1, m_2, m_3, m_4 of the matrix, M are positive integers. Now the determinant of M is

$$Det\ M = m_1 m_4 - m_2 m_3 \tag{13}$$

and the code message, E

$$E = (M \times (Q^n(x)P(x))). \tag{14}$$

So,

$$Det\ E = Det\ (M \times (Q^n(x)P(x))) = Det\ M \times Det\ (Q^n(x)P(x)) \tag{15}$$

$$= Det\ M \times Det\ (Q^n(x))Det\ P(x). \tag{16}$$

This shows that the determinant of the initial message, M is connected with the determinant of the code message, E by the relation (15)-(16). The value of the determinant of the message, E depends on the number n is even or an odd. The essence of the method consists that the sender calculates the determinant of the initial message, M represented in the matrix form (12) and sends it to the channel after the code message, E (14). The receiver calculates the determinant of the code message, E (14) and compares the determinant of the initial message of M (12) received from the channel. If this comparison corresponds to (15)-(16) it means that the code message, E (14) is correct and the receiver can decode the code message, E (14) otherwise the code message, E (14) is not correct. Error detection is the first step in communication of messages.

6.2. Error correction. The possibility of restoration of the code message, E can be done by using the property of the Gaussian Pell polynomials matrix, $Q^n(x)P(x)$. For selecting $n = 1$, Gaussian Pell polynomials matrix, $Q^n(x)P(x)$ will be

$$Q(x)P(x) = \begin{pmatrix} GP_3(x) & GP_2(x) \\ GP_2(x) & GP_1(x) \end{pmatrix} = \begin{pmatrix} 4x^2 + 1 + i2x & 2x + i \\ 2x + i & 1 \end{pmatrix}. \tag{17}$$

Then the Gaussian Pell polynomials coding of the message (12) consists of the multiplication of the initial matrix (17) that is

$$\begin{aligned} M \times Q(x)P(x) &= \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} 4x^2+1+i2x & 2x+i \\ 2x+i & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4m_1x^2+m_1+2m_2x+i(2m_1x+m_2) & 2m_1x+m_2+im_1 \\ 4m_3x^2+m_3+2m_4x+i(2m_3x+m_4) & 2m_3x+m_4+im_3 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E \end{aligned}$$

where

$$\begin{aligned} e_1 &= 4m_1x^2+m_1+2m_2x+i(2m_1x+m_2), \quad e_2 = 2m_1x+m_2+im_1, \\ e_3 &= 4m_3x^2+m_3+2m_4x+i(2m_3x+m_4), \quad e_4 = 2m_3x+m_4+im_3. \end{aligned}$$

After constructing the code matrix, E we calculate the determinant of the initial matrix, M (12). The determinant is sent to the communication channel after the code message, $E = e_1, e_2, e_3, e_4$. Assume that the communication channel has the special means for the error detection in each of the elements e_1, e_2, e_3, e_4 of the code message, E. Assume that the first element e_1 of E is received with the error. Then, we can represent the code message in the matrix form

$$E' = \begin{pmatrix} y & e_2 \\ e_3 & e_4 \end{pmatrix} \tag{18}$$

where y is the destroyed element of the code message, E but the rest matrix entries must be correct and equal to the following:

$$\begin{aligned} e_2 &= 2m_1x+m_2+im_1; \quad e_3 = 4m_3x^2+m_3+2m_4x+i(2m_3x+m_4); \\ e_4 &= 2m_3x+m_4+im_3. \end{aligned}$$

Then, according to the properties of the Gaussian Pell polynomials coding method, we can write the following equation for calculation of y

$$\begin{aligned} ye_4 - e_2e_3 &= y(2m_3x+m_4+im_3) \qquad (19) \\ &\quad -(2m_1x+m_2+im_1)(4m_3x^2+m_3+2m_4x+i(2m_3x+m_4)) \end{aligned}$$

From (19) with the help of (15)-(16), we get

$$y = 4m_1x^2+m_1+2m_2x+i(2m_1x+m_2). \tag{20}$$

Comparing the calculated value (20) with the entry e_1 of the code matrix, E given with (12) we conclude that $y = e_1$. Thus, we have restored the code message, E using the property of determinant of the Gaussian Pell polynomials $Q^n(x)P(x)$ matrix. But in the real situation usually we do not know what element of the code message is destroyed. In this case, we suppose different hypotheses about the possible destroyed elements and then we test these hypotheses. However, we have one more condition for the elements of the code matrix E that all its elements are integers. Our first hypothesis is that we have the case of a single error in the code matrix E received from the communication channel. It is clear that there are four variants of the single errors in the code matrix, E:

$$(a)\begin{pmatrix} y & e_2 \\ e_3 & e_4 \end{pmatrix}(b)\begin{pmatrix} e_1 & z \\ e_3 & e_4 \end{pmatrix}(c)\begin{pmatrix} e_1 & e_2 \\ u & e_4 \end{pmatrix}(d)\begin{pmatrix} e_1 & e_2 \\ e_3 & v \end{pmatrix} \tag{21}$$

where y, z, u, v are destroyed elements. In this case we can check different hypotheses (21). For checking the hypothesis (a), (b), (c), (d) we can write the following algebraic equations based on the checking relation (4):

$$ye_4 - e_2e_3 = Det\ (Q^n(x)P(x))Det\ M, \tag{22}$$

It is a possible single error is in the element e_1;

$$e_1e_4 - ze_3 = Det\ (Q^n(x)P(x))Det\ M, \tag{23}$$

It is a possible single error is in the element e_2;

$$e_1e_4 - e_2u = Det\ (Q^n(x)P(x))Det\ M, \tag{24}$$

It is a possible single error is in the element e_3;

$$e_1v - e_2e_3 = Det\ (Q^n(x)P(x))Det\ M \tag{25}$$

It is a possible single error is in the element e_4.

(26)

It follows from (22)-(25) four variants for calculation of the possible single errors.

$$y = \frac{Det\ (Q^n(x)P(x))Det\ M + e_2e_3}{e_4}, \tag{27}$$

$$z = \frac{-Det\ (Q^n(x)P(x))Det\ M + e_1e_4}{e_3}, \tag{28}$$

$$u = \frac{-Det\ (Q^n(x)P(x))Det\ M + e_1e_4}{e_2}, \tag{29}$$

$$v = \frac{Det\ (Q^n(x)P(x))Det\ M + e_2e_3}{e_1}. \tag{30}$$

The formula (27)-(30) give four possible variants of single error but we have to choice the correct variant only among the cases of the integer solutions

y, z, u, v; besides, we have to choose such solutions, which satisfy the additional checking relations (11). If calculations by formulas (27)-(30) do not give an integer result we have to conclude that our hypothesis about the single error is incorrect or we have an error in the checking element Det M. For the latter case we can use the approximate equalities (11) for checking the correctness of the code matrix E. By analogy we can check all hypotheses of double error in the code matrix. As example let us consider the following case of double errors in the code matrix E

$$\begin{pmatrix} y & z \\ e_3 & e_4 \end{pmatrix} \tag{31}$$

where y, z are the destroyed elements of the code message. Using the first checking relation (4) we can write the following algebraic equation for the matrix (31):

$$ye_4 - ze_3 = Det\ (Q^n(x)P(x))Det\ M. \tag{32}$$

However, according to the second checking relation (11) there is the following relation between y and z :

$$y \approx \mu z. \tag{33}$$

It is important to emphasize that (32) is Diophantine one. As the Diophantine equation (32) has many solutions we have to choice such solutions y, z which satisfy to the checking relation (33). By analogy one may prove that using checking relations (4), (11) by means of solution of the Diophantine equation similar to (32) we can correct all possible double errors in the code matrix. However, we can show by using such approach there is a possibility to correct all possible triple errors in the code matrix E, for example $\begin{pmatrix} y & z \\ u & e_4 \end{pmatrix}$ etc. where y, z, u are destroyed elements. Thus, our method of error correction is based on the verification of different hypotheses about errors in the code matrix by using the checking relations (4), (11) and by using the fact that the elements of the code matrix are integers. If all our solutions do not bring to integer solutions it means that the checking element Det M is erroneous or we have the case of fourfold error in the code matrix, E and we have to reject the code matrix, E as defective and not correctable. Our method allows to correct 14 cases among $({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) = 2^4 - 1 = 15$ cases. It means that correction ability of the method is $\frac{14}{15} = 0.9333 = 93.33\%$.

7. Comparison of the Gaussian Pell polynomials coding method to the other coding method. The Gaussian Pell polynomials coding method is based on matrix approach which possess many peculiarities and advantages in comparison to classical (algebraic) coding method. The use of matrix theory for designing new error-correction codes is the first peculiarity of the Gaussian Fibonacci coding method. The large information units, in particular matrix elements, are objects of detection and correction of errors in the Gaussian Pell polynomials coding method. There is no theoretical restrictions for the value of the numbers that can be matrix elements whereas in algebraic coding theory there are very small information elements, bits and their combinations which are the objects of detection and correction. The Gaussian Pell polynomials coding method has a very high correction ability in comparison to the classical (algebraic) coding method. The Gaussian Pell polynomials coding and decoding method is the extension of the Fibonacci coding and decoding method Stakhov (2006) and it is applicable for a complex plane too.

8. Conclusion. The Gaussian Pell polynomials coding method is the main application of the $Q^n(x)P(x)$ matrix. The Gaussian Pell polynomials coding method reduces to matrix multiplication, a well-known algebraic operation, which is realized very well in modern computers. The main practical peculiarity of this method is that large information units, in particular, matrix elements, are objects of detection and correction of errors. The elements of the initial matrix, M and therefore the elements of the code matrix, E can be the numbers of unlimited value. This means that theoretically the Gaussian Pell polynomials coding method allows to correct the numbers of unlimited value. The correction ability of this method is 93.33%.

9. Open Problem. There is an open problem to define a new matrix $Q^n(x)P(x)$ of order n whose elements are Gaussian Pell polynomials. This matrix will be useful to establish the error detection and correction and relations among the code matrix elements.

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Teoretyczna interpretacja wielomianów Gaussa-Pella jako kodowania.

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Streszczenie W artykule z wykorzystaniem macierzy wielomianów Gaussa Pella, $Q^n(x)P(x)$ (v. Serpil and Sinan (2018)), opracowano nową metodę kodowania. Ta metoda wynika z własności tej macierzy. Uzyskany kod daje możliwość korekcji na poziomie 93:33% .

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