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EXISTENCE AND SMOOTHING EFFECTS OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR $\partial u/\partial t - \Delta\sigma(u) = 0$ IN TIME-DEPENDENT DOMAINS

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Abstract. We show the existence, smoothing effects and decay properties of solutions to the initial-boundary value problem for a generalized porous medium type parabolic equations of the form

$$u_t - \Delta\sigma(u) = 0 \quad \text{in } Q(0,T)$$

with the initial and boundary conditions

$$u(0) = u_0 \quad \text{and} \quad u(t)|_{\partial\Omega(t)} = 0,$$

where $\Omega(t)$ is a bounded domain in R^N for each $t \geq 0$ and

$$Q(0,T) = \bigcup_{0<t<T} \Omega(t) \times \{t\}, \quad T > 0.$$

Our class of $\sigma(u)$ includes $\sigma(u) = |u|^m u$, $\sigma(u) = u\log(1+|u|^m)$, $0 \leq m \leq 2$, and $\sigma(u) = |u|^m u/\sqrt{1+|u|^2}$, $1 \leq m \leq 2$, etc. We derive precise estimates for $\|u(t)\|_{\Omega(t),\infty}$ and $\|\nabla\sigma(u(t))\|^2_{\Omega(t),2}$, $t > 0$, depending on $\|u_0\|_{\Omega(0),r}$ and the movement of $\partial\Omega(t)$.

Keywords: quasilinear parabolic equation, time-dependent domain, smoothing effects.

Mathematics Subject Classification: 35B40, 35K92.

1. INTRODUCTION

In this paper we consider the quasilinear parabolic equation

$$u_t - \Delta\sigma(u) = 0 \quad \text{in } Q(0,T) \tag{1.1}$$

with the initial and boundary conditions

$$u(0) = u_0 \quad \text{and} \quad u(t)|_{\partial\Omega(t)} = 0, \tag{1.2}$$

where $\Omega(t)$ is a bounded domain in R^N for each $t \geq 0$. We set

$$Q(t_1, t_2) = \bigcup_{t_1<t<t_2} \Omega(t) \times \{t\} \quad \text{and} \quad S(t_1, t_2) = \bigcup_{t_1<t<t_2} \partial\Omega(t) \times \{t\}.$$

We assume that

$$\overline{S(0,T)} = \bigcup_{0 \leq t \leq T} \partial\Omega(t) \times \{t\}$$

is of $C^{2+\alpha}$ class, $0 < \alpha < 1$.

Let $\mathbf{n} = (n_x, n_t)$ be the outward normal at $(x,t) \in S(0,T)$. Throughout the paper we assume:

Hypothesis 0. $|n_x| \neq 0$.

By this assumption the so called parabolic boundary of $Q(t_1, t_2)$ coincides to $S(t_1, t_2) \cup \left(\overline{\Omega(t_1)} \times \{t_1\}\right)$ (cf. [7]).

Concerning $\sigma(u)$ we make the following assumptions.

Hypothesis A. $\sigma(u)$ is an odd function in $C^{2+\alpha}(R/\{0\}) \cap C^1(R)$, $0 < \alpha < 1$, and satisfies the conditions:

(1)
$$\sigma(0) = 0, \quad k_0|\sigma(u)| \leq \sigma'(u)|u| \leq k_1|\sigma(u)|,$$

(2) there exist $\nu \geq 0$ and $m \geq 0$ such that for any $K \geq 1$,

$$\sigma'(u) \geq k_0 K^{-\nu}|u|^m \quad \text{if } |u| \leq K,$$

(3)
$$k_0|u|^{l+1} \leq |\sigma(u)| \leq k_1|u|^{L+1} \text{ if } |u| \geq 1$$

with some $0 \leq l \leq L$,

(4)
$$k_0|u|^2 \leq \sigma'(u)|\sigma(u)|^2 \quad \text{if } |u| \geq 1,$$

where in the above k_0, k_1 are positive constants.
(We assume for simplicity that $l \leq m$.)

The condition (4) is required from a technical reason and not necessary when $\Omega(t)$ is independent of t. The typical examples are the following:

$$\sigma(u) = |u|^m u \quad (l = L = m \geq 0, \nu = 0),$$
$$\sigma(u) = u \log(1 + |u|^m) \quad (m \geq 0, l = 0, 0 < L << 1, \nu = m),$$
$$\sigma(u) = |u|^m u/\sqrt{1 + u^2} \quad (m \geq 1, L = l = m - 1, \nu = 1).$$

These ones satisfy Hypothesis A.

When $\Omega(t) = \Omega$, independent of t, the equation (1.1) like $\sigma(u) = |u|^m u$ has been considered by many authors (see DiBenedetto [5], Vázquez [13] and the references cited therein), and in particular, smoothing effects and decay properties including the estimates of $\|u(t)\|_\infty$ and $\|\nabla\sigma(u(t))\|_2$, were investigated by Ohara [12] under the assumption $u_0 \in L^{p_0}$ with some $p_0 > 1$. (Decay property of $\|\nabla\sigma(u(t))\|_2$ is shown in [8].) The object of the present paper is to discuss the existence, smoothing effects and decay properties of the problem (1.1)–(1.2) for $u_0 \in L^r(\Omega(0))$, $r \geq 1$, in time-dependent domains. We derive precise estimates of $\|u(t)\|_{\Omega(t),\infty}$ and $\|\nabla\sigma(u(t))\|_{\Omega(t),2}$, $t > 0$, and these estimations are very delicate for latter examples like $\sigma(u) = u\log(1 + |u|^m)$ and $\sigma(u) = |u|^m u/\sqrt{1 + u^2}$, etc. For a technical reason we must assume $m \leq 2$, but, this assumption is unnecessary if $\Omega(t) = \Omega$ is independent of t and our results are new even for such a cylindrical case. Our results concerning smoothing effects seem to be new even for the nondegenerate case $\sigma'(u) \geq \delta_0 > 0$ when $\Omega(t)$ depends on t. We could consider the equation with a forcing term $f(x,t)$, but to make the essential feature clear we restrict ourselves to the case $f = 0$.

Quite recently we have discussed in [10, 11] the existence and smoothing effects for the equation (1.1) with $-\Delta\sigma(u)$ replaced by $-\mathrm{div}(\sigma(|\nabla u|^2)\nabla u)$ and derived estimate for $\|\nabla u(t)\|_{\Omega(t),\infty}$, and in the present paper we use some techniques developped there. But, we need new devices, in particular, for the estimation of $\|\nabla\sigma(u(t))\|_{\Omega(t),2}$.

We consider the problem (1.1)–(1.2) from mathematical interest. But we expect that our results would have any applications to real phenomena (cf. Díaz [4]).

When $-1 < m < 0$ and $-1 \leq l \leq m \leq L$ in Hypothesis A, we can expect the existence of a finite extinction time of solution, and it is interesting to study the existence and precise behaviours of the solutions, but, these problems are out of scope in the present paper, and we only give some remarks (see Remarks 2.9, 3.4, 5.4 and 7.1). Concerning related problem we refer the interested reader to Antonstsevm, Díaz and Shmarev [2].

2. PRELIMINARIES

Let $u(x,t)$ be a measurable function on

$$\overline{Q(0,T)} = \bigcup_{0 \leq t < T} \overline{\Omega(t)} \times \{t\}.$$

We say u belongs to $L^q([0,T); W_0^{1,p}(\Omega(t))$, $1 \leq p, q \leq \infty$, iff $u(\cdot,t) \in W_0^{1,p}(\Omega(t))$ for each $t \in [0,T)$ and there exists an extended function $\tilde{u}(x,t)$ on $R^N \times [0,T)$ such that $\tilde{u} \in L^q([0,T); W^{1,p}(R^N))$ and $\tilde{u}(x,t) = u(x,t)$ for $x \in \Omega(t)$, $0 \leq t < T$. A similar way of definition will be applied to other function spaces. We denote L^p norm on $\Omega(t)$ by $\|\cdot\|_{\Omega(t),p}$ and L^2 inner product by $(\cdot,\cdot)_{\Omega(t)}$.

We employ the following definitions of solution of the problem (1.1)–(1.2).

Definition 2.1. Let $r \geq 1$. A function $u(t)$ belonging to $L^r([0,T); L^r(\Omega(t))$ is called a Type 1 solution of the problem (1.1)–(1.2) iff

(1)

$$\sigma(u(t)) \in L^1_{loc}((0,T); H^1_0(\Omega(t))) \cap L^1((0,T); L^1(\Omega(t)))$$

and

(2)

$$(u(t),\phi(t))_{\Omega(t)} - \int_0^t \int_{\Omega(s)} u(s)\phi_t(s)dxds - \int_0^t \int_{\Omega(s)} \sigma(u(s))\Delta\phi(s)dxds \\ = (u_0,\phi(0))_{\Omega(0)} \tag{2.1}$$

for any t, $0 < t < T$, and all $\phi(\cdot) \in C^1([0,T); C^2_0(\Omega(t)))$.

Definition 2.2. Let $r > 2N/(N+2)$. A function $u(t) \in L^r([0,T); L^r(\Omega(t)))$ is called a Type 2 solution of the problem (1.1)–(1.2) iff

(1) $\sigma(u(t)) \in L^1((0,T); H^1_0(\Omega(t)))$

and

(2)

$$(u(t),\phi(t))_{\Omega(t)} - \int_0^t \int_{\Omega(s)} u(s)\phi_t(s)dxds + \int_0^t \int_{\Omega(s)} \nabla\sigma(u(s))\cdot\nabla\phi(s)dxds \\ = (u_0,\phi(0))_{\Omega(0)} \tag{2.2}$$

for any t ,$0 < t < T$, and all $\phi(\cdot) \in C([0,T); H^1_0(\Omega(t))) \cap C^1([0,T); L^2(\Omega(t)))$.

For the outward normal $\mathbf{n}(x,t)$ of $Q(0,T)$ at $(x,t) \in S(0,T)$, we set

$$\beta(x,t) = -n_t n_x/|n_x|^2 \in R^N.$$

Since $|n_x| \neq 0$ (Hypothesis 0) and $\overline{S(0,T)}$ is smooth $\beta(x,t)$ can be extended as an R^N valued C^2 class function on $R^N \times [0,T)$. For simplicity of notation we denote this appropriately extended function again by $\beta(x,t)$. We denote $\beta'(t)$ the Jacobian of the map $\beta(\cdot,t) : R^N \to R^N$ for each t. We set

$$\bar{\delta}(t) = \sup_{x\in\Omega(t)} \left(|\beta(x,t)| + |\nabla\beta(x,t)| + |\beta_t(x,t)| + |\beta'(x,t)|\right), \quad t \geq 0,$$

and

$$\bar{\delta}_+(t) = \sup_{0\leq s<t} \bar{\delta}(s).$$

Throughout the paper we assume $T > 1$, $\bar{\delta}_+(T) < \infty$ and the volume $\sup_{0\leq t<T} |\Omega(t)| < \infty$.

Our main results are as follows.

Theorem 2.3. *Let $u_0 \in L^r(\Omega(0))$, $r \geq 1$. We assume in Hypothesis A that*

$$\begin{aligned} &0 \leq m \leq 2, \quad \nu < m + 2r/N, \quad L + 1 + \nu < m + (N+2)r/N, \\ &m(1 - (N+2)r/2N) \leq l + 2r/N. \end{aligned} \tag{2.3}$$

Then there exists a Type 1 solution $u(t)$ of the problem (1.1)–(1.2) *such that $u(t) \in L^\infty_{loc}((0,T); L^\infty(\Omega(t)))$ and $\sigma(u(t)) \in L^\infty_{loc}((0,T); H^1_0(\Omega(t)))$, satisfying*

$$\|u(t)\|_{\infty,\Omega(t)} \leq C_0 \|u_0\|_{\Omega(0),r} t^{-\lambda}, \quad 0 < t \leq 1,$$

with $\lambda = N/(2r + mN - \nu N)$ and

$$\|\nabla\sigma(u(t))\|_{\Omega(t),2} \leq C(A_0^{1/(2-\eta)} t^{-1/(2-\eta)} + \bar{\delta}_+(1)^2), \quad 0 < t \leq 1,$$

where C_0 denotes constants depending continuously on $\|u_0\|_{\Omega(0),r}$ and we set

$$A_0 = \begin{cases} \|u_0\|^{L+2}_{\Omega(0),r} + \|u_0\|^2_{\Omega(0),r} & \text{if } r \geq L+2, \\ \|u_0\|^{L+2-(L+1)\theta}_{\Omega(0),r} + \|u_0\|^{2-\theta}_{\Omega(0),r} & \text{if } 2N/(N+2) \leq r \leq L+2, \\ \|u_0\|^{1-\tilde{\theta}}_{\Omega(0),r} + \|u_0\|^{1-(m+1)\tilde{\theta}/(l+1)}_{\Omega(0),r} & \text{if } 1 \leq r \leq 2N/(N+2), \end{cases} \tag{2.4}$$

and

$$\eta = \begin{cases} 0 & \text{if } r \geq L+2, \\ \theta & \text{if } 2N/(N+2) \leq r \leq L+2, \\ 1 + \tilde{\theta}/(l+1) & \text{if } 1 \leq r < 2N/(N+2). \end{cases} \tag{2.5}$$

with

$$\theta = \frac{L+2-r}{L+1-(N-2)^+ r/2N} \quad \text{and} \quad \tilde{\theta} = \frac{(l+1)(1-(N+2)r/2N}{l+1-(N-2)^+ r/2N}$$

(*when $N = 2$ we replace θ by $\theta + \epsilon$, $0 < \epsilon << 1$*).

Further, if $r > 2N/(N+2)$ the above solution is a Type 2 solution.

Theorem 2.4. *Assume that*

$$\sup_{0 \leq t < T} \{|\Omega(t)| + \bar{\delta}(t)\} \leq k_3 < \infty$$

with a constant k_3 independent of T. Then the solution in Theorem 2.3 satisfies

$$\|u(t)\|_{\Omega(t),\infty} \leq C_0 \|u_0\|^{2N/m(2r+mN)}_{\Omega(0),r} t^{-1/m}, \quad 1 \leq t < T,$$

if $m > 0$ and

$$\|u(t)\|_{\Omega(t),\infty} \leq C_0 \|u_0\|^{N/r}_{\Omega(0),r} e^{-\lambda_0 t}, \quad 1 \leq t < T,$$

with some $\lambda_0 > 0$ if $m = 0$.

We assume further if $m > 0$,

$$\bar{\delta}(t) \leq \bar{\delta}_0 t^{-(3m+2)/4m}, \quad 1 \leq t < T,$$

with some $\bar{\delta}_0 > 0$. *Then*

$$\|\nabla\sigma(u(t))\|_{\Omega(t),2} \le \tilde{C}_0 t^{-(m+1)/m}, \quad 1 \le t < T. \tag{2.6}$$

When $m = 0$ *there exists* $\tilde{\delta}_0 > 0$ *such that if* $\delta(t) \le \tilde{\delta}_0 e^{-\tilde{\lambda} t}$, $\tilde{\lambda} > 0$, *then*

$$\|\nabla\sigma(u(t))\|_{\Omega(t),2} \le \tilde{C}_0 e^{-\lambda_0 t}, \quad 1 \le t < T, \tag{2.7}$$

with some $\lambda_0 > 0$.

Remark 2.5. Hypothesis A(4) and the assumption $m \le 2$ made in the above Theorems are unnecessary when $\Omega(t) = \Omega$, independent of t. Even for such a case of cylindrical domain Theorems 2.3 and 2.4 are new, because we assume only $u_0 \in L^r$, $r \ge 1$, for initial data and also our class of $\sigma(u)$ includes various examples stated in the Introduction. Further we note that our estimates for $\|u(t)\|_\infty$ and $\|\nabla\sigma(u(t))\|_2$ include preciser informations on the dependence on $\|u_0\|_r$. When $\Omega(t) = \Omega$ our results are applied also to the example $\sigma(u) = |u|^{m-m_0} u \log(1 + |u|^{m_0})$, $m \ge m_0 \ge 0$, where we can take $\nu = m_0$, $l = m - m_0$, $L = m - m_0 + \epsilon$, $0 < \epsilon << 1$.

Remark 2.6. The conditions in (2.3) are not so restrictive. Indeed, $\sigma(u) = |u|^m u$, $0 \le m \le 2$, and $\sigma(u) = |u|^m u/\sqrt{1 + u^2}$, $1 \le m \le 2$, satisfy all of the conditions in (2.3) for $r \ge 1$. The example $\sigma(u) = u \log(1 + u^2)$ also satisfies the conditions for all $r \ge \max\{2N/(N + 4), 1\}$.

Remark 2.7. When $\Omega(t) = \Omega$, independent of t, the Type 2 solution in Theorem 2.3 is unique, but for the noncylindrical case the uniqueness problem is very delicate (see Section 8). This is an open problem.

Remark 2.8. Detailed informations on the constants $\tilde{C}_0$ in (2.6) and (2.7) are given in the proofs.

Remark 2.9. The estimates in Theorem 2.3 are formally valid even if $-1 < m < 0$ and $-1 \le l \le m \le L$ without any essential changes when $\Omega(t)$ is independent of t. But, to assure the existence we must construct appropriate smooth approsimate solutions. The way of construction of approximate slutions in the section 7 can not be applied to the case $-1 < m < 0$. When $\Omega(t)$ is time-dependent and $-1 < m < 0$ the situation becomes more complicate because Hypothesis A(4) does not hold.

3. $L^\infty(\Omega(t))$ ESTIMATES FOR $0 < t \le 1$

In this section we derive $L^\infty(\Omega(t))$ estimate for assumed smooth (classical) solution $u(t)$ of the problem (1.1)–(1.2) for $0 < t < T$ which depends on $\|u_0\|_{\Omega(0),r}$, $r \ge 1$. The result in fact will be applied to smooth solutions of an approximate problem. The dependence of $\Omega(t)$ on t does not cause any essential difficulty in this section, and we write often $\|\cdot\|_p$ for $\|\cdot\|_{\Omega(t),p}$. We begin with:

Proposition 3.1.

$$\|u(t)\|_{\Omega(t),r} \le \|u_0\|_{\Omega(0),r}, \quad 0 \le t < T. \tag{3.1}$$

Proof. When $r \geq 2$ we multiply the equation by $|u|^{r-2}u$ and integrationg it to get

$$\begin{aligned} 0 &= \int\limits_{Q(t,t+h)} \frac{1}{r}\frac{\partial}{\partial t}|u|^r dV - \int\limits_{Q(t,t+h)} \Delta(\sigma(u)|u|^{r-2}u dV \\ &= \frac{1}{r}(\|u(t+h)\|_{\Omega(t+h),r} - \|u(t)\|_{\Omega(t),r}) \\ &\quad + (r-1) \int\limits_{Q(t,t+h)} \sigma'(u)|u|^{r-2}|\nabla u|^2 dV, \quad h > 0, \end{aligned}$$

and

$$\|u(t+h)\|_{\Omega(t+h),r} - \|u(t)\|_{\Omega(t),r} \leq 0. \tag{3.2}$$

Dividing (3.2) by $h > 0$ and taking the limit as $h \to 0$ we have

$$\frac{d}{dt}\|u(t)\|^r_{\Omega(t),r} \leq 0$$

which implies (3.1). When $1 \leq r < 2$ we take a function $\rho_\delta(u) \in C^1(R), \delta > 0$, such that $\rho'_\delta(u) \geq 0$, $\rho_\delta(0) = 0$ and $\rho_\delta(u) = |u|^{r-2}u$ if $|u| \geq \delta$, and mutiplying the equation by $\rho_\delta(u(t))$. Repeating a similar argument as above and taking the limit as $\delta \to 0$ we have again (3.2) for $1 \leq r < 2$ and hence (3.1). □

Multiplying the equation by $|u|^{p-2}u$, $p \geq \max\{r, 2\}$, and integrating we see

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|^p_p + (p-1) \int\limits_{\Omega(t)} \sigma'(u(t))|u|^{p-2}|\nabla u|^2 dx = 0. \tag{3.3}$$

First we shall derive the estimate for $\|u(t)\|_\infty$ for t, $0 < t \leq 1$. We can assume that for $K \geq 1$ and $\lambda > 0$,

$$\|u(t)\|_\infty \leq Kt^{-\lambda}, \quad 0 < t \leq \tilde{T} < 1, \tag{3.4}$$

for some $\tilde{T}$. We shall show under (3.4) that a stronger estimate

$$\|u(t)\|_\infty < Kt^{-\lambda}, \quad 0 < t \leq \tilde{T} < 1 \tag{3.5}$$

holds for some $K > 0$ and a specified λ independent of $\tilde{T}$. Then we can conclude by a continuity principle that (3.4) is valid for $t, 0 < t \leq 1$, with such K and λ. (We call such an argument as "loan" method.) The argument deriving (3.5) is essentially included in [10, 11] and we give an outline of it.

By Hypothesis A(2), it follows from (3.3) and (3.4)

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|^p_p + k_0(p-1)K^{-\nu}t^{\nu\lambda} \int\limits_{\Omega(t)} |u|^{p+m-2}|\nabla u|^2 dx \leq 0$$

and hence

$$\frac{d}{dt}\|u(t)\|_p^p + \epsilon_1 K^{-\nu} t^{\nu\lambda} \int_{\Omega(t)} |\nabla(|u|^{(p+m)/2})|^2 dx \le 0 \tag{3.6}$$

with some ϵ_1 independent of p and K. We denote constants depdending on $\sup_{0\le t<T}|\Omega(t)|$ and $\bar{\delta}_+(T)$ by $C(T)$, and $C(1)$ will be written simply by C. By the Gagliardo–Nirenberg inequality, we see (cf. [9, 12, 14])

$$\|u(t)\|_p \le C^{1/(p+m)}\|u(t)\|_r^{1-\theta}\|\nabla|u|^{(p+m)/2}\|_2^{2\theta/(p+m)}, \quad 0 \le t \le 1,$$

with

$$\theta = \left(\frac{p+m}{2}\left(\frac{1}{r}-\frac{1}{p}\right)\right)\left(\frac{1}{N}-\frac{1}{2}+\frac{p+m}{2r}\right)^{-1} = \frac{N(p+m)(1-r/p)}{r(2-N)+N(p+m)}. \tag{3.7}$$

It follows from (3.1), (3.6) and (3.7) that

$$\frac{d}{dt}\|u(t)\|_p^p + \epsilon_1 K^{-\nu} C^{-1/\theta}\|u_0\|_r^{-(1-\theta)(p+m)/\theta} t^{\nu\lambda}\|u(t)\|_p^{(p+m)/\theta} \le 0, \quad 0 \le t \le \tilde{T}. \tag{3.8}$$

We easily see the following lemma.

Lemma 3.2. *Let $y(t)$ be a nonnegative differentiable function on $(0,T)$ satisfying*

$$\frac{dy}{dt} + At^{\tilde{\lambda}\mu-1}y^{1+\mu} \le 0, \quad 0 < t < T,$$

with $A > 0$, $\mu\lambda \ge 1$ and $\mu > 0$. Then

$$y(t) \le \left(\frac{\tilde{\lambda}}{A}\right)^{1/\mu} t^{-\tilde{\lambda}}, \quad 0 < t < T.$$

Applying Lemma 3.2 to (3.8) we have

$$\begin{aligned}\|u(t)\|_p \le \Big((\nu\lambda+1)p\theta/(p(1-\theta)+m)\epsilon_1^{-1}C^{1/\theta}K^{\nu}\Big)^{\theta/(p(1-\theta)+m)} \\ \times \|u_0\|_r^{(1-\theta)(p+m)/(p(1-\theta)+m)} t^{-\theta(\nu\lambda+1)/(p(1-\theta)+m)}, \quad 0 < t \le \tilde{T}.\end{aligned} \tag{3.9}$$

We note that

$$\theta \to 1 \quad \text{and} \quad p(1-\theta) \to \frac{2r}{N} \quad \text{as } p \to \infty,$$

and λ will be taken as $(\nu\lambda+1)(m+2r/N)^{-1} = \lambda$, that is, $\lambda = N/((m-\nu)N+2r)$.

We fix a large p_1 which will be clarified later and define θ_1 by (3.7) with $p = p_1$. We set

$$\eta_1 = (Cp_1(\nu\lambda+1)K^{\nu})^{\theta_1/(p_1(1-\theta_1)+m)}\|u_0\|_r^{(1-\theta_1)(p_1+m)/(p_1(1-\theta_1)+m)} \tag{3.10}$$

and

$$\lambda_1 = \theta_1(\nu\lambda+1)/(p_1(1-\theta_1)+m).$$

Then we see from (3.9) that

$$\|u(t)\|_{p_1} \le \eta_1 t^{-\lambda_1}, \quad 0 < t \le \tilde{T},$$

where the constant C appearing in (3.10) should be chosen appropriately.

We define p_n by $p_n + m = 2p_{n-1}$, that is, $p_n = 2^{n-1}(p_1 - m) + m$, where n denotes natural numbers. For induction argument we assume

$$\|u(t)\|_{p_{n-1}} \le \eta_{n-1} t^{-\lambda_{n-1}}, \quad 0 < t \le \tilde{T}.$$

We return to the inequality (3.6) with $p = p_n$. By the Gagliardo–Nirenberg inequality, we see

$$\|u(t)\|_{p_n} \le C^{1/(p_n+m)} \|u(t)\|_{p_{n-1}}^{1-\theta_n} \|\nabla |u|^{(p_n+m)/2}\|_2^{2\theta_n/(p_n+m)} \tag{3.11}$$

with

$$\begin{aligned}\theta_n &= \left(\frac{p_n+m}{2}\left(\frac{1}{p_{n-1}} - \frac{1}{p_n}\right)\right)\left(\frac{1}{N} - \frac{1}{2} + \frac{p_n+m}{2}\frac{1}{p_{n-1}}\right)^{-1} \\ &= \frac{N(1-m/p_n)}{N+2}, \quad n \ge 2,\end{aligned}$$

and we see from (3.6) and (3.11) that

$$\frac{d}{dt}\|u(t)\|_{p_n}^{p_n} + \frac{\epsilon_1}{CK^\nu}\eta_{n-1}^{-(1-\theta_n)(p_n+m)/\theta_n} t^{\nu\lambda + (1-\theta_n)(p_n+m)\lambda_{n-1}/\theta_n} \|u(t)\|_{p_n}^{(p_n+m)/\theta_n} \le 0. \tag{3.12}$$

Applying Lemma 3.2 to (3.12) we have

$$\begin{aligned}\|u(t)\|_{p_n} \le{}& (C(1+\lambda_{n-1})p_n K^\nu)^{\theta_n/(p_n(1-\theta_n)+m)} \\ &\times \eta_{n-1}^{(1-\theta_n)(p_n+m)/(p_n(1-\theta_n)+m)} t^{-\lambda_n}\end{aligned} \tag{3.13}$$

with some $C > 0$ where we set

$$\lambda_n = \frac{(\nu\lambda+1)\theta_n + (1-\theta_n)(p_n+m)\lambda_{n-1}}{p_n(1-\theta_n)+m}. \tag{3.14}$$

Setting as in [1]

$$\beta_n = \frac{p_n(1-\theta_n)+m}{\theta_n} = \frac{p_n(2p_{n-1}+mN)}{N(p_n - p_{n-1})}$$

we see

$$\lambda_n - \frac{\nu\lambda+1}{m} = \left(1 - \frac{m}{\beta_n}\right)\left(\lambda_{n-1} - \frac{\nu\lambda+1}{m}\right). \tag{3.15}$$

When $m = 0$ we see, instead of (3.15),

$$\lambda_n + \frac{N(\nu\lambda+1)}{2p_n} = \lambda_{n-1} + \frac{N(\nu\lambda+1)}{2p_{n-1}} = \lambda_1 + \frac{N(\nu\lambda+1)}{2p_1}.$$

Since

$$1 - \frac{m}{\beta_n} = \frac{2p_n + mN}{p_n}\frac{p_{n-1}}{2p_{n-1}+mN}$$

and

$$\prod_{k=2}^{n}\left(1 - \frac{m}{\beta_k}\right) = \frac{2p_n+mN}{p_n}\frac{p_1}{2p_1+mN} \to \frac{2p_1}{2p_1+mN} \quad \text{as } n \to \infty$$

we have

$$\begin{aligned}\lambda_n &\to \frac{\nu\lambda+1}{m} + \frac{2p_1}{2p_1+mN}\left(\lambda_1 - \frac{\nu\lambda+1}{m}\right) \\ &= \frac{N(\nu\lambda+1)}{2p_1+mN} + \frac{2p_1\lambda_1}{2p_1+mN} = \frac{N(\nu\lambda+1)}{2r+mN},\end{aligned} \tag{3.16}$$

which is independent of p_1.

When $m = 0$ we see

$$\lambda_n = N(\nu\lambda+1)/2r - N(\nu+1)\lambda/2p_n \to N(\nu\lambda+1)/2r$$

and the conclusion of (3.16) is also valid. We take λ such that

$$\frac{N(\nu\lambda+1)}{2r+mN} = \lambda, \quad \text{i.e., } \lambda = \frac{N}{mN+2r-N\nu},$$

where we make the assumption $\nu < m + 2r/N$.

We know from (3.13) that

$$\|u(t)\|_{p_n} \le \eta_n t^{-\lambda_n},$$

where we can set from the boundedness of λ_n and the choice of λ as

$$\eta_n = (Cp_nK^\nu)^{\theta_n/(p_n(1-\theta_n)+m)}\,\eta_{n-1}^{(1-\theta_n)(p_n(1-\theta_n)+m)} \tag{3.17}$$

with some constant C. Since $(1-\theta_n)(p_n(1-\theta_n)+m) = 1 - m/\beta_n$ we see from (3.17)

$$\begin{aligned}\log\eta_n &\le \frac{\log p_n + \nu\log K + \log C}{\beta_n} + \left(1 - \frac{m}{\beta_n}\right)\log\eta_{n-1} \\ &\le \sum_{k=2}^{n}\frac{\log p_k + \nu\log K + \log C}{\beta_k} + \prod_{k=2}^{n}\left(1 - \frac{m}{\beta_k}\right)\log\eta_1.\end{aligned} \tag{3.18}$$

We note that concerning η_1,

$$\frac{(1-\theta_1)(p_1+m)}{p_1(1-\theta_1)+m} = \frac{(2p_1+mN)N}{(2r+mN)p_1}$$

and

$$\log\eta_1 \le \log C + \mu(p_1)\log K + \frac{(2p_1+mN)N}{(2r+mN)p_1}\log\|u_0\|_r$$

with

$$\mu(p_1) = \frac{\nu\theta_1}{p_1(1-\theta_1)+m} \to \frac{N\nu}{2r+mN} \quad \text{as } p_1 \to \infty.$$

It follows from (3.18)

$$\overline{\lim}_{n\to\infty}\log\eta_n \le \log C + (\nu(p_1)+\mu(p_1))\log K + \frac{2N}{2r+mN}\log\|u_0\|_r$$

with some $C > 0$, where we set $\nu(p_1) = \nu \sum_{k=2}^{\infty} 1/\beta_k$ which tends to 0 as $p_1 \to \infty$, and we obtain

$$\|u(t)\|_\infty \le CK^{\nu(p_1)+\mu(p_1)} \|u_0\|_r^{2N/(2r+mN)} t^{-\lambda}, \quad 0 < t \le \tilde{T}.$$

We see that $\nu(p_1) + \mu(p_1)$ tends to $N\nu/(2r + mN) < 1$ as $p_1 \to \infty$ and hence, for a large p_1 we know $\nu(p_1) + \mu(p_1) < 1$. We fix such a large p_1 to get

$$\|u(t)\|_\infty \le CK^{\gamma} \|u_0\|_r^{2N/(2r+mN)} t^{-\lambda}, \quad 0 < t \le \tilde{T}, \tag{3.19}$$

with $\gamma < 1$. Therefore we can choose a large K which depends on $\|u_0\|_r$ continuously such that

$$\|u(t)\|_\infty < Kt^{-\lambda}, \quad 0 < t \le \tilde{T}. \tag{3.20}$$

We conclude from (3.19) and (3.20) that

$$\|u(t)\|_\infty \le C_0 \|u_0\|_r^{2N/(2r+mN)} t^{-\lambda}, \quad 0 < t \le 1. \tag{3.21}$$

We summarize the above argument.

Proposition 3.3. *Under Hypothesis $A(4)$ with $\nu < m + 2r/N$ any classical solution $u(t)$ of the problem* (1.1)–(1.2), *$T > 1$, satisfies the estimate* (3.21), *where we recall $\lambda = N/(mN + 2r - N\nu)$.*

Remark 3.4. The argument in this section is valid even if $-1 < m < 0$ and $-1 \le l \le L$. In this case we can discuss on the existence of the extinction time. Indeed, (4.2) implies

$$\frac{dy}{dt}(t) + C^{-1}K^{-\nu}t^{\nu\lambda}y^{(p+m)/p}(t) \le 0, \quad 0 < t \le 1 (< T), \tag{3.22}$$

where

$$y(t) = \|u(t)\|_{\Omega(t),p}^p \quad \text{and} \quad K = C_0\|u_0\|_r^{2N/(2r+mN)}.$$

We assume for simplicity that $r > 2 - m$. Then (3.22) is applied to $p = r$ and it implies

$$\|u(t)\|_{\Omega(t),r} \le \left\{ [\|u_0\|_r^{-m} + m\{Cr(\lambda\nu + 1)\|u_0\|_r r^{-m} K^{\nu}\}^{-1} t^{\lambda\nu+1}]^+ \right\}^{-1/m}, \quad 0 < t \le 1.$$

Thus we have an extinction time

$$\begin{aligned} T_{e,0} &= \left(Cr(-m)^{-1}(\lambda\nu + 1)\|u_0\|_r^{-m} K^{\nu}\right)^{1/(\lambda\nu+1)} \\ &= \left(C_0(-m)^{-1}\|u_0\|_r^{-m+2N\nu/(2r+mN)}\right)^{(2r+mN-\nu N)/(2r+mN)} \end{aligned}$$

as far as $T_{e,0} < 1$. Note that $T_{e,0}$ depends only on $\|u_0\|_r$. When $\nu = 0$, for example $\sigma(u) = |u|^m u$, we see that $T_{e,0} = C_0(-m)^{-1}\|u_0\|_r^{-m}$, which is a trivial result directly deduced from (3.6) with $p = r$.

To consider the case $T_{e,0} \ge 1$ and $\nu > 0$ we assume for simplicity,

$$\sup_{0 \le t < T} \{|\Omega(t)| + \bar{\delta}(t)\} < k_3 < \infty$$

with a constant k_3 independent of T. Then we have (see (4.2) and (4.3) below)

$$y(t) \le \Big(y^{-m/r}(1) + m(Cr)^{-1}K^{-\nu}(t-1)\Big)^{-r/m}, \quad 1 \le t < T,$$

where $y(t) = \|u(t)\|^r_{\Omega(t),r}$, and hence

$$\|u(t)\|_{\Omega(t),r}(t) \left(\|u_0\|_r^{-2mN/(2r+mN)} + m(Cr)^{-1}\|u_0\|_r^{-2N\nu/(2r+mN)}(t-1)\right)^{-1/m},$$
$$1 \le t < T.$$

Thus we have an extinction time

$$T_{e,1} = C_0(-m)^{-1}\|u_0\|_r^{2N(\nu-m)/(2r+mN)} + 1.$$

Concerning the extinction time and its estimate we do not want to go far, and further discussions on this problem are left to the interested reader (cf. [2]).

4. DECAY ESTIMATE OF $\|u(t)\|_{\Omega(t),\infty}$ FOR $t \ge 1$

By Propositions 3.1 and 3.3 we see

$$\|u(t)\|_{\Omega(t),\infty} \le \|u(1)\|_{\Omega(1),\infty} \le C_0\|u_0\|_r^{2N/(2r+mN)}, \quad 1 \le t < T. \tag{4.1}$$

Next we derive a decay estimate of $\|u(t)\|_{\Omega(t),\infty}$ for $t, 1 \le t < T$, where T is large (essentially we may consider the case $T = \infty$).

It follows from (3.3), (4.1) and Hypothesis A(2) that

$$\frac{d}{dt}\|u(t)\|^p_{\Omega(t),p} + C^{-1}\bar{C}_0^{-\nu}\|\nabla|u|^{(p+m)/2}\|^2_{\Omega(t),2} \le 0 \tag{4.2}$$

with some $C > 0$, where we set $\bar{C}_0 = C_0\|u_0\|_r^{2N/(2r+mN)}$.

We assume

$$\sup_{0\le t<T}\{|\Omega(t)| + \bar{\delta}(t)\} < k_3 < \infty$$

with a constant k_3 independent of T. Then we have from (4.2)

$$\frac{d}{dt}\|u(t)\|^p_p + C^{-1}\bar{C}_0^{-\nu}\|u\|_p^{p+m} \le 0, \tag{4.3}$$

which implies if $m > 0$,

$$\|u(t)\|_p \le C^{1/p}(m^{-1}p\bar{C}_0^{\nu})^{1/m}(t-1)^{-1/m}, \quad 1 < t < T. \tag{4.4}$$

When $m = 0$ we have instead of (4.4),

$$\begin{aligned}\|u(t)\|_p &\le \|u(1)\|_p \exp\{-(t-1)/p\bar{C}_0\}\\ &\le C_0\|u_0\|_r^{2N/(2r+mN)}\exp\{-(t-1)/p\bar{C}_0\}, \quad 1 \le t < T.\end{aligned}$$

We first consider the case $m > 0$. We fix p_1. Then by (4.4)

$$\|u(t)\|_{p_1} \leq \eta_1 (t-1)^{-\lambda_1}, \quad 1 \leq t < T,$$

where we set

$$\eta_1 = C(m^{-1} p_1 \bar{C}_0^{\nu})^{1/m} \quad \text{and} \quad \lambda_1 = \frac{1}{m}.$$

To derive the estimate for $\|u(t)\|_\infty$, $t \geq 1$, we use a similar, in fact simpler, argument as in deriving (3.19). We set $p_n = 2p_{n-1} - m$, $n \geq 2$, and assume that

$$\|u(t)\|_{p_{n-1}} \leq \eta_{n-1} (t-1)^{-\lambda_{n-1}}, \quad 1 \leq t < T.$$

We know as in (3.12),

$$\frac{d}{dt}\|u(t)\|_{p_n}^{p_n} + \frac{1}{C\bar{C}_0^{\nu}} \eta_{n-1}^{-(1-\theta_n)(p_n+m)/\theta_n} (t-1)^{(1-\theta_n)(p_n+m)\lambda_{n-1}/\theta_n} \|u(t)\|_{p_n}^{(p_n+m)/\theta_n} \leq 0. \tag{4.5}$$

Applying Lemma 3.2 to (4.5) we have (see (3.13) and (3.14))

$$\|u(t)\|_{p_n} \leq \eta_n (t-1)^{-\lambda_n}$$

with

$$\eta_n = (C(1+\lambda_{n-1})p_n)^{1/\beta_n} \, \eta_{n-1}^{1-m/\beta_n}$$

and

$$\lambda_n = \frac{\theta_n + (1-\theta_n)(p_n+m)\lambda_{n-1}}{p_n(1-\theta_n)+m},$$

where we recall $\beta_n = (p_n(1-\theta_n)+m)/\theta_n$. We know (see (3.15)) that

$$\lambda_n - \frac{1}{m} = \Big(1 - \frac{m}{\beta_n}\Big)\Big(\lambda_{n-1} - \frac{1}{m}\Big) = \Big(1 - \frac{m}{\beta_2}\Big)\Big(\lambda_1 - \frac{1}{m}\Big) = 0$$

and hence, $\lambda_n = 1/m$. Therefore we can write η_n as

$$\eta_n = (Cp_n)^{1/\beta_n} \, \eta_{n-1}^{1-m/\beta_n}.$$

We see as in (3.18)

$$\log \eta_n \leq \log C + \frac{2p_1}{2p_1 + mN} \log \eta_1$$

and we conclude that

$$\|u(t)\|_\infty \leq C(m^{-1}\bar{C}_0)^{1/m}(t-1)^{-1/m}, \quad 1 \leq t < T. \tag{4.6}$$

Using the boundedness (4.1) we can rewrite (4.6) as

$$\|u(t)\|_\infty \leq C(m^{-1}\bar{C}_0)^{1/m} t^{-1/m}, \quad 1 \leq t < T. \tag{4.7}$$

Next we consider the case $m = 0$. In this case we see for (4.2),

$$\frac{d}{dt}\|u(t)\|^p_{\Omega(t),p} + C^{-1}\bar{C}_0^{-\nu}\|\nabla |u|^{p/2}\|^2_{\Omega(t),2} \le 0. \tag{4.8}$$

Since $\|u(t)\|_p \le C\|\nabla|u|^{p/2}\|_2^{2/p}$, we first see from (4.8)

$$\begin{aligned}\|u(t)\|_p &\le \|u(1)\|_p \exp\{-C^{-1}\bar{C}_0^{-\nu}p^{-1}(t-1)\}\\ &\le C_0\|u_0\|_r^{N/r}\exp\{-C^{-1}\bar{C}_0^{-\nu}p^{-1}(t-1)\}, \quad t \ge 1,\end{aligned}$$

and fixing p_1 arbitrarily, say $p_1 = 2$, we have

$$\|u(t)\|_{p_1} \le \eta_1 \exp(-\lambda_1(t-1)), \quad t \ge 1,$$

where we set

$$\eta_1 = C_0\|u_0\|_r^{N/r} \quad \text{and} \quad \lambda_1 = 1/C\bar{C}_0^{\nu}p_1.$$

Setting $p_n = 2p_{n-1}, n \ge 2$, we shall derive an estimate

$$\|u(t)\|_{p_n} \le \eta_n \exp\{-\lambda_n(t-1)\}, \quad t \ge 2, \tag{4.9}$$

under the assumption

$$\|u(t)\|_{p_{n-1}} \le \eta_{n-1}\exp\{-\lambda_{n-1}(t-1)\}, \quad t \ge 2.$$

We have, instead of (4.5),

$$\frac{d}{dt}\|u(t)\|^{p_n}_{p_n} + C^{-1}\bar{C}_0^{-\nu}\eta_{n-1}^{-(1-\theta_n)p_n/\theta_n}\exp\{(1-\theta_n)p_n\lambda_{n-1}(t-1)/\theta_n\}\|u(t)\|_{p_n}^{p_n/\theta_n} \le 0 \tag{4.10}$$

with $\theta_n = \theta = N/(N+2)$. Now we prepare a simple Lemma.

Lemma 4.1. *Let $y(t)$ be a nonnegative differentiable function on $[1, T)$ with a large T, satisfying*

$$\frac{d}{dt}y(t) + Ae^{\bar{\lambda}(t-1)}y^{1+\mu} \le 0, \quad 1 \le t < T, \tag{4.11}$$

with $A > 0, \bar{\lambda} > 0$ and $\mu > 0$. Then

$$y(t) \le \min\left\{y(1), \left(\frac{1}{\bar{\epsilon}A\mu}\right)^{1/\mu} e^{-\bar{\lambda}(t-1)/\mu}\right\}, \quad 2 \le t < T,$$

where we set $\bar{\epsilon} = (1 - e^{-\bar{\lambda}})/\bar{\lambda} > 0$.

Proof. Solving (4.11) we see

$$\begin{aligned}y(t) &\le \left(y^{-\mu}(1) + \frac{A\mu}{\bar{\lambda}}(e^{\bar{\lambda}(t-1)} - 1)\right)^{-1/\mu}\\ &\le \min\left\{y(1), \left(\frac{1}{A\mu}\right)^{1/\mu}\left(\frac{e^{\bar{\lambda}(t-1)} - 1}{\bar{\lambda}}\right)^{-1/\mu}\right\}, \quad 1 \le t < T.\end{aligned}$$

Here, if $t \geq 2$ we see further

$$\frac{e^{\bar{\lambda}(t-1)}-1}{\bar{\lambda}} = e^{\bar{\lambda}(t-1)}\frac{1-e^{-\bar{\lambda}(t-1)}}{\bar{\lambda}} \geq e^{\bar{\lambda}(t-1)}\frac{1-e^{-\bar{\lambda}}}{\bar{\lambda}}.$$

Thus we obtain the desired result. □

Applying Lemma 4.1 to (4.10) we have

$$\|u(t)\|_{p_n} \leq \left(\frac{C\bar{C}_0^{\nu}p_n\lambda_{n-1}}{1-e^{-(1-\theta)p_n\lambda_{n-1}/\theta}}\right)^{\theta/(1-\theta)p_n} \eta_{n-1}e^{-\lambda_n(t-1)}$$

with

$$\lambda_n = \lambda_{n-1} = \lambda_1 = \epsilon\tilde{C}_0^{\nu}/2.$$

We can arrange the above as

$$\|u(t)\|_{p_n} \leq (C_0p_n)^{N/2p_n}\eta_{n-1}e^{-\lambda_n(t-1)}$$

and (4.9) has been derived by taking

$$\eta_n = (C_0p_n)^{N/2p_n}\eta_{n-1} \quad \text{and} \quad \lambda_n = \lambda_1 = \epsilon\tilde{C}_0^{\nu}/2. \tag{4.12}$$

It is easy to see from (4.12) that

$$\eta_n \leq C_0\eta_1 = C_0\|u_0\|_r^{N/r}$$

and hence,

$$\|u(t)\|_\infty \leq C_0\|u_0\|_r^{N/r}e^{-\lambda(t-1)}, \quad 2 \leq t < T, \tag{4.13}$$

with $\lambda = \epsilon_0\bar{C}_0^{\nu}/2$. By changing C_0, above (4.13) shows

$$\|u(t)\|_\infty \leq C_0\|u_0\|_r^{N/r}e^{-\lambda t}, \quad 1 \leq t < T. \tag{4.14}$$

We summarize the above argument.

Proposition 4.2. *Let $u(t)$ be a classical solution of the problem* (1.1)–(1.2)*. Then, under Hypothesis $A(2)$ we have the decay estimate* (4.7) *if $m > 0$ and* (4.14) *if $m = 0$.*

5. ESTIMATE FOR $\|\nabla\sigma(u)\|_{\Omega(t),2}, 0 < t \leq 1$

For a function $u(t)$ on $Q(0,T)$ we set

$$\Gamma(t) = \frac{1}{2}\int_{\Omega(t)} |\nabla\sigma(u)|^2 dx.$$

In this section we derive an estimate of $\Gamma(t), 0 < t \leq 1$, depending on $\|u_0\|_r$ for a classical solution $u(t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q(0,T)})$. Combinning the ideas in [6] and [8] we employ $\partial\sigma(u)/\partial t - \beta \cdot \nabla\sigma(u)$ as a multiplyer.

We see that for $0 \le t < t+h < T$,

$$\begin{aligned}
&-\int_{Q(t,t+h)} \Delta\sigma(u)\frac{\partial}{\partial t}\sigma(u)dV \\
&= \frac{1}{2}\int_{Q(t,t+h)} \frac{\partial}{\partial t}|\nabla\sigma(u)|^2 dV - \int_{S(t,t+h)} n_x \cdot \nabla\sigma(u)\frac{\partial}{\partial t}\sigma(u))dS \qquad (5.1)\\
&= \Gamma(t+h) - \Gamma(t) + \frac{1}{2}\int_{S(t,t+h)} n_t|\nabla\sigma(u)|^2 dS - \int_{S(t,t+h)} n_x \cdot \nabla\sigma(u)\frac{\partial}{\partial t}\sigma(u)dS
\end{aligned}$$

$$= \Gamma(t+h) - \Gamma(t) - \frac{1}{2}\int_{S(t,t+h)} n_t|n_x|^2\Big|\frac{\partial}{\partial n}\sigma(u)\Big|^2 dS. \qquad (5.2)$$

The derivation of (5.1) is rather formal because the existence of $\partial|\nabla u|^2/\partial t$ (and hence $\frac{\partial}{\partial t}|\nabla\sigma(u)|^2$) is doubtful, but it is justified through appropriate smooth approximations of $u(t)$. Indeed, we take a ball $B(T)$ in R^N such that $\bigcup_{0\le t<T}\overline{\Omega(t)} \subset B(T)$ and extend $u(x,t)$ to a function $\tilde{u}(x,t) \in C^{2,1}(B(T)\times(-\delta,T)$, $\delta > 0$. Next, we take a mollifier $\rho_\epsilon(t)$, $0 < \epsilon << 1$, with respect to t and set

$$u_\epsilon(t) = \tilde{u} * \rho_\epsilon(t), \quad 0 \le t < t+h < T-\epsilon.$$

Then (5.1) is valid if we replace $u(t)$ by $u_\epsilon(t)$. Taking the limt as $\epsilon \to 0$ we get (5.1) for $u(t)$, and consequently (5.2) holds.

Next,

$$\begin{aligned}
&-\int_{Q(t,t+h)} \Delta\sigma(u)\beta\cdot\nabla\sigma(u)dV \\
&= \int_{Q(t,t+h)} \nabla\sigma(u)\cdot\beta\Delta\sigma(u)dV + \int_{Q(t,t+h)} \nabla\sigma(u)\cdot\beta'(t)\nabla\sigma(u)dV \\
&\quad - \int_{S(t,t+h)} n_x\cdot\beta|n_x|^2\frac{\partial}{\partial n}|\sigma(u)|^2 dS,
\end{aligned}$$

which implies

$$\begin{aligned}
&-\int_{Q(t,t+h)} \Delta\sigma(u)\beta\cdot\nabla\sigma(u)dV \\
&= \frac{1}{2}\int_{Q(t,t+h)} \nabla\sigma(u)\cdot\beta'(t)\nabla\sigma(u)dV - \frac{1}{2}\int_{S(t,t+h)} n_x\cdot\beta|n_x|^2\Big|\frac{\partial}{\partial n}\sigma(u)\Big|^2 dS.
\end{aligned} \qquad (5.3)$$

Therefore we have from (5.2) and (5.3),

$$
\begin{aligned}
&-\int\limits_{Q(t,t+h)} \Delta\sigma(u))\left(\frac{\partial}{\partial t}\sigma(u)+\beta\cdot\nabla\sigma(u)\right)dV\\
&=\Gamma(t+h)-\Gamma(t)+\frac{1}{2}\int\limits_{Q(t,t+h)} \nabla\sigma(u)\cdot\beta'\nabla\sigma(u)dV+A,
\end{aligned}
\tag{5.4}
$$

where A is the boundary integral

$$
A=-\frac{1}{2}\int\limits_{S(t,t+h)}(n_t+\beta\cdot n_x)|n_x|^2\left|\frac{\partial}{\partial n}\sigma(u)\right|^2 dS=0.
$$

By (5.4), we see that $\Gamma(t)$ is differentiable and

$$
\begin{aligned}
\frac{d}{dt}\Gamma(t)=-\int\limits_{\Omega(t)} &\Delta\sigma(u)\left(\frac{\partial}{\partial t}\sigma(u)+\beta\cdot\nabla\sigma(u)\right)dx\\
&-\frac{1}{2}\int\limits_{\Omega(t)}\nabla\sigma(u)\beta'\nabla\sigma(u)dx.
\end{aligned}
\tag{5.5}
$$

We see also

$$
\int\limits_{\Omega(t)} u_t\left(\frac{\partial}{\partial t}\sigma(u)+\beta\cdot\nabla\sigma(u)\right)dx=\int\limits_{\Omega(t)}(\sigma'(u)|u_t|^2+u_t\sigma'(u)\beta\cdot\nabla u)dx. \tag{5.6}
$$

It follows from (5.5) and (5.6)

$$
\begin{aligned}
\frac{d}{dt}\Gamma(t)+\int\limits_{\Omega(t)}\sigma'(u)|u_t|^2dx&=-\int\limits_{\Omega(t)}u_t\beta\cdot\nabla\sigma(u))dx-\frac{1}{2}\int\limits_{\Omega(t)}\nabla\sigma(u)\beta'\nabla\sigma(u)dx\\
&\le\frac{1}{2}\int\limits_{\Omega(t)}|\beta'||\nabla\sigma(u)|^2+\int\limits_{\Omega(t)}|u_t||\beta|\sigma'(u)|\nabla u|dx.
\end{aligned}
\tag{5.7}
$$

The last term in (5.7) is treated as follows.

$$
\int\limits_{\Omega(t)}|u_t||\beta|\sigma'(u)|\nabla u|dx\le\frac{1}{2}\int\limits_{\Omega(t)}|u_t|^2\sigma'(u)dx+\frac{1}{2}\int\limits_{\Omega(t)}|\beta|^2\sigma'(u)|\nabla u|^2dx.
$$

Thus we can summarize the above argument as follows.

Proposition 5.1. *Let $u(t)$ be a classical solution of* (1.1)–(1.2)*. Then we have*

$$
\frac{d}{dt}\Gamma(t)+\frac{1}{2}\int\limits_{\Omega(t)}\sigma'(u)|u_t|^2dx\le\bar{\delta}(t)\Gamma(t)+\frac{1}{2}\bar{\delta}^2(t)\int\limits_{\Omega(t)}\sigma'(u)|\nabla u|^2dx. \tag{5.8}
$$

To control the last term in (5.8) we multiply the equation (1.1) by u to get

$$\begin{aligned}\bar{\delta}^2(t)\int_{\Omega(t)}\sigma'(u)|\nabla u|^2dx &= -\bar{\delta}^2(t)\int_{\Omega(t)}u_t u dx\\ &\le \bar{\delta}^2(t)\int_{\Omega(t)}\sqrt{\sigma'(u)}|u_t|\frac{|u|}{\sqrt{\sigma'(u)}}dx\\ &\le \frac{1}{2}\int_{\Omega(t)}\sigma'(u)|u_t|^2dx+\frac{1}{2}\bar{\delta}^4(t)\int_{\Omega(t)}\frac{|u|^2}{\sigma'(u)}dx.\end{aligned}\tag{5.9}$$

To estimate the second term of the right-hand side of (5.9) we assume $m \le 2$ in Hypothesis A(2). (This is unnecessary if $r(t) = 0$, that is, $\Omega(t) = \Omega$, independent of t.) Then by Hypothesis A(2) (with $K = 1$) and (4),

$$\begin{aligned}&\int_{\Omega(t)}\frac{|u|^2}{\sigma'(u)}dx \le \int_{\Omega_1(t)}\frac{|u|^2}{\sigma'(u)}dx+\int_{\Omega_2(t)}\frac{|u|^2}{\sigma'(u)}dx\\ &\le C|\Omega(t)|+C\int_{\Omega(t)}|\sigma(u)|^2dx \le C(T)+C(T)\Gamma(t),\end{aligned}\tag{5.10}$$

where we set

$$\Omega_1(t)=\{x\in\Omega(t)||u(x,t)|\le 1\}\quad\text{and}\quad \Omega_2(t)=\{x\in\Omega(t)||u(x,t)|\ge 1\}.$$

It follows from (5.8), (5.9) and (5.10) that

$$\frac{d}{dt}\Gamma(t)+\frac{1}{4}\int_{\Omega(t)}\sigma'(u)|u_t|^2dx \le C(T)(\bar{\delta}(t)\Gamma(t)+\bar{\delta}^4(t))\tag{5.11}$$

Next, multiplying the equation by $\sigma(u)$ we see

$$\int_{\Omega(t)}|\nabla\sigma(u)|^2dx = -\int_{\Omega(t)}u_t\sigma(u)dx \le \left(\int_{\Omega(t)}|u_t|^2\sigma'(u)dx\right)^{1/2}\left(\int_{\Omega(t)}\frac{\sigma^2}{\sigma'}\right)^{1/2}.\tag{5.12}$$

We shall derive a bound of the last integral term in (5.12) such as $C(\|u_0\|_r)\Gamma(t)^\kappa$, $0 \le \kappa < 1$. By Hypothesis A(1), we have

$$\begin{aligned}\int_{\Omega(t)}\frac{\sigma^2(u)}{\sigma'(u)}dx &\le C\int_{\Omega(t)}|u||\sigma(u)|dx\\ &\le C\left(\int_{\Omega_1(t)}|u||\sigma(u)|dx+\int_{\Omega_2(t)}|u||\sigma(u)|dx\right).\end{aligned}\tag{5.13}$$

Thus if $r \geq L+2$, we see by Hypothesis A(3) that

$$\int_{\Omega(t)} |u\sigma(u)|dx \leq C\left(\int_{\Omega_1(t)} |u|^2 dx + C(T)\int_{\Omega_2(t)} |u|^{L+2}dx\right)$$
$$\leq C(T)(\|u_0\|_r^2 + \|u_0\|_r^{L+2}).$$

When $r < L+2$, we divide two cases: $r \geq 2N/(N+2)$ and $1 \leq r < 2N/(N+2)$. Consider the case $2N/(N+2) \leq r < L+2$.

We take $\theta, 0 \leq \theta \leq 1$, which will be specified later. Then by Hypothesis A(3),

$$\begin{aligned}
&\int_{\Omega_2(t)} |u\sigma(u)|dx = \int_{\Omega_2(t)} |u\|\sigma(u)|^{1-\theta}|\sigma(u)|^{\theta} \\
&\leq C\int_{\Omega_2(t)} |u|^{1+(L+1)(1-\theta)}|\sigma(u)|^{\theta}dx \\
&\leq C\left(\int_{\Omega(t)} |u|^q dx\right)^{1-\theta(N-2)^+/2N}\left(\int_{\Omega(t)} |\sigma(u)|^{2N/(N-2)^+}dx\right)^{\theta(N-2)^+/2N},
\end{aligned} \tag{5.14}$$

where we set
$$q = \frac{2N(1+(L+1)(1-\theta))}{2N-\theta(N-2)^+}.$$

(The cases $N = 1, 2$ should be modified.)

We can take $q = r$ if we choose

$$\theta = \frac{L+2-r}{L+1-(N-2)^+r/2N},$$

and by (5.14) we have

$$\int_{\Omega_2(t)} |u\sigma(u)|dx \leq C\|u_0\|_r^{1+(L+1)(1-\theta)}\|\nabla\sigma(u)\|_{\Omega(t),2}^{\theta}.$$

Similarly, with the same θ,

$$\begin{aligned}
\int_{\Omega_1(t)} |u\sigma(u)|dx &= \int_{\Omega_1(t)} |u\sigma(u)|^{1-\theta}|\sigma(u)|^{\theta}dx \leq C\int_{\Omega_1(t)} |u|^{2-\theta}|\sigma(u)|^{\theta}dx \\
&\leq C\|u_0\|_r^{2-\theta}\|\nabla\sigma(u(t))\|_2^{\theta}.
\end{aligned} \tag{5.15}$$

When $N = 1, 2$ we should replace

$$\left(\int_{\Omega(t)} |\sigma(u)|^{2N/(N-2)^+}dx\right)^{\theta(N-2)^+/2N)}$$

in (5.14) by $\|\sigma(u)\|^{\theta}_{\Omega(t),\infty}$ and $\|\sigma(u)\|^{\theta}_{\Omega(t),p}$ for sufficiently large p, respectively. In the case $N = 2$ the right-handside of (5.15) should be replaced by

$$C\|u_0\|_r^{1+(L+1)(1-\theta-\epsilon)}\|\nabla\sigma(u)\|^{\theta+\epsilon}_{\Omega(t),2}.$$

Next, we consider the case $1 \le r < 2N/(N+2)$. We see first by Hypothesis A(3),

$$\begin{aligned}\int\limits_{\Omega_2(t)} u\sigma(u)dx &= \int\limits_{\Omega_2(t)} |u|^{1-\tilde{\theta}}|u|^{(l+1)\tilde{\theta}/(l+1)}|\sigma(u)|dx \\ &\le C\int\limits_{\Omega_2(t)} |u|^{1-\tilde{\theta}}|\sigma(u)|^{1+\tilde{\theta}/(l+1)}dx \\ &\le C\left(\int\limits_{\Omega_2(t)} |u|^{\tilde{q}}dx\right)^{1-(N-2)^+(l+1+\tilde{\theta})/2N(l+1)} \\ &\quad\times\left(\int\limits_{\Omega_2(t)} |\sigma(u)|^{2N/(N-2)^+}\right)^{(N-2)(l+1+\tilde{\theta})/2N(l+1)}\end{aligned} \tag{5.16}$$

for $\tilde{\theta}$, $0 \le \tilde{\theta} \le 1$, where we set

$$\tilde{q} = \frac{1-\tilde{\theta}}{1-(N-2)(l+1+\tilde{\theta})/2N(l+1)}.$$

We take $\tilde{\theta}$ such that $\tilde{q} = r$, that is,

$$\tilde{\theta} = \frac{(l+1)(1-(N+2)r/2N}{l+1-(N-2)r/2N}.$$

Then we have from (5.16)

$$\int\limits_{\Omega_2(t)} u\sigma(u)dx \le C\|u_0\|^{1-\tilde{\theta}}_{\Omega(0),r}\|\nabla\sigma(u)\|^{1+\tilde{\theta}/(l+1)}_{\Omega(t),2}.$$

Similarly, by use of Hypothesis A(2) with $K = 1$,

$$\begin{aligned}\int\limits_{\Omega_1(t)} |u\sigma(u)|dx &= \int\limits_{\Omega_1(t)} |u|^{1-(m+1)\tilde{\theta}/(l+1)}(|u|^{m+1})^{\tilde{\theta}/(l+1)}|\sigma(u)|dx \\ &\le C \int\limits_{\Omega_1(t)} |u|^{1-(m+1)\tilde{\theta}/(l+1)}|\sigma(u)|^{1+\tilde{\theta}/(l+1)}dx \\ &\le C \left(\int\limits_{\Omega_1(t)} |u|^{\hat{q}}dx \right)^{1-(N-2)^+(l+1+\tilde{\theta})/(l+1)} \\ &\quad \times \left(\int\limits_{\Omega(t)} |\sigma(u)^{2N/(N-2)^+}dx \right)^{(N-2)^+(l+1+\tilde{\theta})/2N(l+1} ,\end{aligned} \tag{5.17}$$

where we assume $l + 1 \ge (m + 1)\tilde{\theta}$ and set

$$\hat{q} = \frac{1 - (m + 1)\tilde{\theta}/(l + 1)}{1 - (N - 2)^+(l + 1 + \tilde{\theta})/2N(l + 1)}.$$

Since $m \ge l$, we see that $\hat{q} \le \tilde{q} = r$ and we have from (5.17)

$$\int\limits_{\Omega_1(t)} u\sigma(u)dx \le C\|u_0\|_{\Omega(0),r}^{1-(m+1)\tilde{\theta}/(l+1)} \|\nabla\sigma(u)\|_{\Omega(t),2}^{1+\tilde{\theta}/(l+1)}.$$

The assumption $l + 1 \ge (m + 1)\tilde{\theta}$ above becomes

$$m(1 - (N + 2)r/2N) \le l + 2r/N.$$

To summarize the above arguments about the integral $\int_{\Omega(t)} u\sigma(u)dx$ we set A_0 and η as in (2.4) and (2.5), respectively.

Then we obtain the estimate

$$\int\limits_{\Omega(t)} u\sigma(u)dx \le CA_0\Gamma(t)^{\eta/2}. \tag{5.18}$$

Now we return to (5.12). By (5.11), (5.13) and (5.18), we have

$$\Gamma(t) \le C \left(C\bar{\delta}(t)\Gamma(t) + C\bar{\delta}^4(t) - \frac{d}{dt}\Gamma(t) \right)^{1/2} A_0^{1/2}\Gamma(t)^{\eta/4}, \quad 0 < t \le 1,$$

and hence,

$$\frac{d}{dt}\Gamma(t) + C^{-1}A_0^{-1}\Gamma(t)^{2-\eta/2} \le C(\bar{\delta}_+(1)\Gamma(t) + \bar{\delta}_+^4(1)), \quad 0 < t \le 1. \tag{5.19}$$

We use the following lemma due to Ohara [12] which is a generalization of Lemma 3.2.

Lemma 5.2. *Let $y(t)$ be a nonnegative differentiable function on $(0,T)$ satisfying*

$$\frac{d}{dt}y(t) + At^{\tilde{\lambda}\mu-1}y^{1+\mu} \leq By(t) + C, \quad 0 < t < T,$$

with constants $A > 0$, $B \geq 0$, $C \geq 0$, $\mu > 0$ and $\tilde{\lambda}\mu \geq 1$. Then

$$y(t) \leq \left(\frac{2\tilde{\lambda} + 2BT}{A}\right)^{1/\mu} t^{-\tilde{\lambda}} + \frac{2Ct}{\tilde{\lambda} + BT}, \quad 0 < t < T.$$

Applying this lemma to (5.19) we arrive at the following assertion.

Proposition 5.3. *Let $u(t)$ be a classical solution of* (1.1)–(1.2) *with $u(0) = u_0 \in L^r(\Omega(0))$ with $r \geq 1$. We assume Hypothesis A*(1)–(4) *with $m \leq 2$ and in addition, if $1 \leq r < 2N/(N+2)$ we assume*

$$m(1 - (N+2)r/2N) \leq l + 2r/N.$$

Then

$$\Gamma(t) \leq C(A_0^{2/(2-\eta)} t^{-2/(2-\eta)} + \bar{\delta}_+^4(1)), \quad 0 < t \leq 1,$$

with a constant C depending on $r^+(1)$ continuously, where A_0 and η are given by (2.4) *and* (2.5)*, respectively.*

The estimate of $\Gamma(t), 1 \leq t < T$, depending on T is easy. Indeed, we return to (5.11) to get

$$\Gamma(t) \leq C(T)(\Gamma(1) + \bar{\delta}_+^4(T)) \leq C(T)\left(A_0^{2/(2-\eta)} + \bar{\delta}_+^4(T)\right), \quad 1 \leq t < T.$$

Finally, in this section we note that from (5.11) and the estimates for $\Gamma(t)$ we see

$$\begin{aligned} &\int_\delta^T \int_{\Omega(t)} \left|\frac{\partial}{\partial t}\sigma(u(t))\right|^2 dxdt \\ &\leq 2 \sup_{\delta \leq t < T} \|\sigma'(u(x,t))\|_{\Omega(t),\infty} \int_\delta^T \int_{\Omega(t)} |\sigma'(u(t))||u_t|^2 dxdt \leq C_0(\delta, T) < \infty, \end{aligned} \tag{5.20}$$

with $0 < \delta << 1$.

Remark 5.4. The argument in this section is valid even for the case $-1 < m < 0$ and $-1 \leq l \leq m \leq L$ when $\Omega(t)$ is independent of t, though some trivial modifications are required.

6. BOUNDEDNESS AND DECAY OF $\|\nabla\sigma(u(t))\|_{\Omega(t),2}$

In this section we show boundedness and decay estimates of an assumed classical solution $u(t)$ for large t. We assume

$$|\Omega(t)| \le k_3 < \infty \quad \text{and} \quad \bar{\delta}^+(T) \le \bar{\delta}_0 < \infty, \quad 0 \le t < T. \tag{6.1}$$

We first derive a boundedness estimate of $\Gamma(t)$, $1 \le t < T$, independent of T under (6.1).

We return to the inequalities (5.11) and (5.12). We see, by Hypothesis A(1)–(2) and the estimate (4.1),

$$|\sigma(u(t))| \ge k_1^{-1}\sigma'(u(t))|u(t)| \ge C_0^{-1}\|u_0\|^{-2N\nu/(2r+mN)}|u|^{m+1}, \quad 1 \le t < T,$$

and

$$\begin{aligned}\int\limits_{\Omega(t)} \frac{\sigma^2(u)}{\sigma'(u)}dx &\le k_0 \int\limits_{\Omega(t)} |u\sigma(u)|dx \le \tilde{A}_0 \int\limits_{\Omega(t)} |\sigma(u)|^{(m+2)/(m+1)}dx \\ &\le \tilde{A}_0\|\nabla\sigma(u)\|_{\Omega(t),2}^{(m+2)/(m+1)}, \quad 1 \le t < T,\end{aligned} \tag{6.2}$$

where we set

$$\tilde{A}_0 = C_0\|u_0\|^{2N\nu/(m+1)(2r+mN)}.$$

It follows from (5.11), (5.12) and (6.2)

$$\Gamma(t) \le \tilde{A}_0^{1/2}\left(C(\bar{\delta}(t)\Gamma(t) + \bar{\delta}^4(t)) - \frac{d}{dt}\Gamma(t)\right)^{1/2}\Gamma(t)^{(m+2)/4(m+1)}$$

and

$$\frac{d}{dt}\Gamma(t) + \tilde{A}_0^{-1}\Gamma(t)^{(3m+2)/2(m+1)} \le C(\bar{\delta}(t)\Gamma(t) + \bar{\delta}^4(t)), \quad 1 \le 1 < T. \tag{6.3}$$

When $m > 0$ we take $M_0 > 0$ such that

$$\tilde{A}_0^{-1}M_0^{(3m+2)/2(m+1)} - C(\bar{\delta}_0 M_0 + \bar{\delta}_0^4) > 0,$$

say,

$$M_0 = (2C\tilde{A}_0\bar{\delta}_0)^{2(m+1)/m} + (2C\tilde{A}_0\bar{\delta}_0^4)^{2(m+1)/(3m+2)}.$$

Then, if $\Gamma(t_0) > M_0$ for some $t = t_0$ we have $d\Gamma(t)/dt < 0$ at $t = t_0$. This implies that

$$\Gamma(t) \le \max\{\Gamma(1), M_0\} \le M_0 + C(\tilde{A}_0^{2/(2-\eta)} + \bar{\delta}_0^4), \quad 1 \le t < T,$$

which shows the boundedness of $\Gamma(t)$, $1 \le t < T$, independent of T.

When $m = 0$ we have from (6.3)

$$\frac{d}{dt}\Gamma(t) + C_0^{-1}\|u_0\|^{-N\nu/r}\Gamma(t) \le C(\bar{\delta}_0\Gamma(t) + \bar{\delta}_0^4), \quad 1 \le 1 < T.$$

Thus if $C\bar{\delta}_0 \le C_0^{-1}\|u_0\|^{-N\nu/r}/2$, that is, if

$$2CC_0\bar{\delta}_0\|u_0\|^{N\nu/2r} \le 1 \tag{6.4}$$

we see

$$\frac{d}{dt}\Gamma(t) + (2C_0)^{-1}\|u_0\|^{-N\nu/r}\Gamma(t) \le C\bar{\delta}_0^4$$

and hence,

$$\Gamma(t) \le \max\{\Gamma(1), CC_0\|u_0\|_r^{N\nu/r}\bar{\delta}_0^4\} \le \max\{\Gamma(1), C\bar{\delta}_0^3\}, \quad 1 \le t < T. \tag{6.5}$$

We proceed to the decay estimate for $\Gamma(t)$. For this we assume

$$\bar{\delta}(t) \le \bar{\delta}_0 t^{-(3m+2)/4m}, \quad 1 \le t < T, \tag{6.6}$$

where the case $m > 0$ is now considered.

We set $y(t) = \Gamma(t)t^{2(m+1)/m}$. Then (6.3) is changed to

$$\begin{aligned} &\frac{d}{dt}y(t) + t^{-1}\left(\tilde{A}_0^{-1}y(t)^{(3m+2)/2(m+1)} - \frac{2(m+1)}{m}y(t)\right) \\ &\le C(\bar{\delta}(t)y(t) + \bar{\delta}^4(t)t^{2(m+1)/m}), \quad 1 \le t < T. \end{aligned} \tag{6.7}$$

Then we have from (6.6) and (6.7)

$$\begin{aligned} &\frac{d}{dt}y(t) + t^{-1}\left(\tilde{A}_0^{-1}y(t)^{(3m+2)/2(m+1)} - \frac{2(m+1)}{m}y(t) - C\bar{\delta}_0 y(t) - C\bar{\delta}_0^4\right) \le 0, \\ &1 \le t < T. \end{aligned} \tag{6.8}$$

We take M_1 such that

$$\tilde{A}_0^{-1}M_1^{(3m+2)/2(m+1)} - \left(\frac{2(m+1)}{m} + C\bar{\delta}_0\right)M_1 - C\bar{\delta}_0^4 > 0,$$

say,

$$M_1 = \left(2\tilde{A}_0\bar{\delta}_0((m+1)/m + \bar{\delta}_0)\right)^{2(m+1)/m} + (C\tilde{A}_0\bar{\delta}_0^4)^{2(m+1)/(3m+2)}.$$

Then (6.8) gives

$$y(t) \le \max\{y(1), M_1\} \le M_1 + C(A_0^{2/(2-\eta)} + \bar{\delta}_+(1)^4) \equiv \tilde{M}_1$$

and hence,

$$\Gamma(t) \le \tilde{M}_1 t^{-2(m+1)/m}, \quad 1 \le t < T. \tag{6.9}$$

When $m = 0$ we see from (6.3) that

$$\frac{d}{dt}\Gamma(t) + (C_0\|u_0\|_r^{N\nu/r})^{-1}\Gamma(t) \le C(\bar{\delta}(t)\Gamma(t) + \bar{\delta}^4(t)), \quad 1 \le t < T.$$

Therefore under the assumption

$$\bar{\delta}(t) \le \bar{\delta}_0 e^{-\lambda t}, \quad 1 \le t < T, \tag{6.10}$$

with $\lambda > 0$ and under (6.5), we have

$$\frac{d}{dt}\Gamma(t) + (2C_0\|u_0\|_r)^{-N\nu/r}\Gamma(t) \le \bar{\delta}_0^4 e^{-4\lambda t},$$

which implies

$$\Gamma(t) \le C(\Gamma(1) + \bar{\delta}_0^4)e^{-2\lambda_0 t} \tag{6.11}$$

with some $\lambda_0 > 0$ such that

$$\lambda_0 \le \min\{(2C_0\|u_0\|_r^{N\nu/r}\bar{\delta}_0)^{-1}, 4\lambda\}.$$

We summarize the result in this section.

Proposition 6.1. *In addition to the assumptions in Proposition 5.1 we assume* (6.1)*. Then we have the boundedness estimate* (6.5) *for* $\Gamma(t)$*. Further, under the condition* (6.6) *we have the decay estimate* (6.9)*. When* $m = 0$ *we have the boundeness estimates* (6.5) *under the conditions* (6.1) *and* (6.5)*, and the decay estimate* (6.11) *holds under the assumption* (6.10)*.*

7. EXISTENCE AND THE PROOFS OF THEOREMS 2.3 AND 2.4

Let $0 < \epsilon << 1$ and we set

$$\sigma_\epsilon(u) = \frac{\sigma(\sqrt{u^2+\epsilon})u}{\sqrt{u^2+\epsilon}}. \tag{7.1}$$

(If $\sigma(u) = u\log(1+u^2)$, then $\sigma_\epsilon(u) = u\log(1+\epsilon+u^2)$.)

We see that $\sigma_\epsilon(u) \to \sigma(u)$ as $\epsilon \to 0$ and

$$\sigma'_\epsilon(u) = \frac{\epsilon\sigma(\sqrt{u^2+\epsilon})}{(u^2+\epsilon)^{3/2}} + \frac{u^2\sigma'(\sqrt{u^2+\epsilon})}{u^2+\epsilon}.$$

Using this it is easy to check that $\sigma_\epsilon(u)$ satisfies all of the conditions in Hypothesis A with essentially the same k_0, k_1. We consider the modified problem (1.1)-(1.2) with $\sigma(u)$ replaced by $\sigma_\epsilon(u)$ which we call problem P_ϵ.

We first assume $u_0 \in C_0^3(\Omega(0))$. Since $\sigma'_\epsilon(u)$ belongs to $C^{1+\alpha}(R)$ and $\sigma'_\epsilon(u) \ge k(\epsilon) > 0$ by Hypothesis A(1) and A(3) the problem P_ϵ admits a unique classical solution $u_\epsilon(t) \in C^{2+\alpha,1+\alpha/2}$ (see the remark after Theorem 12.14 in [7]). We discuss convergency properties of $u_\epsilon(x,t)$as $\epsilon \to 0$ along a subsequence, and for a sequence $\{\epsilon_n\}$ we write $u_n(x,t)$ for $u_\epsilon(x,t)$ with $\epsilon = \epsilon_n$. All of the estimates established so far are applied to $u_n(x,t)$ and they are independents of ϵ_n. We can take a ball $B(T)$ in R^N such that $\bigcup_{0\le t<T}\Omega(t) \subset B(T)$. Also we take an extended function $\tilde{u}_n(x,t)$ on $B(T)\times[0,T)$ such that

$$\tilde{u}_n(x,t) = \begin{cases} u_n(x,t) & \text{if } x \in \Omega(t), \\ 0 & \text{if } x \notin \Omega(t). \end{cases}$$

Since $u_n(x,t)=0$ for $x\in\partial\Omega(t)$, we note that $\tilde{u}_{n,t},\nabla\tilde{u}_n\in L^\infty((B(T)\times[0,T))$ and $\tilde{u}_{n,t}(x,t)=\nabla\tilde{u}_{n,t}(x,t)=0$ for $x\notin\Omega(t)$. By Hypothesis A(2), (3.19) and (4.1) we see

$$\int_{\bar\delta}^{T}\int_{B(T)}\left(\left|\frac{\partial}{\partial t}(|\tilde{u}_n|^m\tilde{u}_n)\right|^2+|\nabla(|\tilde{u}_n|^m\tilde{u}_n)|^2\right)dxds$$
$$\le C_0(\bar\delta,T)\int_{\bar\delta}^{T}\int_{\Omega(s)}\left(|\sigma_\epsilon'(u_n)||u_{nt}|^2+|\nabla\sigma_\epsilon(u_n)|^2\right)dxds\le C_0(\delta,T)<\infty,\quad 0<\delta<<1. \tag{7.2}$$

Hence there exist a sequence $\{\epsilon_n\}$ and a function $\tilde{u}(x,t)$ on $B(T)\times[0,T)$ such that

$$\tilde{u}_n(x,t)\to\tilde{u}(x,t)\quad\text{a.e. } (x,t)\in B(T)\times[0,T),$$
$$\sigma_{\epsilon_n}(\tilde{u}_n)\to\sigma(\tilde{u})\quad\text{strongly in } L^2_{loc}((0,T);B(T)) \tag{7.3}$$

and

$$|\tilde{u}(t)|^m\tilde{u}(t)\in C_{loc}((0,T);L^2(B(T)))). \tag{7.4}$$

Note that (7.4) implies easily

$$\tilde{u}(t)\in C_{loc}((0,T);L^{2(m+1)}(B(T))).$$

Under the assumption $u_0\in C_0^3(\Omega(0))$ we see

$$\|\tilde{u}_n(t)\|_{B(T),\infty}\le C(\|u_0\|_\infty,T)$$

and

$$|\nabla\sigma(\tilde{u}_n(t))|^2_{B(T),2}\le C(\|u_0\|_\infty,\Gamma(0),T)<\infty.$$

Hence, at this stage we can take $\delta=0$ in (7.2) with $C_0(\delta,T)$ replaced by $C(\|u_0\|_\infty,\Gamma(0))$ and hence,

$$\tilde{u}(t)\in C([0,T);L^{2(m+1)}(B(T)))),\quad \tilde{u}(t)\in L^\infty([0,T);L^\infty(B(T))),$$
$$\|\nabla\tilde{u}(t)\|_{B(T),2}\le C(\|u_0\|_\infty,\Gamma(0),T)<\infty.$$

Setting $u(x,t)=\tilde{u}(x,t)$ for $x\in\Omega(t)$ all of the estmates in previous propositions and (5.20) with $\delta=0$ are valid for $u(x,t)$ and it is easy to see that the identities (2.1) hold.

For later use we give an addtional estimate. Let u_n and v_n be the approximate solutions with $u_n(0)=u_0\in C_0^3(\Omega(0))$ and $v_n(0)=v_0\in C_0^3(\Omega(0))$, respectively. Introducing the functions $U_n(t),V_n(t)\in C^2(\overline{\Omega(t)})$ such that

$$-\Delta U_n(t)=u_n(t)\quad\text{and}\quad -\Delta V_n=v_n(t)$$

with the boundary condition $U_n(t)|_{\partial\Omega(t)} = V_n(t)|_{\partial\Omega(t)} = 0$. Then by the equations for u_n and v_n we have

$$\int_0^t \int_{\Omega(s)} \frac{\partial}{\partial s}\Delta(U_n(s) - V_n(s))(U_n(s) - V_n(s))dxds$$
$$+ \int_0^t \int_{\Omega(s)} (\sigma_\epsilon(u_n) - \sigma_\epsilon(v_n))(u_n(s) - v_n(s))dxds = 0$$

and hence

$$\frac{1}{2}\int_{Q(0,t)} \frac{\partial}{\partial s}|\nabla(U_n(s) - V_n(s))|^2 dxds \leq 0,$$

which implies

$$\|\nabla(U_n(t) - V_n(t))\|^2_{\Omega(t),2} - \|(\nabla U_n(0) - V_n(0))\|^2_{\Omega(0),2}$$
$$+ \int_{S(0,t)} n_s|\nabla(U_n(s) - V_n(s)|^2 ds \leq 0.$$

Therefore, under the assumption $n_t \geq 0$ we have

$$\|u_n(t) - v_n(t)\|_{H^{-1}(\Omega(t))} \leq \|\nabla(U_n(t) - V_n(t)\|_{\Omega(t),2} \leq \|\nabla(U_0 - V_0)\|_{\Omega(0),2}$$
$$\leq C(T)\|u_0 - v_0\|_{H^{-1}(\Omega(0))}, \quad 0 < t < T,$$

and we may assume for the solutions $u(t), v(t)$ with $u(0) = u_0$, $v(0) = v_0$ just proved above that

$$\|u(t) - v(t)\|_{H^{-1}(\Omega(t))} \leq C(T)\|u_0 - v_0\|_{H^{-1}(\Omega(0)}, \quad 0 \leq t < T. \tag{7.5}$$

Next, we let $u_0 \in L^r(\Omega(0))$, $r \geq 1$, and take a sequence $u_{0,n} \in C_0^3(\Omega(0))$ such that $u_{0,n} \to u_0$ in $L^r(\Omega(0))$. We write $u_n(x,t)$ the Type 1 solution of (3.1)–(3.2) with $u_n(0) = u_{0,n}$. Then all of the estimates stated previous propositions hold for $u_n(x,t)$, and hence the convergency properties (7.3) and (7.4) hold for this $\{u_n(x,t)\}$. We see that for $\phi(t) \in C^1([0,T); C_0^2(\Omega(t))$,

$$\int_0^t \int_{\Omega(s)} \sigma(u_n(x,s)\Delta\phi(x,s)dxds$$
$$= \int_\delta^t \int_{\Omega(s)} \sigma(u_n(x,s))\Delta\phi(x,t)dxds + \int_0^\delta \int_{\Omega(s)} \sigma(u_n(x,s))\Delta\phi(x,s)dxds$$

for any δ, $0 < \delta < t$, and by the estimate $\|u_n(t)\|_{\Omega(t),\infty} \le C_0(\delta, T) < \infty$, $\delta \le t < T$, we have

$$\lim_{n\to\infty} \int_\delta^t \int_{\Omega(s)} \sigma(u_n(x,s))\Delta\phi(x,s)dxds = \int_\delta^t \int_{\Omega(s)} \sigma(u(x,s))\Delta\phi(x,s)dxds.$$

Further we note that

$$\begin{aligned}
&\int_0^\delta \int_{\Omega(s)} |\sigma(u_n(x,s))|dxds \\
&\le C \int_0^\delta \int_{\Omega(s)} (1 + |u_n(s)|^{L+1})dxds \le C \int_0^\delta (1 + \|u_n(s)\|_\infty^{L+1-\tilde{r}} \|u_n(0)\|_r^{\tilde{r}})ds \\
&\le C_0(T) \int_0^\delta s^{-(L+1-\tilde{r})N/(mN+2r-N\nu)}ds
\end{aligned}$$

with $\tilde{r} = \min\{L+1, r\}$, where we have used the estimate (3.20) stated in Proposition 3.3. Therefore, under the additional assumption $(L + 1 - \tilde{r})N/(mN + 2r - N\nu) < 1$, i.e., $L + 1 + \nu < m + (N + 2)r/N$, we see

$$\left(\int_0^\delta \int_{\Omega(s)} |\sigma(u(x,s))\Delta\phi(x,s)|dxds \le \right) \lim_{n\to\infty} \int_0^\delta \int_{\Omega(s)} |\sigma(u_n(x,s))\Delta\phi(x,t)|dxds \to 0$$

as $\delta \to 0$. Thus we know $\sigma(u) \in L^1((0,T); L^1(\Omega(t)))$ and

$$\lim_{n\to\infty} \int_0^t \int_{\Omega(s)} \sigma(u_n(x,s))\Delta\phi(x,s)dxds = \int_0^t \int_{\Omega(s)} \sigma(u(x,s))\Delta\phi(x,s)dxds,$$

and $u(x,t)$ is a desired Type 1 solution of the problem (1.1)–(1.2) with $u_0 \in L^r(\Omega(0))$.

We proceed to the existence of Type 2 solution when $r > 2N/(N + 2)$. By Proposition 5.3, we know that

$$\begin{aligned}
\int_0^t \int_{\Omega(s)} |\nabla\sigma(u(x,s))||\nabla\phi(x,t)|dxds &\le C \int_0^t \sqrt{\Gamma(s)}ds \int_0^T \|\nabla\phi(s)\|_2 ds \\
&\le C_0(T) \int_0^t s^{-1/(2-\eta)}ds < C_0(T) < \infty
\end{aligned}$$

because $\eta < 1$ if $r > 2N/(N+2)$. Hence,

$$\int_0^t \int_{\Omega(s)} \sigma(u(x,s))\Delta\phi(x,t)dxds = -\int_0^t \int_{\Omega(s)} \nabla\sigma(u(x,s))\nabla\phi(x,s)dxds.$$

Further we note that if $r > 2N/(N+2)$,

$$H_1^0(\Omega(t)) \subset L^{2N/(N-2)^+}(\Omega(t)) \subset L^{r/(r-1)}(\Omega(t)).$$

($L^{r/(r-1)}(\Omega(t)) = L^\infty(\Omega(t))$ if $r = 1$ or $N = 1$.) Then

$$|(u(t), \phi(t))| \leq \|u(t)\|_r \|\phi(t)\|_{r/(r-1)} \leq C(T)\|u_0\|_r \|\phi(t)\|_{H_0^1(\Omega(t))}.$$

Therefore, by standard density argument we see that the identity (2.2) hold for any $\phi \in C([0,T]; H_0^1(\Omega(t))) \cap C^1([0,T]; L^2(\Omega(t)))$. Thus $u(t)$ is a Type 2 solution.

Finally in this section we show that the Type 2 solution $u(t)$ above belongs to $C([0,T]; H^{-1}(\Omega(t)))$ if $n_t \geq 0$. Indeed, take a sequence $u_{n,0} \to u_0$ in $L^r(\Omega(0)))$ and let $u_n(t)$ be the solution shown above with $u_n(0) = u_{n,0}$. By the estimate (7.5), we have

$$\begin{aligned} \|u_m(t) - u_n(t)\|_{H^{-1}(\Omega(t))} &\leq C(T)\|u_{m,0} - u_{n,0}\|_{H^{-1}(\Omega(0)} \\ &\leq C(T)\|u_{0,m} - u_{0,n}\|_r, \quad 0 \leq t < T. \end{aligned}$$

Further we know $u_n(t) \in C([0,T]; L^{2(m+1)}(\Omega(t))) \subset C([0,T]; H^{-1}(\Omega(t)))$. Thus $u_n(t)$ is uniformly convergent to $u(t) \in C([0,T]; H^{-1}(\Omega(t))$ as $n \to \infty$.

Remark 7.1. The approximate function $\sigma_\epsilon(u)$ in (7.1) can not be applied to the case $-1 < m < 0$ because $(u^2+\epsilon)^{m/2} \geq |u|^m$ does not hold. It is an interesting problem to construct appropriate approximate function for the case $-1 < m < 0$ to show the existence of solutions as in Theorem 2.3.

8. A REMARK ON THE UNIQUENESS OF TYPE 2 SOLUTION

We shall discuss on the uniqueness of Type 2 solutions for the case $r > 2N/(N+2)$. We know that the Type 2 solution $u(t)$ proved in the previous section belongs also to $C([0,T]; H^{-1}(\Omega(t)))$ if $n_t \geq 0$. We shall discuss on the uniqueness of such a solution.

Let $u(t)$ and $v(t)$ be two possible solutions and take $U(t), V(t) \in H_0^1(\Omega(t)) \cap H_2(\Omega(t))$ such that

$$-\Delta U(t) = u(t) \quad \text{and} \quad -\Delta V(t) = v(t).$$

We set also

$$w(t) = u(t) - v(t) \quad \text{and} \quad W(t) = U(t) - V(t).$$

Then by the identities (2.2) for $u(t)$ and $v(t)$ we see that for any $\bar{\delta}, 0 < \bar{\delta} << 1$,

$$\begin{aligned}(w(t),\phi(t)) - (w(\delta),\phi(\delta)) - \int_\delta^t \int_{\Omega(s)} w(t)\phi_t(s)dxds \\ + \int_\delta^t \int_{\Omega(s)} \nabla(\sigma(u(s)) - \sigma(v(s)))\nabla\phi(s)dxds = 0,\end{aligned} \tag{8.1}$$

for all $\phi(t) \in C^1([\delta, T); L^2(\Omega(t))) \cap L^2([\delta, T); H_0^1(\Omega(t)))$. We know, in particular, that

$$\frac{\partial}{\partial t} w(t) = \Delta(\sigma(u(t)) - \sigma(v(t)))$$

in the sense of distribution on $Q(0, T)$.

We consider the case $\Omega(t) = \Omega$, independent of t. We set $w_\epsilon(t) = w * \rho_\epsilon(t)$ and $W_\epsilon(t) = W * \rho_\epsilon(t)$, where $\rho_\epsilon(t)$, $\epsilon > 0$, is the mollifier with respect to t. Then we see

$$-\Delta \frac{\partial}{\partial t} W_\epsilon(t) = \Delta\left(\sigma(u)_\epsilon(t) - \sigma(v)_\epsilon(t)\right).$$

Since $\frac{\partial}{\partial t} W_\epsilon(t), (\sigma(u)_\epsilon(t) - \sigma(v)_\epsilon(t)) \in H_0^1(\Omega)$ we have

$$-\frac{\partial}{\partial t} W_\epsilon(t) = \sigma(u)_\epsilon(t) - \sigma(v)_\epsilon(t)$$

and hence,

$$-\frac{\partial}{\partial t} W(t) = \sigma(u(t)) - \sigma(v(t)) \in L^\infty_{loc}((0, T); H_0^1(\Omega)).$$

Therefore we can take $\phi(t) = W(t)$ in (8.1) to get

$$\begin{aligned}\|\nabla W(t)\|_2^2 - \|\nabla W(\delta)\|^2 + \int_\delta^t \int_\Omega \Delta W W_t dxds \\ + \int_\delta^t \int_\Omega \left(\sigma(u(s)) - \sigma(v(s))\right)(u(s) - v(s))dxds = 0\end{aligned} \tag{8.2}$$

which implies easily

$$\frac{1}{2}(\|\nabla W(t)\|_2^2 - \|\nabla W(\delta)\|_2^2) + \int_\delta^t \int_\Omega \left(\sigma(u(s)) - \sigma(v(s))\right)(u(s) - v(s))dxds = 0. \tag{8.3}$$

The integral term in (8.3) is nonnegative and we have

$$\|\nabla W(t)\|_2 \le \|\nabla W(\delta)\|_2. \tag{8.4}$$

Since

$$\|w(t)\|_{H^{-1}} \le \|\nabla W(t)\|_2 \le C\|w(t)\|_{H^{-1}},$$

we see from (8.2) that

$$\|w(t)\|_{H^{-1}} \le C\|w(\bar{\delta})\|_{H^{-1}} \to 0 \quad \text{as } \bar{\delta} \to 0.$$

Thus we conclude that $w(t) = 0$, i.e., $u(t) = v(t)$.

When $\Omega(t)$ depends on t we can not take $W(t)$ as a test function in the ogiginal sense. But if we could use it as a test function in any wider sense, we can formally calculate

$$\begin{aligned}\int_\delta^t \int_{\Omega(s)} \Delta W W_t dx ds &= -\frac{1}{2} \int_{Q(\delta,t)} \frac{\partial}{\partial s} \|\nabla W(s)\|^2_{\Omega(s)} dV + \int_{S(\delta,t)} n_x \nabla W W_t dS \\ &= -\frac{1}{2}\left(\|\nabla W(t)\|^2_{\Omega(t),2} - \|\nabla W(\delta)\|^2_{\Omega(\delta),2} \right) \\ &\quad - \frac{1}{2} \int_{S(\delta,t)} n_s |\nabla W|^2 dS + \int_{S(\delta,t)} n_x \nabla W W_t dS \\ &= -\frac{1}{2}\left(\|\nabla W(t)\|^2_{\Omega(t),2} - \|\nabla W(\delta)\|^2_{\Omega(\bar{\delta}),2} \right) \\ &\quad + \frac{1}{2} \int_{S(\delta,t)} n_s |n_x|^2 \left| \frac{\partial}{\partial n} W(s) \right|^2 dS.\end{aligned}$$

Therefore under the assumption $n_t \ge 0$ we formally get (8.4) and the uniqueness follows. However it is never trivial to justfy the above argument. It may be useful to use a mollifier after locally changing the variables as in Cooper[3]. This uniqueness problem is left as an open problem. Concerning uniqueness problem we also refer the interested reader to [4], where nonuniqueness is discussed for an equation arising in climatology.

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REFERENCES

[1] N.D. Alikakos, R. Rostamian, *Gradient estimates for degenerate diffusion equations. I*, Math. Ann. **259** (1982), no. 1, 53–70.

[2] S. Antontsev, J.I. Díaz, S. Shmarev, *Energy Methods for Free Boundary Problems Applications to Nonlinear PDEs and Fluid Mechanics*, Ser. Progress in Nonlinear Differential Equations and Their Applications, vol. 48, Birkhäuser, Boston, 2002.

[3] J. Cooper, *Local decay of solutions of the wave equation in the exterior of a moving body*, J. Math. Anal. Appl. **49** (1975), 130–153.

[4] J.I. Díaz, *Mathematical analysis of some diffusive energy balance models in climatology*, Mathematics, Climate and Environment, RMA Res. Notes Appl. Math., vol. 27, Masson, Paris, 1993, 28–56.

[5] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, NY, 1993.

[6] K. Lee, *A mixed problem for hyperbolic equations with time-dependent domain*, J. Math. Anal. Appl. **16** (1966), 455–471.

[7] G.M. Lieberman, *Second Order Parabolic Differential Equations*, Revised ed., World Scientific, Singapore, 2005.

[8] M. Nakao, *On solutions to the initial-boundary value problem for* $\partial u/\partial t - \Delta\beta(u) = f$, J. Math. Soc. Japan **35** (1983), no. 1, 71–83.

[9] M. Nakao, *Global solutions for some nonlinear parabolic equations with nonmonotonic perturbations*, Nonlinear Anal. **10** (1986), no. 3, 299–314.

[10] M. Nakao, *Smoothing effects of the initial-boundary value problem for logarithmic type quasilinear parabolic equations*, J. Math. Anal. Appl. **462** (2018), no. 2, 1585–1604.

[11] M. Nakao, *Existence and smoothing effect of the initial-boundary value problem for quasilinear parabolic equations in time-dependent domains*, submitted.

[12] Y. Ohara, L^∞*-estimates of solutions of some nonlinear degenerate parabolic equations*, Nonlinear Anal. **18** (1992), no. 5, 413–426.

[13] J.L. Vázquez, *The Porous Medium Equation*, Oxford University Press, 2007.

[14] L. Véron, *Coercivité et propriétés régularisantes des semi-groupes non-linéaires dans les espaces de Banach*, Faculte des Sciences et Techniques, Université Francois Rabelais, Tours, France, 1976.

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