Parallel Scott-Blair fractional model of viscoelastic biological materials

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Abstract. Fractional calculus is a mathematical approach dealing with derivatives and integrals of arbitrary and also complex orders. Therefore, it adds a new means to understand and describe the nature and behavior of complex dynamical systems. Here we use the fractional calculus for modeling mechanical viscoelastic properties of materials. In the present work, after reviewing some of the main viscoelastic fractional models, a new parallel model is employed, connecting in parallel two Scott-Blair models with additional multiplicative weight functions. The model is presented in terms of two power functions weighted by Debye-type functions extend representation, understanding and description of complex systems viscoelastic properties. Monotonicity of the model relaxation modulus is studied and some upper bounds for the minimal time value, above which the model relaxation modulus is monotonically decreasing are given and compared both analytically and numerically. The comparison with the results of relaxation tests executed on some real phenomena has shown that the parallel Scott-Blair model involving fractional derivatives has been in a good agreement.
Key words: fractional calculus, viscoelasticity, relaxation modulus, Scott-Blair fractional model.

INTRODUCTION

Fractional calculus is a branch of mathematical analysis that generalizes the derivative and integral of a function to non-integer order [6]. Application of fractional calculus in classical and modern physics greatly contributed to the analysis and our understanding of physical, chemical and bio-physical complex dynamical systems, since it provides excellent instruments for the description of memory and properties of various materials and processes. Models involving fractional derivatives and operators have been found to better describe some real phenomena than integer-order differential equations [11, 15-17,], whence there are many new exciting areas of fractional models and fractional calculus applications, such as the automatic control [12, 17], engineering [15, 16, 23] the modeling of biological, medical and environmental systems [7]. A historical review of applications can be found in [13]. In recent decades fractional derivatives have been found to be quite flexible also in the rheology [9, 20, 29], where fractional calculus constitutes a valuable mathematical tool to handle viscoelastic aspects of systems and materials mechanics. A general survey to the viscoelastic models constructed via fractional calculus is provided in the paper [14], where the analysis is given of the basic fractional models as far as their creep, relaxation and viscosity are considered in particular. The Kelvin-Voight, Maxwell and Zener fractional order models are considered.

Viscoelastic materials present a behavior that implies dissipation and storage of mechanical energy. In an attempt to describe the viscoelasticity phenomenon mathematically, several constitutive laws have been proposed which describe the stress–strain relations in terms of the quantities like creep compliance, relaxation modulus, storage and loss moduli and dynamic viscosity. Some of these constitutive laws have been developed with the aid of mechanical models consisting of combinations of springs and viscous dashpots.

For over five decades classical exponential behavior models have been widely applied to describe the viscoelastic properties of biological materials. Maxwell, Kelvin-Voight and Zener models have been used to the mathematical modeling of stress relaxation and creep processes [2, 18, 25]. For these models the relationship between the stress and deformation of the material is approximated though an ordinary differential or integral equations.

However, relaxation or creep processes deviating from the exponential Debye decay behavior are often encountered in the dynamics of biological complex materials [2, 20]. For such materials a stretched exponential decay KWW model (Kohlrausch-Williams-Watts) [25], hyperbolic type decay Peleg model [2] or power type behavior models [3] are used to approximate the experimentally obtained relaxation modulus or creep compliance data. Also, the experimental results obtained by other authors have shown that the behavior of some viscoelastic biological materials agrees well with that of the fractional models [7, 21, 26, 27, 29]. By replacing the springs and dashpots of the classical viscoelastic models by the Scott-Blair elements, several fractional models, including the fractional Maxwell, fractional Voigt and fractional Kelvin models, have been proposed [20, 28]. For details concerning the construction of fractional differential constitutive equations on the basis of classical mechanical viscoelastic models, such as the Maxwell, Kelvin-Voight and Zener ones, see also [20].

In this paper we introduce a new viscoelestic model. This model is based on fractional Scott-Blair elements combined in parallel with multiplicative exponential type weight function. In result, the parallel Scott-Blair model admits the closed form of linear exponential type weighted combination of two power functions of elementary fractional model type. The main properties of the parallel Scott-Blair model are summarized. An important task in understanding the rheology of viscoelastic materials is to construct models that allow one to describe the behavior of a material using the minimal number of physical parameters [22]. A new model is described by means of only five parameters.

FRACTIONAL DERIVATIVE

The Caputo's fractional derivative of a function $f(x)$ of non-integer order α with respect to variable t and with starting point at $t = 0$ is defined by [17]:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-1)^{n-\alpha-1} \frac{d^n}{dt^n} f(t) dt, \quad (1)$$

where: $n - 1 < \alpha < n$ and $\Gamma(n)$ is Euler's gamma function defined by the integral:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \quad (2)$$

FRACTIONAL MODELS

We consider a linear viscoelastic material subjected to small deformations for which the uniaxial, unaging and izotropic stress-strain equation can be represented by a Boltzmann superposition integral [5]:

$$\sigma(t) = \int_{-\infty}^t G(t-\lambda) \dot{\varepsilon}(\lambda) d\lambda, \quad (3)$$

where: $\sigma(t)$ and $\varepsilon(t)$ denotes the stress and strain, respectively, and $G(t)$ is the linear time-dependent relaxation modulus. The modulus $G(t)$ is the stress, which is induced in the viscoelastic material described by equation (3) when the unit-step strain $\varepsilon(t)$ is imposed.

Fractional Scott-Blair model [9, 21] is described by the fractional differential equation:

$$\sigma(t) = E\tau^\alpha \frac{d^\alpha \varepsilon(t)}{dt^\alpha}, \quad (4)$$

where: E and τ are the elastic modulus and relaxation time, respectively, α is non-integer positive order of fractional derivative of the strain $\varepsilon(t)$. Here, $d^\alpha/dt^\alpha = D_t^\alpha$ means the fractional derivative operator in the sense of Caputo derivative (1). Fractional Scott Blair model is intermediate model between ideal spring (see Fig. 1a) Hooke's model:

$$\sigma(t) = E\varepsilon(t) \quad (5)$$

and Newton's model:

$$\sigma(t) = \eta \frac{d\varepsilon(t)}{dt}, \quad (6)$$

of ideal fluids represented by means of an ideal dashpot (see Fig. 1b). To illustrate the structure of fractional models, a fractional element, in addition to the standard purely elastic (5) and purely viscous (6) elements, must be introduced – see Fig. 1c [4]. Assuming unit-step strain $\varepsilon(t)$ the uniaxial stress response of elementary fractional element (4), i.e. the time-dependent relaxation modulus $G(t)$ is given by [9, 20]:

$$G(t) = \frac{E}{\Gamma(1-\alpha)} \left(\frac{t}{\tau}\right)^{-\alpha}. \quad (7)$$

Thus the elementary fractional element is uniquely described by three parameters (E, τ, α), as shown in Fig. 1c. A deep insight into the complex properties of elementary fractional model gives the infinite hierarchical three structure of spring-dashpot fractal network model derived in [10] and presented in Fig. 2. It is shown to be $\alpha = \frac{1}{2}$ order model (4). Other hierarchical arrangements of springs and dashpot, which compose the elements of classical Maxwell model, such as ladders, trees or fractal structures can be found in the literature [10, 19].

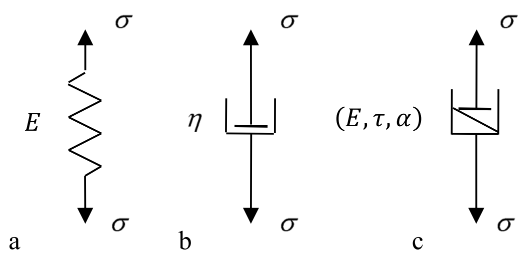


Fig. 1. From left to right: Hooke (a), Newton (b) and elementary fractional (c) elements

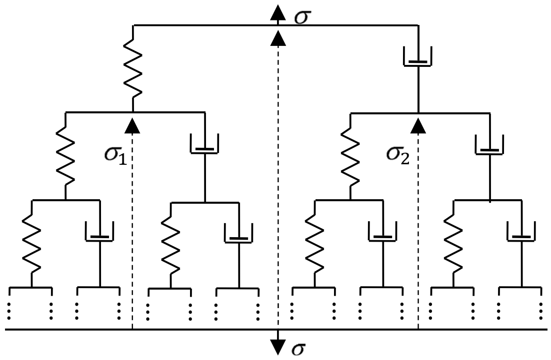


Fig. 2. Tree model of fractional Scott-Blair element (4) for $\alpha = \frac{1}{2}$ [10]

The classic Maxwell model is a viscoelastic body that stores energy like a linearized elastic spring and dissipates energy like a classical fluid dashpot. Precisely, classic viscoelastic Maxwell model is the arrangement of ideal spring in series with a dashpot (see Fig. 3a) described by the first order differential equation:

$$\frac{d\sigma(t)}{dt} + \frac{E}{\eta}\sigma(t) = E\frac{d\varepsilon(t)}{dt}, \tag{8}$$

which for unit-step strain $\varepsilon(t)$ has the exponential type response:

$$G(t) = Ee^{-t/\tau},$$

with the relaxation time $\tau = \eta/E$.

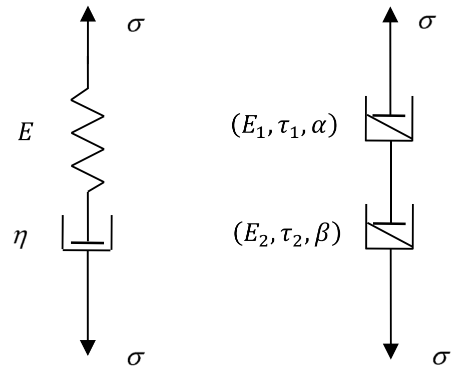
Connecting in series, by analogy to classic Maxwell model, two elementary fractional Scott-Blair elements (E_1, τ_1, α) and (E_2, τ_2, β) – see Fig. 3b – we obtain fractional Maxwell model described by the fractional differential equation [9, 20, 26, 28]:

$$\tau^{\alpha-\beta}\frac{d^{\alpha-\beta}\sigma(t)}{dt^{\alpha-\beta}} + \sigma(t) = E\tau^{\alpha}\frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}}, \tag{9}$$

where the parameters of the fractional Maxwell model (9) are functions of the parameters (E_1, τ_1, α) and (E_2, τ_2, β) of the model components given by [26]:

$$\tau = \left[\frac{E_1(\tau_1)^{\alpha}}{E_2(\tau_2)^{\beta}}\right]^{\frac{1}{\alpha-\beta}},$$

$$E = \left[\frac{(E_1\tau_1)^{-\beta}(\tau_1)^{\alpha(1-\alpha)}}{\left[E_2(\tau_2)^{\beta}\right]^{-\alpha}}\right]^{\frac{1}{\alpha-\beta}}.$$



a b

Fig. 3. Classic Maxwell model (a) followed by the fractional Maxwell model (b)

The assumption $\alpha \geq \beta$ is taken, usually [9, 20, 26, 28]. For details of the fractional Maxwell model (9) construction see [20] or [9]. For the unit-step strain the solution $\sigma(t) = G(t)$ of the fractional Maxwell model (9) is known for an arbitrary $1 \geq \alpha \geq \beta \geq 0$ and given by the formula [9, 20, 28]:

$$G(t) = E\left(\frac{t}{\tau}\right)^{-\beta} E_{\alpha-\beta,1-\beta}\left(-\left(\frac{t}{\tau}\right)^{\alpha-\beta}\right), \tag{10}$$

where: $E_{\kappa,\mu}(x)$ is the generalized Mittag-Leffler function defined by series representation, convergent in the whole z-complex plane [8, 17]:

$$E_{\kappa,\mu}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\kappa n+\mu)}. \tag{11}$$

The course of the relaxation modulus $G(t)$ (10) for fixed α parameter and a few values of the β order are shown in Fig. 4a, for fixed β and a few values of α the respective modulus $G(t)$ are plotted in Fig. 4b.

The classic Kelvin-Voight model consisting in a pure-

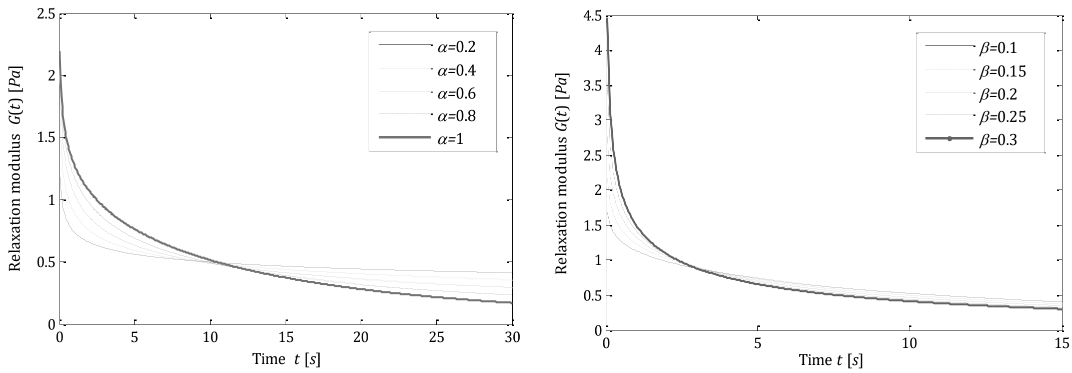


Fig. 4. The fractional Maxwell model relaxation modulus for $E = 1$ [Pa], $\tau = 20$ [s] and: (a) fixed $\beta = 0.15$, (b) fixed $\alpha = 0.9$

ly elastic element, i.e. spring, arranged in parallel with a purely viscous element, a dashpot (see Fig 5a), is described by the well-known first order differential equation:

$$\sigma(t) = E\varepsilon(t) + \eta \frac{d\varepsilon(t)}{dt}.$$

However, the Kelvin-Voight model exhibits an exponential strain creep, but not stress relaxation, which is described by the relaxation modulus:

$$G(t) = E + \eta\delta(t),$$

here $\delta(t)$ denotes the Dirac impulse.

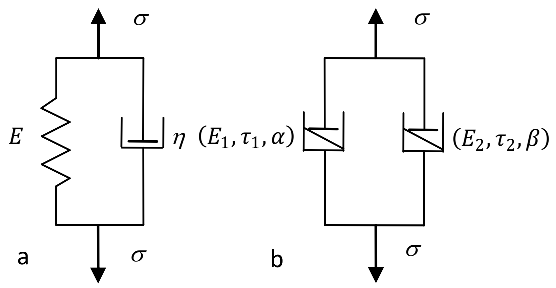


Fig. 5. Classic Kelvin-Voight model (a) followed by the fractional Kelvin-Voight model (b)

By replacing the spring and dashpot of the classical Kelvin-Voight model by two fractional Scott–Blair elements (E_1, τ_1, α) and (E_2, τ_2, β) – see Fig. 5b – we obtain fractional Kelvin-Voight model described by the non-integer order differential equation. We assume that $1 > \alpha \geq 0$ and $1 > \beta \geq 0$. In view of (4) for the same strain $\varepsilon(t)$ the stresses for these sub-elements are described by:

$$\sigma_1(t) = E_1(\tau_1)^\alpha \frac{d^\alpha \varepsilon(t)}{dt^\alpha},$$

$$\sigma_2(t) = E_2(\tau_2)^\beta \frac{d^\beta \varepsilon(t)}{dt^\beta}.$$

Thus, the sum stress $\sigma(t) = \sigma_1(t) + \sigma_2(t)$ of the parallel connection of two elementary fractional elements is described by the fractional differential equation:

$$\sigma(t) = E_1(\tau_1)^\alpha \frac{d^\alpha \varepsilon(t)}{dt^\alpha} + E_2(\tau_2)^\beta \frac{d^\beta \varepsilon(t)}{dt^\beta}.$$

The respective relaxation modulus is given by:

$$G(t) = \frac{E_1}{\Gamma(1-\alpha)}\left(\frac{t}{\tau_1}\right)^{-\alpha} + \frac{E_2}{\Gamma(1-\beta)}\left(\frac{t}{\tau_2}\right)^{-\beta}. \quad (12)$$

The course of the relaxation modulus (12) for fixed α and a few values of the β order are shown in Fig. 6a, for fixed β and a few values of α the modulus are plotted in Fig. 6b.

As can be seen, the performance of the three models: classic Maxwell, fractional Maxwell and fractional Kelvin-Voight models, although non identical, are comparable to typical shape of the relaxation modulus of different physical viscoelastic materials.

In order to make quantitative predictions about the behavior of viscoelastic materials, the rheological models should be chosen in a form that is convenient not only for mathematical analysis but also for derivation from experimental data.

Note, that the fractional Kelvin-Voight model (12) is uniquely defined by six parameters (E_1, τ_1, α) and (E_2, τ_2, β). The relaxation modulus (12) can be rewritten as follows:

$$G(t) = \frac{E_1(\tau_1)^\alpha}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{E_2(\tau_2)^\beta}{\Gamma(1-\beta)} t^{-\beta}, \quad (13)$$

thus the parameters (E_1, τ_1) can not be determined uniquely, since only the quotient $\frac{E_1(\tau_1)^\alpha}{\Gamma(1-\alpha)}$ can be uniquely determined if no additional conditions are imposed. Similarly, the uniqueness of the elastic modulus E_2 and relaxation time τ_2 identifiability is unattainable. If we shall allow that $E_1 = E_2 = E$ and $\tau_1 = \tau_2 = \tau$ in (13), which yields the simplified model:

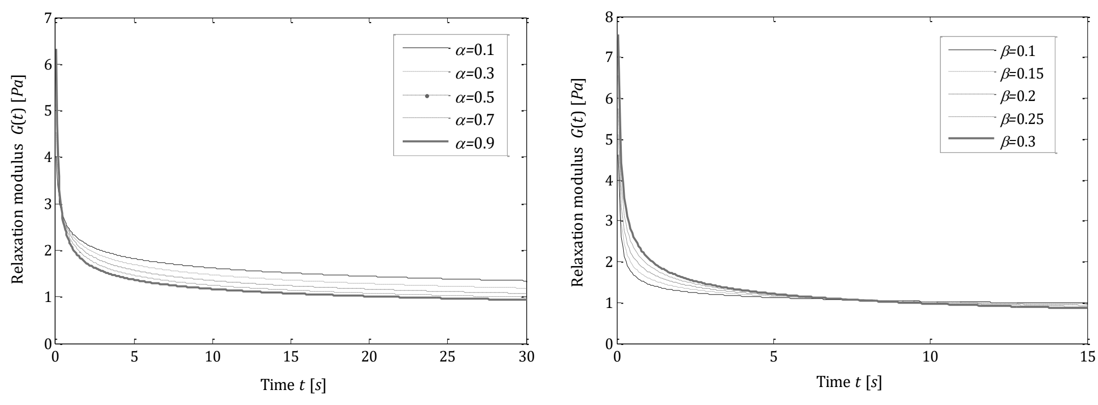


Fig. 6. The relaxation modulus of the fractional Kelvin-Voight models for $E_1 = E_2 = 1$ [Pa], $\tau_1 = 2$ [s], $\tau_2 = 20$ [s] and: (a) fixed $\beta = 0.15$, (b) fixed $\alpha = 0.9$

$$G(t) = \frac{E\tau^{\alpha}}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{E\tau^{\beta}}{\Gamma(1-\beta)} t^{-\beta},$$

the unique identification of the model parameters (E,τ,α,β) can be readily obtained. However, this relaxation modulus description is poorer than (12). The parallel structure fractional model with two Scott-Blair elements of common elastic modulus E and relaxation time τ but arbitrary non-integer order α and β parameters is proposed in the next section, for which this drawback is overcome.

PARALLEL SCOTT-BLAIR MODEL

Let us consider now the arrangement of two elementary fractional Scott-Blair elements (E,τ,α) and (E,τ,β) in parallel (Fig. 7) with additional multiplicative elements of product operation using the weight function $\varphi(t)$, the relaxation modulus of which is described by:

$$G(t) = \frac{E}{\Gamma(1-\alpha)}\left(\frac{t}{\tau}\right)^{-\alpha}\varphi(t) + \frac{E}{\Gamma(1-\beta)}\left(\frac{t}{\tau}\right)^{-\beta}\left(1-\varphi(t)\right). \quad (14)$$

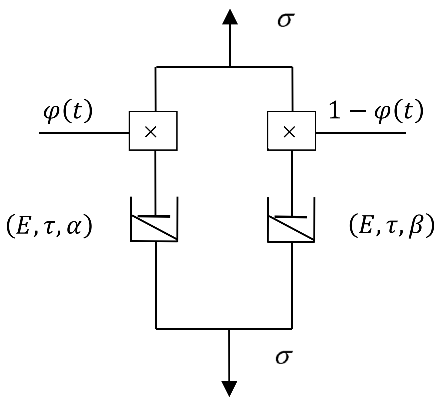


Fig. 7. Parallel Scott-Blair model

In Fig. 7 the product operations in (14) are symbolically marked by the product blocks in two parallel branches. We assume the Debye type weight function:

$$\varphi(t) = e^{-\gamma t}$$

of exponential decay represented by the parameter $\gamma > 0$. Thus, the relaxation modulus (14) has the following exact form:

$$G(t) = \frac{E}{\Gamma(1-\alpha)}\left(\frac{t}{\tau}\right)^{-\alpha} e^{-\gamma t} + \frac{E}{\Gamma(1-\beta)}\left(\frac{t}{\tau}\right)^{-\beta}\left(1-e^{-\gamma t}\right). \quad (15)$$

We assume that $1 > \alpha \geq 0$ and $1 > \beta \geq 0$. We also allow one, but not both, of the α and β values to be zero.

MONOTONICITY OF THE MODEL

It is assumed in the rheology literature that the relaxation modulus is positive definite and non-increasing (or decreasing) function. To study the relaxation modulus (15) monotonicity, let us rewrite it as follows:

$$G(t) = \frac{E\tau^{\alpha}}{\Gamma(1-\alpha)} t^{-\alpha} e^{-\gamma t} + \frac{E\tau^{\beta}}{\Gamma(1-\beta)} t^{-\beta}\left(1-e^{-\gamma t}\right). \quad (17)$$

Now, differentiation it with respect to the time yields:

$$\frac{dG(t)}{dt} = \frac{-\alpha E\tau^{\alpha}}{\Gamma(1-\alpha)} t^{-\alpha-1} e^{-\gamma t} - \frac{\gamma E\tau^{\alpha}}{\Gamma(1-\alpha)} t^{-\alpha} e^{-\gamma t} - \frac{\beta E\tau^{\beta}}{\Gamma(1-\beta)} t^{-\beta-1}\left(1-e^{-\gamma t}\right) + \frac{E\tau^{\beta}\gamma}{\Gamma(1-\beta)} t^{-\beta} e^{-\gamma t}.$$

We examine whether $\frac{dG(t)}{dt} = G'(t) < 0$, thus whether $G(t)$ decreases. By multiplying $G'(t)$ by $e^{\gamma t} t^{\beta+1}$ this requirement is equivalent to:

$$\frac{-\alpha\tau^{\alpha}}{\Gamma(1-\alpha)} t^{-\alpha+\beta} - \frac{\tau^{\alpha}\gamma}{\Gamma(1-\alpha)} t^{\beta-\alpha+1} - \frac{\tau^{\beta}\beta}{\Gamma(1-\beta)}\left(e^{\gamma t}-1\right) + \frac{\tau^{\beta}\gamma t}{\Gamma(1-\beta)} < 0,$$

whence, after some rearrangement of terms, it is easy to see that the condition $G'(t) < 0$ holds if and only the following inequality is satisfied:

$$\Phi(t) = \frac{\tau^{\alpha}}{\Gamma(1-\alpha)}[\alpha+\gamma t]t^{\beta-\alpha} + \frac{\tau^{\beta}}{\Gamma(1-\beta)}[\beta e^{\gamma t} - \beta - \gamma t] > 0. \quad (18)$$

Since $\beta - \alpha + 1 > 0$, then $[\alpha+\gamma t]t^{\beta-\alpha} \to \infty$ as $t \to \infty$. Similarly, $e^{\gamma t} \to \infty$ if $t \to \infty$. Thus there exists such t, call it t_{min}, that $\Phi(t) > 0$ for every $t \geq t_{min}$, by (18) above. Then we have the following.

Property 1. There exists $t_{min} \geq 0$ such that for any $t \geq t_{min}$ the relaxation modulus $G(t)$ (15) of the parallel Scott-Blair model is monotonically decreasing function.

In view of the above, the relaxation modulus may be non-decreasing (increasing) function in some time interval. For examples of relaxation modulus of such courses see, e.g. [24] and the example of the relaxation modulus recorded in the relaxation test for the real biological material given herein (sample 2 in the Example below). We propose now some upper bounds of minimal time t_{min}, after which $G(t)$ is decreasing in time.

Property 2. The parallel Scott-Blair model relaxation modulus $G(t)$ (15) is a monotonically decreasing function for any time such that:

$$t \geq \frac{2(1-\beta)}{\beta\gamma} = t_1. \quad (19)$$

Proof. The inequality (18) is used. Since, based on the Taylor's expansion of the exponential function for any positive t, we have:

$$e^{\gamma t} > 1 + \sum_{i=1}^{p} \frac{(\gamma t)^i}{i!},$$

if we set $p = 2$, then for the function $\Phi(t)$ defined by (18) the following upper bound is obtained:

$$\Phi(t) > \frac{\tau^\alpha}{\Gamma(1-\alpha)}(\alpha + \gamma t)t^{\beta-\alpha} + \frac{\tau^\beta}{\Gamma(1-\beta)}\left(\beta - 1 + \beta\frac{\gamma t}{2}\right)\gamma t. \quad (20)$$

The first component of the right-hand side of (20) is positive for any $t > 0$ and the second component is nonnegative for any $t \geq t_1$, where t_1 is defined in (19). Thus, (18) gives $G'(t) < 0$ and the result is proven.

We now present the next upper bound for t_{min}, which will improve on upper bound given in Property 2. To prove this result, we use similar arguments. The third degree Taylor's expansion of $e^{\gamma t}$, $p = 3$, gives:

$$\Phi(t) > \frac{\tau^\alpha}{\Gamma(1-\alpha)}(\alpha + \gamma t)t^{\beta-\alpha} + \frac{\tau^\beta}{\Gamma(1-\beta)}\left[\beta - 1 + \beta\frac{\gamma t}{2} + \beta\frac{(\gamma t)^2}{6}\right]\gamma t.$$

Thus, $\Phi(t) > 0$ for any t such that:

$$t \geq \frac{-3\beta + \sqrt{24\beta - 15\beta^2}}{2\beta\gamma} = t_2, \quad (21)$$

and we established more restrictive condition.

Property 3. The parallel Scott-Blair model relaxation modulus $G(t)$ (15) is a monotonically decreasing function for any time such that $t \geq t_2$, where t_2 is defined in (21).

We shall now give another upper bound of t_{min} and next compare it to t_1 and t_2.

Property 4. The parallel Scott-Blair model relaxation modulus $G(t)$ (15) is a monotonically decreasing function for any time:

$$t \geq \frac{1}{\beta\gamma} - \frac{\beta}{\gamma} = t_3. \quad (22)$$

Proof. Using the following upper bound [1; Theorem (1)]:

$$e^x > \left(1 + \frac{x}{a}\right)^{[a(a+x)]^{1/2}},$$

valid for any positive a and x, we establish the following estimation of the function $\Phi(t)$:

$$\Phi(t) > \frac{\tau^\alpha}{\Gamma(1-\alpha)}(\alpha + \gamma t)t^{\beta-\alpha} + \frac{\tau^\beta}{\Gamma(1-\beta)}\left[\beta\left(1 + \frac{\gamma t}{a}\right)^{[a(a+\gamma t)]^{1/2}} - \beta - \gamma t\right].$$

Let $a = \beta$. Then, we have:

$$\Phi(t) > \frac{\tau^\alpha}{\Gamma(1-\alpha)}(\alpha + \gamma t)t^{\beta-\alpha} + \frac{\tau^\beta(\beta+\gamma t)}{\Gamma(1-\beta)}\left[\beta^{1-[\beta(\beta+\gamma t)]^{1/2}}(\beta + \gamma t)^{[\beta(\beta+\gamma t)]^{1/2}-1} - 1\right]. \quad (23)$$

The second summand of the right-hand side of (23) is nonnegative if and only if:

$$\beta^{1-[\beta(\beta+\gamma t)]^{1/2}}(\beta + \gamma t)^{[\beta(\beta+\gamma t)]^{1/2}-1} \geq 1,$$

which is equivalent to:

$$\left(1 + \frac{\gamma t}{\beta}\right)^{[\beta(\beta+\gamma t)]^{1/2}-1} \geq 1. \quad (24)$$

The logarithm of both sides of (24) gives:

$$\left\{[\beta(\beta + \gamma t)]^{1/2} - 1\right\}log\left(1 + \frac{\gamma t}{\beta}\right) \geq 0.$$

So for $t > 0$ the second factor is positive, the following condition follows immediately:

$$[\beta(\beta + \gamma t)]^{1/2} > 1,$$

whence, we obtain $[\beta(\beta + \gamma t)] > 1$, which immediately implies (22) and proves the property.

To show that for $\beta > 0$ this upper bound of t_{min} is better than (19), note that the obvious inequality $(1-\beta)^2 > 0$ is equivalent to $1 - \beta^2 < 2(1-\beta)$, which for $\beta > 0$ yields:

$$\frac{1}{\beta\gamma} - \frac{\beta}{\gamma} < \frac{2(1-\beta)}{\beta\gamma},$$

whence:

$$t_3 = \frac{1}{\beta\gamma} - \frac{\beta}{\gamma} < \frac{2(1-\beta)}{\beta\gamma} = t_1$$

directly results.

Next, we prove that for $\beta > 0$ we have $t_2 < t_3$. Since $\beta^2 - \beta + 1 > 0$, the inequality $0 < (\beta - 1)^2(\beta^2 - \beta + 1)$ implies:

$$0 < 4 + 16\beta^2 - 12\beta + 4\beta^4 - 12\beta^3,$$

and hence, after simple algebra, we obtain:

$$24\beta - 15\beta^2 < (2 + 3\beta - 2\beta^2)^2.$$

Since both sides of the above are positive, we have:

$$\sqrt{24\beta - 15\beta^2} < 2 + 3\beta - 2\beta^2,$$

whence:

$$t_2 = \frac{-3\beta + \sqrt{24\beta - 15\beta^2}}{2\beta\gamma} < \frac{1}{\beta\gamma} - \frac{\beta}{\gamma} = t_3$$

results after algebraic manipulations. Thus, the upper bound t_3 is better that t_1 and is sometimes at least as good as t_2 - see Fig. 8, in which the times t_1 (19), t_2 (21) and t_3 (22) as a functions of the order β are summarized. In Fig. 8 the logarithmic scale is used for the times scale. However, t_1 and t_3 are more useful, they provide less restrictive estimation of the time interval for which the relaxation modulus monotonically decreases. The estimations t_1, t_2 and t_3 depend only on β and γ. They are α and τ independent, since only the second summand of $\Phi(t)$ (18) has been taken into account in the above reasoning. The analysis here is fundamental for the model $G(t)$ (15) monotonicity, but the results obtained are, however, sufficient (but not necessary, of course) conditions for the relaxation modulus monotonicity. Note finally, that if $\beta \to 1$, then the upper bound t_2 (21):

$$t_2 = \frac{-3}{2\gamma} + \frac{\sqrt{\frac{24}{\beta} - 15}}{2\gamma} \to 0.$$

Similarly $t_3 = \frac{1}{\beta\gamma} - \frac{\beta}{\gamma} \to 0$ and $t_1 = \frac{2(1-\beta)}{\beta\gamma} \to 0$ as $\beta \to 1$.

The course of the relaxation modulus $G(t)$ (15) for the fixed α parameter and a few values of the β order are shown in Fig. 9a, for fixed β and a few values of α the respective $G(t)$ are plotted in Fig. 9b. The following rule holds: the greater the parameter α is, the shorter the relaxation times are for the order β being fixed. A similar rule holds for the model order β.

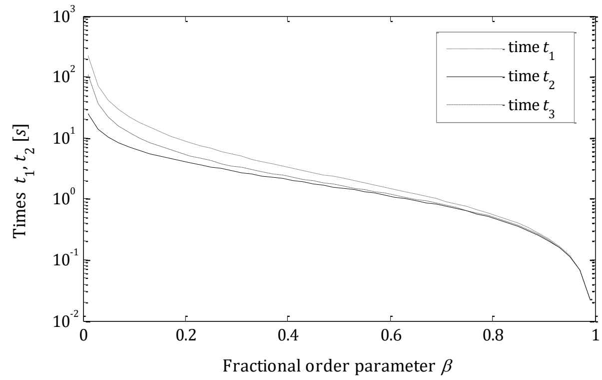


Fig. 8. The times t_1 (19), t_2 (21) and t_3 (22) as functions of the order parameter $\beta, \gamma = 0.9$

EXAMPLE

In Fig. 10 the relaxation modulus $G(t)$ (15) of the parallel Scott-Blair model is plotted for sample 1 of the root of sugar beet Janus variety in the state of uniaxial deformation (for details concerning the respective stress relaxation test and measurement data see, e.g., [24]), where the measurements $\bar{G}(t_i)$ are also marked, $i = 1, \dots, N$ and the number of measurements $N = 400$. The relaxation modulus of the classic four-parameter, optimal in the least-squares sense, Maxwell model:

$$G_M(t) = E_1 e^{-v_1 t} + E_2 e^{-v_2 t}, \qquad (25)$$

where: E_j and v_j represent the elastic modulus and relaxation frequencies, respectively, are also depicted in Fig. 10. The parameters of parallel Scott-Blair model and Maxwell model (25) are given in Table 1; the relaxation times $\tau_j = 1/v_j$. The course of relaxation modulus (15) and (25) for sample 2 are plotted in Fig. 11, the model parameters are given in Table 1, as above.

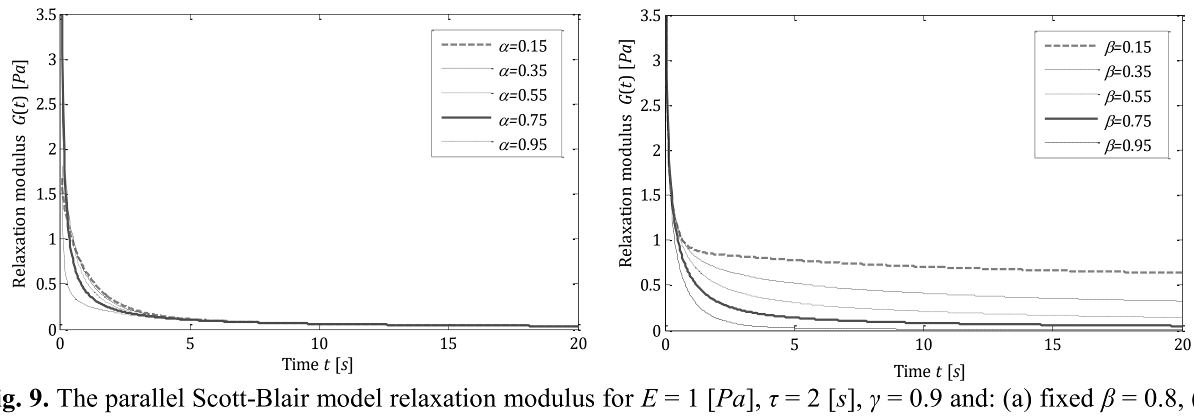


Fig. 9. The parallel Scott-Blair model relaxation modulus for $E = 1$ [Pa], $\tau = 2$ [s], $\gamma = 0.9$ and: (a) fixed $\beta = 0.8$, (b) fixed $\alpha = 0.5$

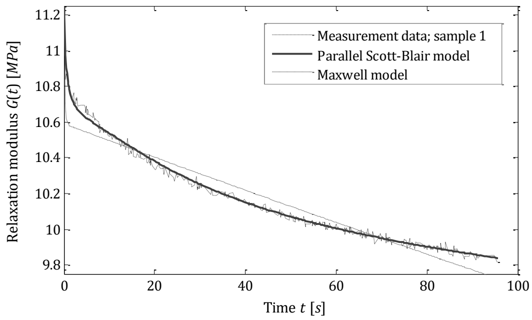


Fig. 10. The fit of parallel Scott-Blair model (15) and four-parameter optimal Maxwell model (25) to the experimental data in the state of uniaxial deformation for sample 1 of the root of sugar beet Janus variety

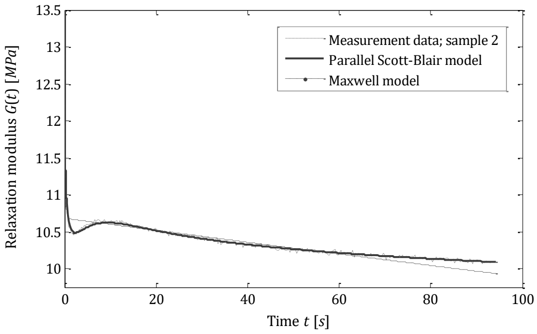


Fig. 11. The fit of parallel Scott-Blair model (15) and four-parameter optimal Maxwell model (25) to the experimental data in the state of uniaxial deformation for sample 2 of the root of sugar beet Janus variety

Note, that for sample 1 the parallel Scott-Blair model relaxation modulus decreases for any time $t > 0$, while for sample 2 this monotonicity is obtained only for $t > t_{min} = 8.01$ $[s]$. However, the parallel Scott-Blair model is adequate for description of the real stress relaxation, which is also non monotonic.

It is also easy to observe, that the Maxwell model is inappropriate for description of the relaxation processes for both the samples, i.e., the classic Maxwell models do not fully characterize the true viscoelastic behavior of the biological material. A better fit to the experimental data can be obtained if the fractional model proposed is used.

Table 1. Parallel Scott-Blair (15) and classic Maxwell (25) models parameters for two samples of the root of sugar beet Janus in the state of uniaxial deformation

Sample 1			
Four-parameter Maxwell model		Parallel Scott-Blair fractional model	
E_1 $[MPa]$	10.5899	E $[MPa]$	9.2346
ν_1 $[s^{-1}]$	0.000892	τ $[s]$	1.0232e+3
τ_1 $[s]$	1.1205e+3	α $[-]$	0.0232
E_2 $[MPa]$	1.4208	β $[-]$	0.0359
ν_2 $[s^{-1}]$	6.100486	γ $[s^{-1}$	0.1387
τ_2 $[s]$	0.1639		

Sample 2			
E_1 $[MPa]$	10.68289	E $[MPa]$	12.4696
ν_1 $[s^{-1}]$	0.000769	τ $[s]$	0.0687
τ_1 $[s]$	1.29933e+3	α $[-]$	0.0563
E_2 $[MPa]$	2.52574	β $[-]$	0.0271
ν_2 $[s^{-1}]$	7.16436	γ $[s^{-1}$	0.2805
τ_2 $[s]$	0.13958		

FINAL REMARKS

A new fractional model, which combines the Debye decay of Maxwell model with the time power decay of Scott-Blair elementary fractional model, has been proposed. It can be treated as a generalization of classic Kelvin-Voight model to non-integer order derivatives.

Some important advantage of the new model must be emphasized. First, it describes the complex phenomena with only five parameters, secondly, it has a simple analytical form, especially if compared with the infinite series of Mittag-Leffler function (11) response of fractional Maxwell model (9) and thirdly, simple identification procedure can be proposed for the fitting of relaxation modulus data. It has been demonstrated here that the relaxation modulus of real biological material is more accurately modeled using parallel Scott-Blair model than by the classic Maxwell model approximation.

Future work will deal with the model identification. Two asymptotic approximate models for short times and for large times will be found in the next paper, where we shall provide a two-stage, two-interval near optimal identification routine for the determination of a parallel Scott-Blair model with arbitrary order parameters. The effectiveness of the new model and the two-stage identification scheme will be demonstrated for sugar beet root.

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