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THE TRANSITION EFFECT IN RAIL TRACKS – ASSESSMENT OF TRANSIENT ENERGY FOR LOW FREQUENCY VIBRATIONS

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The transition effect between different track-foundation systems is examined from the point of view of energy that is produced during the passage of load. Analytical solution is given. A model of beam on elastic foundation with damping is used as the base model. It is developed into a model composed of two parts that represent the track-subgrade system with an abrupt change in mechanical parameters: bending stiffness, foundation stiffness, damping, and mass. Several calculations are carried out including examples of comparative calculations with the Finite Difference Model and the Finite Element Model. Transient rail deflections and energy are determined, which may serve to estimate the rate of track-subgrade deterioration.

Keywords: transition effect, rail track modeling, transient analysis, energy approach

1. Introduction

Ballasted tracks, i.e. conventional tracks, have been in use since the early days of railways. Their bearing structure consists of a rail-sleeper grid embedded in crushed-stone layer called ballast. Nowadays, they constitute about 95% of all track types used worldwide. Such tracks require continuous maintenance in order to prevent track deterioration and settlement.

The non-conventional tracks or ballastless tracks, as they are also called, exhibit quite different mechanical properties. Whenever there is a need to connect these two types of tracks, a special transition must be constructed. This is especially important when the ballastless tracks are laid on bridges, and a ballasted track is laid on the embankment. Less severe situation is when both types of tracks are laid on one common subgrade, which is the case for level crossings and stations. Nevertheless, transitions must be constructed even if the same ballasted track is laid on the embankment and on the bridge. This is the most common situation.

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Due to the change in the mechanical parameters of tracks and their foundation, the so-called transition effect appears. It is the main cause for excessive track deterioration. One of the first attempts to construct a proper transition structure is presented (Eisenmann [1]), where special rails were used to strengthen the connection of the two tracks. This structure was later refined and nowadays it is considered a standard structure in Germany (Anforderungskatalog [2]), which with some modifications is followed by Austrian and Swiss railways.

The first Polish slab structure was used on the Warsaw Main Station in 1973. It was designed by prof. Tadeusz Basiewicz and consisted of shortened wooden sleepers equipped with rubber pads and embedded in a reinforced concrete slab (Sysak [3]). The ballastless railway tracks that can currently be found on Polish railway network are shown in Figure 1.

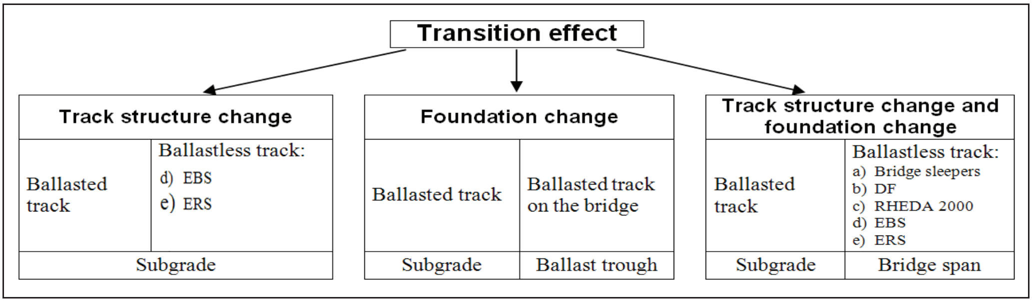


Fig. 1. Types of track structures used in Poland which result in the emergence of the transition effect: a) bridge sleepers structure, b) DF – direct fastening to the reinforced concrete slab with a layer of elastic pad under the fastening plate (in-situ poured or prefabricated pads), c) RHEDA 2000 – rail support blocks embedded in reinforced concrete slab, d) EBS (Embedded Block System) – reinforced concrete blocks embedded in a reinforced concrete slab with a tray filled with elastic material (polyurethane resin), e) ERS (Embedded Rail System) – reinforced concrete slab with channels in which the rails are placed and embedded continuously with elastic, in-situ poured material (polyurethane resin), without fasteners.

An example of modern structure is RHEDA 2000 built in Kraków (E30 line, Kraków-Płaszów section, year of construction 2003) (Czyczuła [4]). The DF or ERS track structures (see Figure 1) are present on several viaducts in Poland and EBS is applied on a few stations (recently Wrocław Main Station). The wooden bridge sleeper structure is very common – just in Małopolska Region they constitute nearly 50% of all tracks on bridges.

Polish regulations concerning railway bridges (Id-2 [5]) state that all newly built bridges are to be equipped with a ballast trough. In special cases it is possible to get an exemption from this requirement. Since the construction height of the bridge span with slab tracks is much lower than in the case of ballast trough spans, they are still in use and the trend seems to be continued in the future.

The transition effect resulting from the application of slab tracks on bridges is especially strong. It is still present even in the case of bridges equipped with ballasted tracks. According to (Regulation by MTiGM [6]) the stiffness of the tracks in the transition zone and on the bridge must be close to each other and the required length of the transition zone over which the stiffness may vary smoothly is 20m. There are no standard Polish solutions within the track structure to diminish the transition effect, and the regulations concerning the subgrade (Id-3 [7]) give only examples of solutions.

The transition effect has been analysed almost exclusively using numerical models – e.g. (Lei [8], Zhai [9]), where the stiffness difference between two tracks or geometrical rail irregularity are analysed. Analytical models are very rare – see (Dimitrovová [10]). In the paper, a beam model on elastic foundation was assumed and the effect of an abrupt change in foundation stiffness on the critical speed was determined. Research combining modeling and experiments may be represented by (Coelho [11], Davis [12], Hunt [13], Plotkin 14]), where more effects were taken into account: gaps under sleepers and track settlement. On the basis of an extensive review of the literature and author's work (Sołkowski [15]) it may be stated that the transition effect has to be analyzed in a wider context of all track-foundation parameters including wave phenomena and may be divided into:

- primary effect which results from mechanical differences between the track-foundation structure including the constraints between the rail and the support structure,
- secondary effect which results from geometrical deformation such as vertical rail irregularity, track twist, gaps under the sleepers, etc.; it is a consequence of the primary effect and results from natural and increased track settlement and deformation.

2. Aim of the work, assumptions and way of solution

The aim of the work is to present a model of the transition effect with the emphasis put on the primary effect, and to show its consequences for the secondary effect. The model is considered from the point of view of energy changes in the track-subgrade system. The track is modelled as a beam on elastic, non-linear foundation with damping, where the beam and the foundation may exhibit abrupt changes in their mechanical parameters along the track. It is assumed that the parameters are independent, i.e. the resultant vibrating mass of the beam is not related to the stiffness and damping of the foundation. This relation, also depending on the frequency of vibrations, has been analysed in many papers – e.g. (Bukowski [16], Czyczuła [17]). In the present study, as in (Vostroukhov [18]), the assumption of independency will be used. Also, the classical Winkler model was compared in the paper to the elastic half-space model with good agreement of results, and where it was also stated that the continuous model is in good agreement with the discrete support model for analysing bending waves.

The junction of the two track-subgrade structures is considered to constitute an "elastic barrier" (Sołkowski [19], Wolfert [20]) which may cause reflections and interferences of the bending wave that travels with the load. In the latter, a string resting on elastic foundation was analysed. It was composed of two parts having different material density and the foundation might exhibit an abrupt change in its stiffness. It was assumed that the displacement of the string might be expressed as the sum of the stationary vibration and transient vibration. The same approach is applied below to the beam model. As a starting point for the beam model, the approach presented in (Kerr [21]), where a bi-linear foundation was assumed, will be used. This model for the static case is extended to a dynamic case by adding the beam inertia and viscous damping (Sołkowski [15]).

The beam-foundation model will be loaded with a constant point load moving along the system at a relatively small speed (up to 200 km/h) assuming that only the low frequency vibrations are taken into account and that the surface wave velocities (the shear waves or the *Rayleigh* waves) are much greater than the load speed. First, the primary transition effect will be calculated. Next, on the basis of the calculated rail deflections and energy in the transition zone, a settlement model will be suggested from the literature, so as to estimate the possible rail irregularity and track settlement. These will then constitute the principal cause for the secondary transition effect.

The base model (stationary model) consists of a Bernoulli-Euler beam on the Winkler foundation with damping:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+m\frac{\partial^2 w(x,t)}{\partial t^2}+c\frac{\partial w(x,t)}{\partial t}+U\cdot w(x,t)=P\cdot\delta(x-vt) \tag{2.1}$$

where: $w(x,t)$ – beam (rail) deflection in space x and time t, EI – beam bending stiffness, E – Young modulus, I – moment of inertia of the cross-section, m – unit mass of the beam (includes the fasteners and sleepers), $c = 2m\omega_b$ – viscous damping of the foundation, where ω_b – circular natural frequency of the beam, U – rail support modulus, assumed continuous and defined as $U = C \cdot b_z$, where C the foundation modulus including ballast and the subgrade, b_z – width of the substitute rail support, the fastener pads are considered to have an infinite stiffness, P – constant point load that moves along the beam, v – constant speed of the load, δ – Dirac delta. The solution is:

$$w(x,t)=w_0\cdot w(s,t) \tag{2.2}$$

where: $w_0=\frac{P\cdot\lambda}{2\cdot U}$, $s=\lambda(x-vt)$, $\lambda=\sqrt[4]{\frac{U}{4EI}}$, and $w(s,t)$ is the solution of equation (1) in the load coordinate system.

3. TRANSIENT FUNCTIONS

The track model consisting of two parts having different mechanical parameters is shown in Figure 2. In order to simplify calculations, the load parameter (corresponding to time) is introduced as τ, which takes on value zero at the junction of the two tracks (Wolfert [20]).

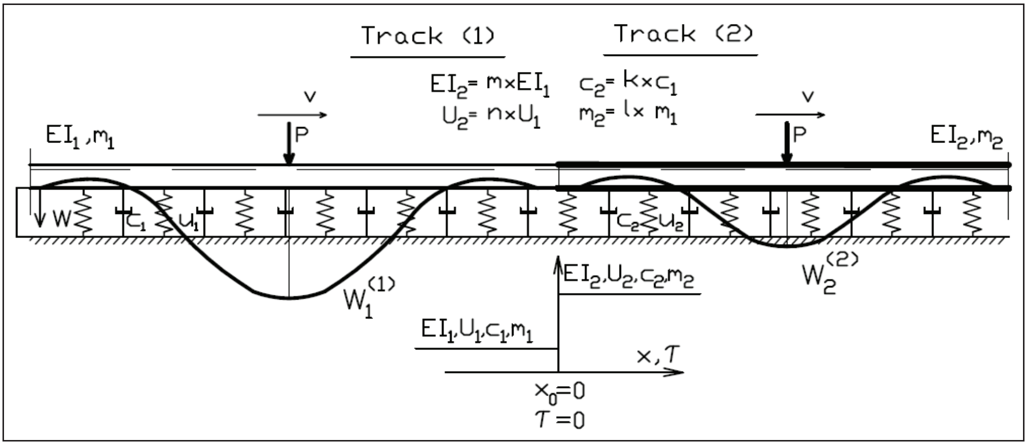


Fig. 2. Model of the junction of two tracks: *m*, *n*, *k*, *l* – multipliers for mechanical parameters

Assuming that the vibration of the beam may be treated as the sum of the stationary and the transient vibration, we have the following expressions for two positions of the load:

$$w_1^{(1)}(x,\tau) = w_{st}^{(1)}(x,\tau) + w_{tr}^{(1)}(x,\tau) \quad \text{for } \tau < 0 \text{ - track (1)}$$
$$w_2^{(2)}(x,\tau) = w_{st}^{(2)}(x,\tau) + w_{tr}^{(2)}(x,\tau) \quad \text{for } \tau \geq 0 \text{ - track (2)} \tag{3.1}$$

where: τ is used to indicate the distance between the load and the border between the two tracks. Unknown functions $w_{tr}^{(1)}(x,\tau)$ and $w_{tr}^{(2)}(x,\tau)$ are additional deflection lines due to the change of the beam-foundation parameters. Functions $w_{st}^{(1)}(x,\tau)$ and $w_{st}^{(2)}(x,\tau)$ are determined for the beam and foundation having uniform parameters. Superscripts (1) and (2) denote the position of the load on the tracks. Functions $w_{tr}^{(1)}(x,\tau)$ and $w_2^{(1)}(x,\tau)$ are assumed to be in the following form:

$$w_n(x,s_n) = A_n(x,s_n) \cdot e^{-b_n|s_n|}\left[\cos(a_1^n s_n) + \sin(a_1^n |s_n|)\right] \tag{3.2}$$

where $n = 1{,}2$ and for the sake of simplicity the functions were denoted as $w_1(x,s_1) = w_{tr}^{(1)}(x,\tau)$, $w_2(x,s_2) = w_2^{(1)}(x,\tau)$, and a_1^n, b_n denote parameters for track n, $A_n(x,s_n)$ is a transient function defining the amplitude of the additional deflection for track n dependent on $s_n = \lambda_n(x-vt)$ and $\lambda_n = \sqrt[4]{U_n/4EI_n}$:

$$A_n(x,s_n) = A_n^0(s_n)\cdot e^{-b_n\lambda_n|x|}\left[\cos(a_1^n\lambda_n x) + \sin(a_1^n\lambda_n|x|)\right] \tag{3.3}$$

where: $A_n^0(s_n)$ are constants for the tracks – two constants for either track depending on the position of the load. As was shown in (Sołkowski [22]) on the basis of the continuity conditions at $x = x_0$ the constants are determined as

$$\begin{cases} A_1^0(s_1) = A_2^0(s_2) - w_0\varphi_1^{(1)}(s_1) \\ A_2^0(s_2) = w_0\varphi_1^{(1)}(s_1)\dfrac{\left(\dfrac{1}{D_1^{(1)}D_2^{(1)}}+1\right)}{\left(\dfrac{\lambda_2}{\lambda_1}\right)^2\dfrac{\left(D_3^{(2)2}+D_4^{(2)2}\right)}{\left(D_1^{(1)2}+D_2^{(1)2}\right)D_3^{(2)}D_2^{(1)}}+\dfrac{1}{D_1^{(1)}D_2^{(1)}}} \qquad \text{for } \tau<0 \\ \varphi_1^{(1)}(s_1) = \dfrac{2}{a_1(D_1^{(1)2}+D_2^{(1)2})} \end{cases} \tag{3.4}$$

where: $D_i^{(n)}$ are new constants for $i = 1{,}2{,}3{,}4$, and $n = 1{,}2$ denoting track (1) or (2). Constants $D_i^{(n)}$ for track (1) are determined in (Fryba [23]) as: $D_1^{(1)} = a_1 b_1$, $D_2^{(1)} = b_1^2 - \frac{1}{4}\left(a_1^2 - a_2^2\right)$,, $D_3^{(1)} = a_2 b_1$, $D_4^{(1)} = b_1^2 + \frac{1}{4}\left(a_1^2 - a_2^2\right)$, in which

$$\begin{aligned} a_{1,2} &\approx \sqrt{1+\alpha_1^2 \pm \frac{2\alpha_1\beta_1}{b_1}} \\ b_1 &= \sqrt{(1-\alpha_1^2)} \end{aligned} \tag{3.5}$$

Coefficients $a_{1'2}$ and b_1 relate the dependence on the speed through $\alpha_1 = v/v_{cr(1)}$ and damping $c_1 = \beta_1 \cdot c_{cr(1)}$, in which $v_{cr(1)}$ and $c_{cr(1)}$ are the critical speed and damping on track (1) respectively. Coefficients a take on subscripts 1,2 depending on whether the sign under the root in Eq.(3.5) is (+) or (–). By analogy it is possible to obtain constants $D_i^{(n)}$ for track (2).

If there is no damping ($\beta_1 = 0$), we get: $a_1 = \sqrt{1+\alpha_1^2}$, $b_1 = \sqrt{1-\alpha_1^2}$, $a_2 = \sqrt{1+\alpha_2^2}$, $b_2 = \sqrt{1-\alpha_2^2}$, $D_1^{(n)} = D_3^{(n)}$, $D_2^{(n)} = D_4^{(n)}$, $\varphi_1(s_1) = 1$. Therefore:

(3.6)
$$\begin{cases} A_1^0(s_1) = A_2^0 - \dfrac{w_0^{(1)}}{a_1 b_1} \\ A_2^0(s_2) = w_0^{(1)} \dfrac{\dfrac{1}{a_1 b_1} + 1}{\left(\dfrac{\lambda_2}{\lambda_1}\right)^2 \dfrac{\left(a_2^2 + b_2^2\right) b_1}{\left(a_1^2 + b_1^2\right) b_2} + 1} \end{cases} \quad \text{for } \tau < 0$$

where: $w_0^{(1)}$ – rail deflection for track (1). In a case of small speeds $\alpha_1 = v / v_{cr(1)} \to 0$, we get $a_i = b_i \approx 1$. Thus:

(3.7)
$$\begin{cases} A_1^0(s_1) = \left(\dfrac{2}{\left(\dfrac{\lambda_2}{\lambda_1}\right)^2 + 1} - 1 \right) w_0^{(1)} \\ A_2^0(s_2) = \dfrac{2}{\left(\dfrac{\lambda_2}{\lambda_1}\right)^2 + 1} w_0^{(1)} \end{cases} \quad \text{for } \tau < 0$$

By analogy the amplitudes for the load position on track (2) may be obtained. In the case of an abrupt mass change according to (Sołkowski [15]) we have:

(3.8)
$$A_1^0 = \frac{C(m_1 - m_2)}{2 + C(m_1 - m_2)} \cdot w_0^{(1)}$$

where $w_0^{(1)}$ is rail deflection on track (1) with uniform parameters and

(3.9)
$$C = \frac{v^2}{4\lambda^2} \frac{1}{EI},$$

where $EI_1 = EI_2 = EI$ and $\lambda_1 = \lambda_2 = \lambda$. The maximum transient amplitudes are determined for $x = 0$ and $\tau = 0$. Because $A_1^0 = A_2^0 - w_0^{(1)}$, we have:

(3.10)
$$\begin{cases} w_1^{(1)}(0, 0^{(-)}) = w_0^{(1)} + A_1^0 \\ w_2^{(1)}(0, 0^{(-)}) = A_2^0 \\ w_{tr}^{(2)}(0, 0^{(+)}) = A_2^0 - w_0^{(2)} \end{cases}$$

where $w_{tr}^{(2)}(0, 0^{(+)})$ is the transient deflection for load position on track (2) whereas $0^{(-)}$ means that the load is on track (1).

4. ENERGY BALANCE

The considered cases of the behavior of the system are shown in Figure 3. Additionally, the comparison between the analytical solution (AM), the Finite Element Model and Finite Difference Model is presented according to (Sołkowski [15]). It was shown that all the models behave similarly. E.g., for the speed v = 20 km/h and for the assumed rail modulus difference $U_2 = 3 \times U_1$, the calculated rail deflections for both tracks are respectively: 1.198/0.528 mm (AM), 1.201/0.530 mm (FDM), 1.219/0.539 mm (FEM), and for the bending stiffness difference $EI_2 = 2 \times EI_1$ it was obtained: 1.198/0.996 mm (AM), 1.201/1.003 mm (FDM), 1.219/1.075 mm (FEM). In the case

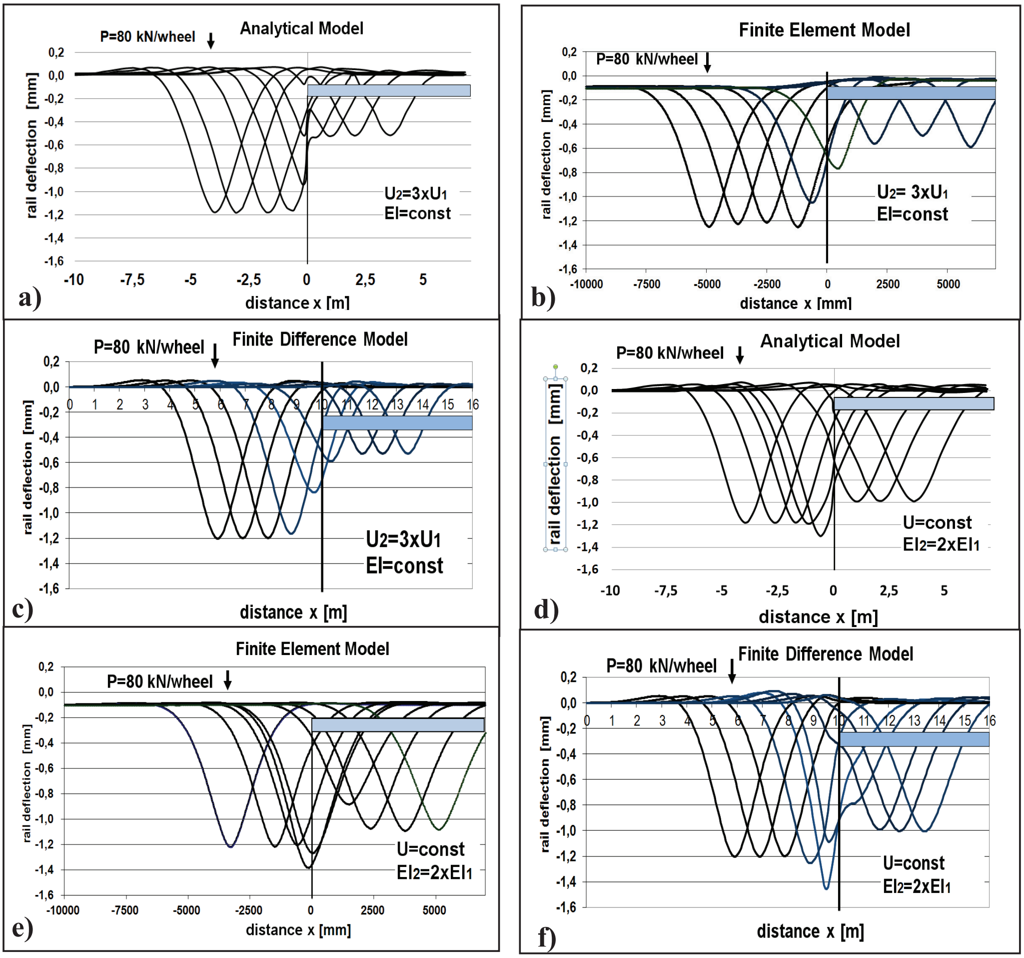


Fig. 3. Rail deflections in the cases of rail modulus change: a,b,c, and bending stiffness change: d,e,f

of the bending stiffness change the maximum values are: 1.309/1.455/1.382 mm (AM/FDM/FEM). Thus the calculated transient increments are: 9.4%, 13.4% and 21.1% respectively.

The energy balance of the beam-foundation system is assumed as:

$$E_{KP} = W_P \tag{4.1}$$

where: E_{KP} – kinetic and potential energy of the beam-foundation system, W_P – work performed by force P inducing deflection $w(x,\tau)$. For the load position on track (1) we have:

$$\begin{cases} W_{P(1)}^{(1)} = W_{st(1)}^{(1)} + \widetilde{W}_{tr(1)}^{(1)} & dla \quad x < x_0 \wedge \tau < 0 \\ W_{P(2)}^{(1)} = W_{st(2)}^{(1)} + \widetilde{W}_{tr(2)}^{(1)} & dla \quad x \geq x_0 \wedge \tau < 0 \end{cases} \tag{4.2}$$

where: $W_{st(n)}^{(1)}$, $\widetilde{W}_{tr(n)}^{(1)}$ – work performed on the beam of uniform parameters (stationary) and on the transient deflection respectively. The superscript (1) denotes the load position and the subscripts (1,2) refer to the track number. Because for the uniform beam-foundation parameters $W_{st(1)}^{(1)} = E_{KP(1)}^{(1)}$, $W_{st(2)}^{(1)} = E_{KP(2)}^{(1)}$, for the load position close to $(x = x_0)$ we obtain the following balance for the total energy of the beam-foundation system for the two tracks and for the two load positions (Figure 4):

$$\begin{cases} E_C^{(1)} = E_{KP(1)}^{(1)} + \widetilde{W}_{tr(1)}^{(1)} + E_{KP(2)}^{(1)} + \widetilde{W}_{tr(2)}^{(1)} & dla \quad \tau < 0 \\ E_C^{(2)} = E_{KP(1)}^{(2)} + \widetilde{W}_{tr(1)}^{(2)} + E_{KP(2)}^{(2)} + \widetilde{W}_{tr(2)}^{(2)} & dla \quad \tau \geq 0 \end{cases} \tag{4.3}$$

where, for $\tau < 0$:

$$\begin{cases} E_{KP(1)}^{(1)} = \dfrac{1}{2}\displaystyle\int_{-\infty}^{0}\left[EI_1\left(\frac{\partial^2 w_{st}^{(1)}(s_1,\tau)}{\partial s_1^2}\right)^2 + m_1\left(\frac{\partial w_{st}^{(1)}(s_1,\tau)}{\partial \tau}\right)^2 + c_1\frac{\partial w_{st}^{(1)}(s_1,\tau)}{\partial \tau} + U_1 w_{st}^{(1)}(s_1,\tau)\right]dx \\ E_{KP(2)}^{(1)} = \dfrac{1}{2}\displaystyle\int_{0}^{\infty}\left[EI_2\left(\frac{\partial^2 w_{st}^{(2)}(s_2,\tau)}{\partial s_2^2}\right)^2 + m_2\left(\frac{\partial w_{st}^{(2)}(s_2,\tau)}{\partial \tau}\right)^2 + c_2\frac{\partial w_{st}^{(2)}(s_2,\tau)}{\partial \tau} + U_2 w_{st}^{(2)}(s_2,\tau)\right]dx \end{cases} \tag{4.4}$$

and the work performed by the force set is

(4.5)
$$\begin{cases} \widetilde{W}^{(1)}_{tr(1)} = P\int\limits_{-\infty}^{0}\int\limits_{-\infty}^{0}\delta(s_1)w^{(1)}_{tr(1)}(x,\tau)dxd\tau \\ \widetilde{W}^{(1)}_{tr(2)} = P\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\delta(s_2)w^{(1)}_{tr(2)}(x,\tau)dxd\tau \end{cases}$$

The integration with respect to x is performed in the load self-coordinate system and with respect to τ it is performed for each position of the load.

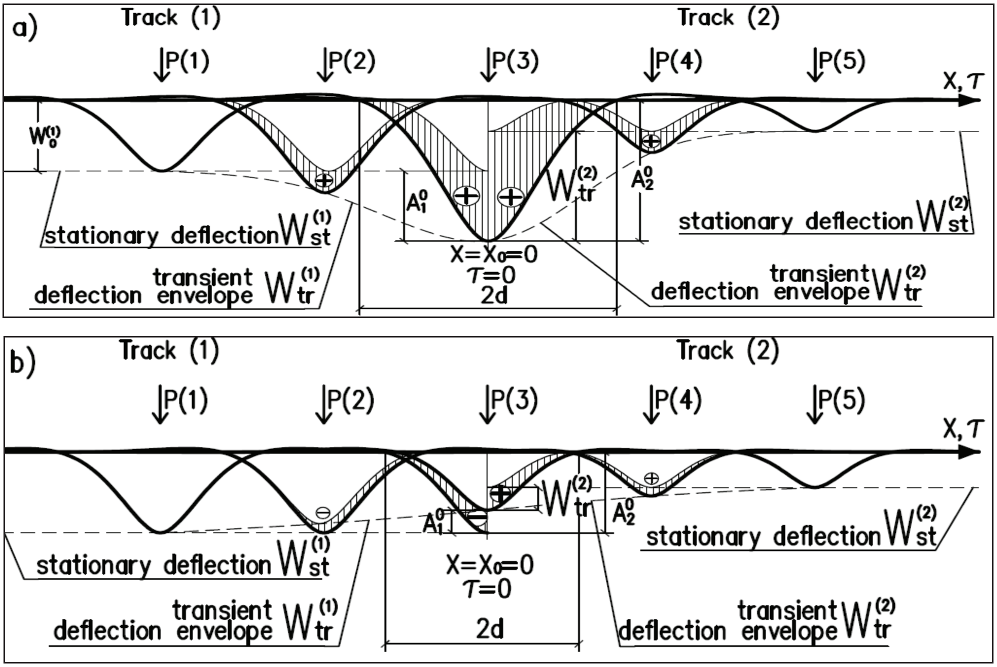


Fig. 4. Integration procedure of transient deflections: a) in the cases of increased deflections at the junction point, b) in the cases of decreased deflections at the junction point. $P(j)$, j = 1,2,3,4,5 denote the subsequent load positions

The transient energy is then calculated as:

(4.6)
$$E_{tr} = \widetilde{W}^{(1)}_{tr(1)} + \widetilde{W}^{(1)}_{tr(2)} + \widetilde{W}^{(2)}_{tr(1)} + \widetilde{W}^{(2)}_{tr(2)}$$

The transient deflection is assumed as

(4.7)
$$w^{(1)}_{tr(1)}(x,t) = w^{(1)}_{tr(1)}(s_1, x_s) = A_1(x_s, s_1)\cdot e^{-b_1|s_1|}\left[\cos(a_1 s_1) + \sin(a_1|s_1|)\right],$$

in which

(4.8) $$A_1(x_s, s_1) = A_1^0(s_1) \cdot e^{-b_1|x_s|}\left[\cos(a_1 x_s) + \sin(a_1 |x_s|)\right]$$

where: $x_s = \lambda_1 x$ is the dimensionless distance from the load position to the junction point and constant $A_1^0(s_1)$ is given by Eq.(3.4, 3.6, 3.7, 3.8). Substitution of Eq.(4.7, 4.8) into Eq.(4.5) for $\tau < 0$, and assuming the symmetry of the transition effect over $2d$ (Figure 4), yields the densities of energy in case of no damping:

(4.9) $$W_{tr(1)}^{(1)} = \frac{P\,A_1^0(s_1)}{2d}\left[\int_{-\infty}^{0}\delta(s_1)\,e^{-b_1|s_1|}\left[\cos(a_1 s_1) + \sin(a_1|s_1|)\right]ds_1 \int_{-\infty}^{0} e^{-b_1|x_s|}\left[\cos(a_1 x_s) + \sin(a_1|x_s|)\right]dx_s\right]$$

(4.10) $$W_{(2)}^{(1)} = \frac{P\,A_2^0(s_2)}{2d}\left[\int_{0}^{\infty}\delta(s_2)\,e^{-b_2|s_2|}\left[\cos(a_2 s_2) + \sin(a_2|s_2|)\right]ds_2 \int_{0}^{\infty} e^{-b_2|x_s|}\left[\cos(a_2 x_s) + \sin(a_2|x_s|)\right]dx_s\right]$$

where: a_2, b_2 are defined for track (2) and $W_{(2)}^{(1)}$ is the work done on track (2) due to the action of the force on track (1). Because the integrals are dimensionless $W_{tr(1)}^{(1)}$, $W_{(2)}^{(1)}$ are expressed in $[N \cdot m / m_{track}]$. Because $\int_{-\infty}^{0}\delta(s_1) f(s_1) ds_1 = \frac{1}{2} f(0) = \frac{1}{2}$, we have:

(4.11) $$\begin{cases} W_{tr(1)}^{(1)} = \dfrac{1}{2}\dfrac{P\,A_1^{0(1)}(s_1)}{2d}\left(\dfrac{a_1^1 + b_1}{(a_1^1)^2 + (b_1)^2}\right) \\ W_{(2)}^{(1)} = \dfrac{1}{2}\dfrac{P\,A_2^{0(1)}(s_2)}{2d}\left(\dfrac{a_2^1 + b_2}{(a_2^1)^2 + (b_2)^2}\right) \end{cases} \quad \text{for} \quad \tau < 0$$

where $A_1^{0(1)}(s_1)$ and $A_2^{0(1)}(s_2)$ are introduced to indicate that the load is on track (1). In the same way the work is found for the load position on track (2):

(4.12) $$\begin{cases} W_{(1)}^{(2)} = \dfrac{1}{2}\dfrac{P\,A_1^{0(2)}(s_1)}{2d}\left(\dfrac{a_1^1 + b_1}{(a_1^1)^2 + (b_1)^2}\right) \\ W_{tr(2)}^{(2)} = \dfrac{1}{2}\dfrac{P\,A_2^{0(2)}(s_2)}{2d}\left(\dfrac{a_2^1 + b_2}{(a_2^1)^2 + (b_2)^2}\right) \end{cases} \quad \text{for} \quad \tau \geq 0$$

The amplitudes are now given by:

$$
(4.13)\qquad \begin{cases} A_1^{0(2)}(s_1) = w_0^{(2)} \dfrac{\dfrac{1}{a_2 b_2} + 1}{\left(\dfrac{\lambda_1}{\lambda_2}\right)^2 \dfrac{\left(a_1^2 + b_1^2\right) b_2}{\left(a_2^2 + b_2^2\right) b_1} + 1} \\ A_2^{0(2)}(s_2) = A_1^0 - \dfrac{w_0^{(2)}}{a_2 b_2} \end{cases} \qquad \text{for } \tau \geq 0
$$

The average energy density over the transition effect is

$$
(4.14)\quad E_{tr} = \frac{P}{2d}\left[\frac{\left[A_1^{0(1)} + \left(A_1^{0(2)} - w_0^{(2)}\right)\right]}{2}\left(\frac{a_1^1 + b_1}{\left(a_1^1\right)^2 + \left(b_1\right)^2}\right) + \frac{\left[\left(A_2^{0(1)} - w_0^{(1)}\right) + A_2^{0(2)}\right]}{2}\left(\frac{a_2^1 + b_2}{\left(a_2^1\right)^2 + \left(b_2\right)^2}\right)\right]
$$

For the sake of simplicity arguments (s_1), (s_2) are omitted. For small speeds $(a_i = b_i \approx 1)$ we get:

$$
(4.15)\qquad E_{tr} = \frac{P}{2d}\left[\frac{\left(A_1^{0(1)} + A_1^{0(2)} - w_0^{(2)}\right)}{2} + \frac{\left(A_2^{0(1)} - w_0^{(1)} + A_2^{0(2)}\right)}{2}\right]
$$

Due to the symmetry with respect to $x = x_0$, the average values of the above amplitudes have to be equal to each other and so the following amplitudes are equal $A_1^{0(1)} = A_1^{0(2)}$ and $A_2^{0(1)} = A_2^{0(2)}$. Because these amplitudes are constant during the force motion we get one formula:

$$
(4.16)\qquad E_{tr} = \frac{P}{2d}\left[A_1^0 + A_2^0 - w_0^{(2)}\right] \quad \text{for } \begin{cases} \tau < 0 \\ \tau \geq 0 \end{cases}
$$

where notation A_1^0 and A_2^0 is used again to indicate that it is enough to consider only the first position of the load (for $\tau < 0$) to get the transient energy estimation. If damping is introduced, Eq. (3.4) should be used, and for the varied mass Eq. (3.8, 3.9) should be applied.

5. Energy calculations in cases of abrupt changes in beam-foundation parameters

Special cases of the change in the beam-foundation parameters are:

1) Uniform parameters: $EI_1 = EI_2$, $U_1 = U_2$, $c_1 = c_2$, $m_1 = m_2$, then we get

$$\lambda_1 = \lambda_2 \;\Rightarrow\; \begin{cases} A_1^0 = 0 \\ A_2^0 = w_0^{(1)} = w_0^{(2)} \end{cases} \tag{5.1}$$

So $E_{tr}^{(2)} = E_{KP(2)}^{(1)}$ – no additional action on the foundation will be exerted.

2) Change in bending stiffness: $EI_1 < EI_2$, $U_1 = U_2$, $c_1 = c_2$, $m_1 = m_2$, then we get

$$\lambda_1 > \lambda_2 \;\Rightarrow\; \begin{cases} A_1^0 > 0 \\ A_2^0 - w_0^{(2)} > 0 \end{cases} \tag{5.2}$$

So $E_{tr} > 0$, which means that more energy will flow to the foundation than in the stationary case.

3) Change in foundation stiffness: $EI_1 = EI_2$, $U_1 < U_2$, $c_1 = c_2$, $m_1 = m_2$, then we get

$$\lambda_1 < \lambda_2 \;\Rightarrow\; \begin{cases} A_1^0 < 0 \\ A_2^0 - w_0^{(2)} > 0 \end{cases} \tag{5.3}$$

So the energy balance is zero: $E_{tr} = 0$ – this means that energy on track (1) decreases at the same rate as the energy on track (2) increases. This is due to $A_1^0 = A_2^0 - w_0^{(1)}$ and symmetry with respect to $x = x_0 = 0$, which results in $A_2^0 = A_1^0 - w_0^{(2)}$.

4) Change in damping: $EI_1 = EI_2$, $U_1 = U_2$, $c_1 < c_2$, $m_1 = m_2$. Using Eq.(3.4) we get:

$$A_1^0 \approx \left(\frac{2 - \alpha_1\beta_1}{\left(\dfrac{1 - \alpha_2\beta_2}{1 - \alpha_1\beta_1} + 1 \right)} - 1 \right) \frac{1}{1 - \alpha_1\beta_1} \cdot w_0^{(1)} \tag{5.4}$$

The above may be written as $A_1^0 = \psi \cdot w_o^{(1)}$. Because the coefficient of viscous damping is $c = 2\beta\sqrt{mU} = \beta \cdot c_{cr}$, in which $c_{cr} = 2\sqrt{mU}$, so

(5.5) $$c_1 < c_2 \quad \Rightarrow \quad \beta_1 < \beta_2$$

If we assume for the simplicity that damping does not influence the critical speed we may approximately take $\alpha_1 = \alpha_2$. Therefore we have:

(5.6) $$c_1 < c_2 \quad \Rightarrow \quad \begin{cases} \psi > 1 \Rightarrow \ A_1^0 > w_o^{(1)} \\ A_2^0 > w_o^{(2)} \end{cases}$$

where: $w_o^{(1)} = w_o^{(2)} = w_o$. This result means that in the case of the change in viscous damping from a smaller coefficient (here c_1) to a larger coefficient (here c_2), the rail deflection on the track with smaller coefficient will be increased in comparison to the deflection for the system with uniform parameters. Therefore energy $E_{tr} > 0$. Moreover, the energy will also increase in the case of inverse configuration of coefficients or if the direction of load motion is changed.

5) Change in mass: $EI_1 = EI_2$, $U_1 = U_2$, $c_1 = c_2$, $m_1 < m_2$, then we get

(5.7) $$m_1 < m_2 \Rightarrow \begin{cases} A_1^0 < 0 \\ A_2^0 > w_0^{(2)} \end{cases}$$

This result means that the rail deflection will increase on that part of the track which has a greater mass. Therefore energy $E_{tr} > 0$. This will also hold in the case of inverse configuration of masses or the direction of load motion.

6) change in foundation characteristics in the central zone of the bending wave and in the lift-off zone: $U_1^{central} = U_2^{central}$, $U_1^{lift-off} < U_2^{ift-off}$. Other parameters are constant. We have (Sołkowski [15]):

(5.8) $$w_{trans} = \left(\frac{2}{\left(\frac{\kappa_s^*}{\kappa_w^*} \right)^2 + 1} - 1 \right) w_0^{(1)}$$

where: κ_s^* and κ_w^* are functions of the ratio $U_1^{lift-off} / U_2^{ift-off}$. If this is lesser than unity then $\kappa_s^* < \kappa_w^*$ so the transient deflection $w_{trans} > 0$ and the energy $E_{tr} > 0$. This is similar to case 2 above.

Below selected calculations are presented using the derived formulas. The data are collected in Table 1. Calculations are carried out for varied parameters of the beam-foundation system, one at a time leaving the multipliers for other parameters equal to one.

Table 1

Data of the tracks per unit length and per one rail with the support distance of 0.6m.
Speed of the load v = 0÷200 km/h

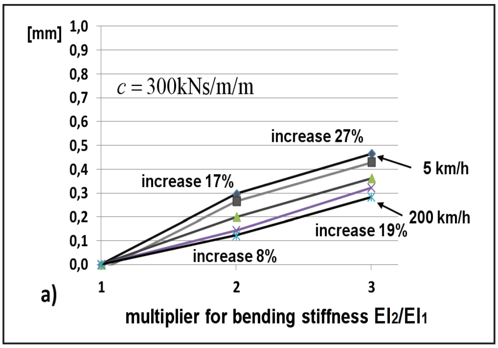
Parameters		Track (1) (*)	Track (2)		
			Multiplier		
Bending stiffness	$EI\,[MN \cdot m^2]$	6.42	1	2	3
Rail foundation modulus, Czyczuła [16], Bukowski [17]	$U\,[MPa]$	30	1	2	3
Unit mass	$m\,[kg]$	327	1(***)	0.183 (**)	2
Viscous damping, Czyczuła [16] (critical damping)	$c\,[kNs/m/m]$ c_{kr}	30 (626)	1	5	10

(*) Track 60E1, SB-4, PS-94 (mass m = 60 + 320/2/0.6 = 327 kg per meter)

(**) Mass of the rail, (DF) or bridge sleeper structure (m = 60 kg per meter),

(***) Mass of the rail and support block (EBS), (approximately m = 60 + 260/0.6 = 493 kg per meter)

Calculations of the case with the change in the bending stiffness are shown in Figure 5. The rail deflection on both tracks increases. On the part with smaller bending stiffness it increases by about 30% and on the part with greater stiffness it increases by about 60%. Viscous damping influences the results considerably. As was shown in (Sołkowski [15]), two cases of this type of behavior are possible: one with "real" change in bending stiffness due to the use of additional rails as in the RHEDA 2000 structure, or "apparent" change in the case of bi-linear foundation characteristics in the central zone and in the lift-off zone, which may have different (possibly adverse) characteristics for the tracks connected (e.g. slab track and ballasted track).



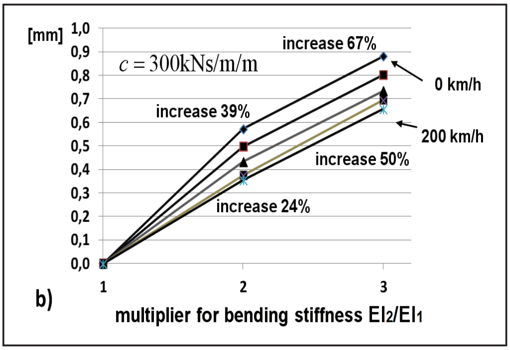


Fig. 5. Influence of the change in bending stiffness on rail deflections: a) for track (1), b) for track (2), deflections are $w_0^{(1)} = 1.733mm$, $w_0^{(2)} = 1.457mm$ (multiplier=2), $w_0^{(2)} = 1.317mm$ (multiplier =3). Damping cases: $c_1 = c_2 = c = \beta \cdot c_{cr} = 0,48 \cdot c_{cr} = 300\ kN$ s/m/m

This result means that the beam-foundation system seems stiffer if the stiffness in the lift-off region is greater. There is some analogy to the beam with unmovable supports, which is stiffer than a simply supported beam of the same length because the supports can carry bending moments and decrease the maximum bending moment in the middle of the beam.

The influence of the foundation stiffness is shown in Figure 6. As can be observed the rail deflection on track (1) decreases and on track (2) (which has stiffer foundation) increases.

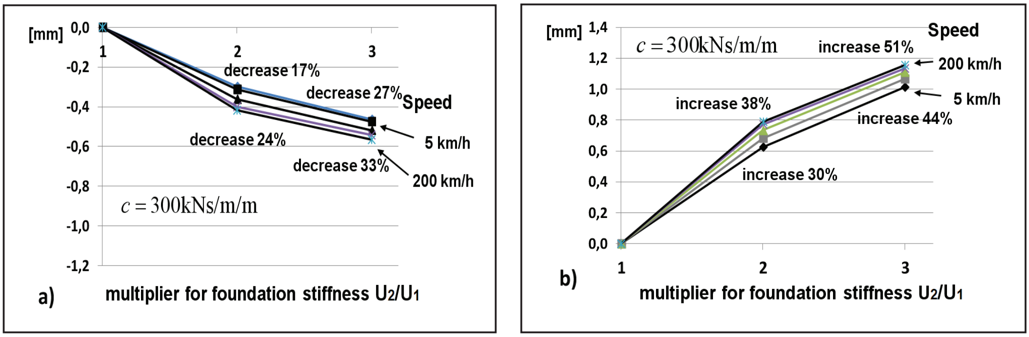


Fig. 6. Influence of the change in foundation stiffness: a) for track (1), b) for track (2), rail deflections are $w_0^{(1)} = 1.733mm$, $w_0^{(2)} = 1.030mm$ (multiplier =2), $w_0^{(2)} = 0.760mm$ (multiplier =3)

The influence of the change in the viscous damping is shown in Figure 7. It reaches 7% on the track with smaller damping and 18% on the track with greater damping (at the level of 80% of critical damping). In a normal case (40% of critical damping) and relatively small speeds (120km/h) this influence, in terms of increased rail deflection, reaches 2-6%.

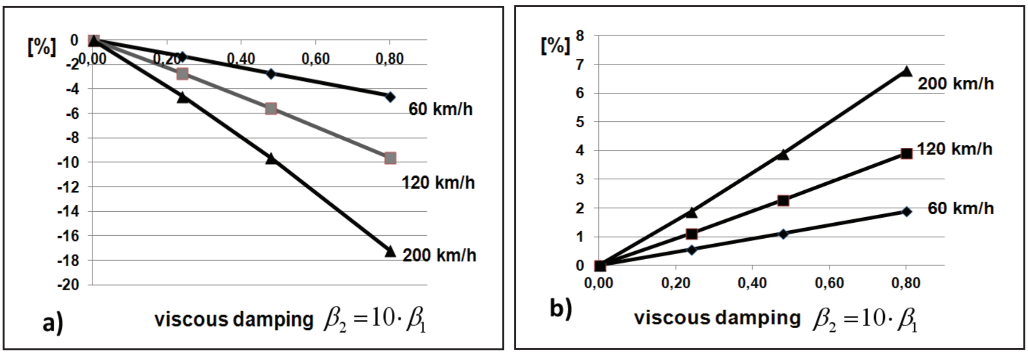


Fig. 7. Influence of damping on rail deflections: a) increase for track (1), b) decrease for track (1), assumed coefficients $c_1 = 0 - 50\ kNs/m^2$, $c_2 = 0 - 500\ kNs/m^2$. If $c_1 = c_2 = 0$ then $w_0^{(1)} = w_0^{(2)} = 1.733mm$

The influence of the change in the mass of the beam was determined in (Sołkowski [15]) at a level of a few percent in terms of increased deflections for speeds in the order of 200 km/h.

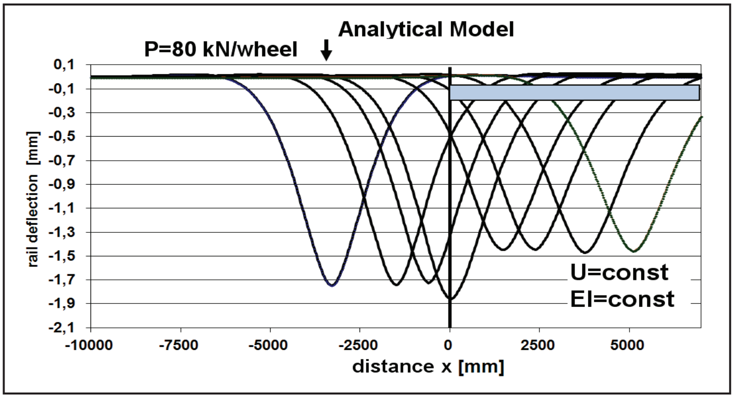


Fig. 8. Rail deflections due to the change in viscous damping (v = 200 km/h, $\beta_2 = 0{,}8 = 10 \cdot \beta_1$). Calculated deflections are respectively 1.733/1.856/1.473 mm

The energy density in the stationary case for track (1) and (2), taking into account Eq.(2.2), is:

$$\begin{cases} e(s_1,\tau) = \dfrac{P}{d} w_0^{(1)} \cdot w(s_1,\tau) \quad \text{for } \tau < 0 \text{ - track (1)} \\ e(s_2,\tau) = \dfrac{P}{d} w_0^{(2)} \cdot w(s_2,\tau) \quad \text{for } \tau \ge 0 \text{ - track (2)} \end{cases} \tag{5.9}$$

The averaged energy density over the distance $2d$ is:

$$\tilde{e} = \frac{1}{2}[e(s_1,\tau) + e(s_2,\tau)] \tag{5.10}$$

and the relative increments of transient energy is then defined by

$$\Delta E = \frac{E_{tr}}{\tilde{e}} \quad \text{for} \begin{cases} \tau < 0 \\ \tau \ge 0 \end{cases} \tag{5.11}$$

In the case of uniform beam-foundation parameters $\Delta E = 0$. In Figure 9 energy increments Eq.(5.11) are shown for a few cases of changes in the beam-foundation parameters. Using Eq. (5.9, 5.10) the relative energy increment Eq. (5.11) can be expressed by:

(5.12) $$\Delta E = \frac{\eta_1 w_{st}^{(1)} + \eta_2 w_{st}^{(2)}}{w_{st}^{(1)} + w_{st}^{(2)}} \quad \text{for} \begin{cases} \tau < 0 \\ \tau \geq 0 \end{cases}$$

where: η_1, η_2 are non-linear functions of λ_1, λ_2, c_1, c_2, m_1 and m_2 so that $w_{tr}^{(n)} = \eta_n w_{st}^{(n)}$ at $x = x_0$ ($n = 1,2$). The functions may take on (+/-) values. Introducing a new variable which is the overall stiffness ratio between the connected beam-foundation systems for low frequency vibrations:

(5.13) $$\kappa = \frac{w_{st}^{(2)}}{w_{st}^{(1)}}$$

we obtain the following expression for transient energy increments:

(5.14) $$\Delta E = \frac{\eta_1 + \eta_2 \kappa}{1 + \kappa} \quad \text{for} \begin{cases} \tau < 0 \\ \tau \geq 0 \end{cases}$$

In the stationary case $A_1^0 = 0$, $A_2^0 = w_0^{(2)} \Rightarrow w_{tr}^{(1)} = w_{tr}^{(2)} = 0 \Rightarrow \eta_1 = \eta_2 = 0 \Rightarrow \Delta E = 0$. An example of calculation of energy Eq.(5.11) or Eq.(5.14) is given in Figure 9.

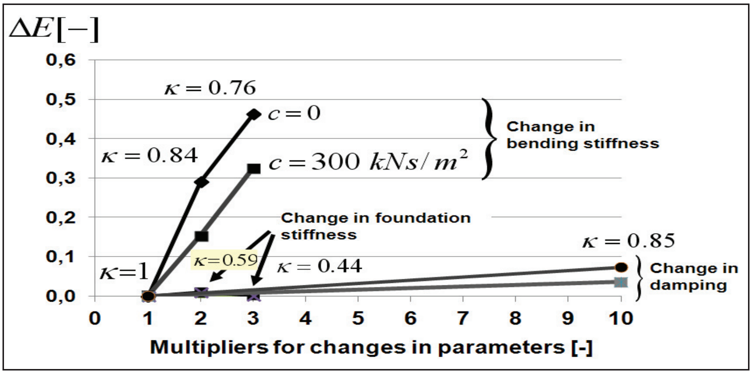


Fig. 9. Relative transient energy increment Eq. (5.11, 5.14) depending on the change in the beam-foundation parameters: multipliers for bending stiffness and foundation stiffness = 1,2,3; multipliers for damping = 1,10 (Table 1); only one parameter is varied at a time. For stationary case, the stiffness ratio Eq. (5.13) is $\kappa = 1$. The constant wheel load is $P = 80$ kN, speed $v = 200$ km/h

6. Summary

As a result of the calculations using the proposed model of transient energy estimation the following conclusions may be formulated:

- influence of the bending stiffness change due to "real" (structural) change or "apparent" (beam-foundation constraints) change reaches 30-60% in terms of increased rail deflection and the transient energy increment exceeds 30% ($\Delta E > 0.3$ for v = 200 km/h) relative to the stationary energy, however the constraint problem needs further detailed investigation,
- influence of the foundation stiffness is almost neutral for the energy of the beam-foundation,
- influence of the viscous damping in terms of increased rail deflection reaches 7-18%, and the transient energy is less than 10% of the stationary energy ($\Delta E < 0.1$),
- since, as was shown in (Sołkowski [15]), the influence of mass change on rail deflection is less than 10% it is expected that the transient energy will also be less than 10% of the stationary energy.

The estimated frequencies are due to the speed of the bending wave travelling with the load (up to 10 Hz for v = 200 km/h) (Fröhling [24]). As was shown in (Auersch [25,26]), the track response supported by stiff foundation characterized by large surface wave, velocities may be treated as quasi-static. The relevance of the results obtained above for the secondary transition effect may be shown using a settlement model. Such models are usually based on the calculation of the under-sleeper stresses (Dahlberg [27], Iwnicki [28], Lundqvist [29]) or the energy flux into the subgrade (Czyczuła[17]). In (Dahlberg [27]) the ballast is represented by non-linear springs, and the settlement results form accumulated plastic strain. Analyses of special cases of track settlements near bridges are, however, very rare (Hunt[13], Jaup [30]). The latter one gives an analytical formula for settlement near a bridge abutment. In (Demharter [31]) we have:

(6.1) $$S_p = \eta \cdot p \cdot \Delta N + 3.04 \cdot p^{1.21} \ln N$$

where: ΔN – number of axles in the first phase of settlement, p – averaged under-sleeper stress, η – constant. In (Sato[32]) there is an empirical formula given:

(6.2) $$S_J = 2.09 \cdot 10^{-3} \cdot T^{0.31} \cdot V^{0.98} \cdot M^{1.1} \cdot L^{0.21} \cdot P^{0.26}$$

where: S_J – average settlement (mm/100 days), T – annual gross tonnage borne, V – average speed [km/h], M, L – structural and track factors, P – subgrade factor (1 = good, 10 = bad). In (Czyczuła [7]) the settlement results from under-sleeper stresses $\sigma_{\max}$ and the sleeper vertical speed v_p^{RMS}:

(6.3) $$S_C = \alpha(N) \cdot \left(\sigma_{\max} \cdot v_p^{RMS}\right)^{\beta(N)}$$

where: functions $\alpha(N)$ and $\beta(N)$ depend on the number of axles N.

The sleeper vibrations may be determined in time domain or in frequency domain on the basis of the low frequency (quasi-static) bending lines for each load position near the junction point after expanding them into the Fourier series (Bogacz [33]). To obtain a more reliable approach, higher frequency vibrations due to periodical track stiffness change on discrete sleeper support (Shaer [34]) or due to under-sleeper gaps (Grassie [35]), or wheel-rail irregularities should be included in the track model and also in the settlement model.

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