Opuscula Math. 35, no. 1 (2015), 21–36
http://dx.doi.org/10.7494/OpMath.2015.35.1.21 Opuscula Mathematica

ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF A SEMILINEAR DIRICHLET PROBLEM IN THE ANNULUS

Safa Dridi and Bilel Khamessi

Communicated by Vicentiu D. Radulescu

Abstract. In this paper, we establish existence and asymptotic behavior of a positive classical solution to the following semilinear boundary value problem:

$$-\Delta u = q(x)u^{\sigma} \text{ in } \Omega, \quad u_{|\partial\Omega} = 0.$$

Here Ω is an annulus in $\mathbb{R}^n$, $n \geq 3$, $\sigma < 1$ and q is a positive function in $\mathcal{C}^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$, satisfying some appropriate assumptions related to Karamata regular variation theory. Our arguments combine a method of sub- and supersolutions with Karamata regular variation theory.

Keywords: asymptotic behavior, Dirichlet problem, Karamata function, subsolution, supersolution.

Mathematics Subject Classification: 31C15, 34B27, 35K10.

1. INTRODUCTION

The general nonlinear following problem

$$\begin{cases} -\Delta u = q(x)f(u), & x \in \Omega, \\ u > 0 \quad \text{in } \Omega, & u_{|\partial\Omega} = 0, \end{cases}$$

has been extensively studied by many authors in general bounded and unbounded domains Ω in $\mathbb{R}^n$, $n \geq 3$, where q is a nonnegative continuous function in Ω and f is a nonnegative function in $(0, +\infty)$ allowed to be singular at $u = 0$ or satisfying $f(0) = 0$. For related results, we refer to [2–7, 9–12, 14, 15, 18]. In this paper, we undertake the study of the following semilinear Dirichlet problem

$$\begin{cases} -\Delta u = q(x)u^{\sigma}, & x \in \Omega, \\ u > 0 \quad \text{in } \Omega, & u_{|\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega := \{x \in \mathbb{R}^n; 0 < a < |x| < b < \infty\}$ is an annulus in $\mathbb{R}^n$, $n \geq 3$, $\sigma < 1$ and q is a positive function in $\mathcal{C}^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$, which may be singular at the boundary of Ω. Unlike the case of general domains, there is little literature dealing with problems of type (1.1), in an annulus of $\mathbb{R}^n$. We refer to [1,8,13,17,19]. Namely, for $0 < \sigma < 1$ and $q(x) = g(|x|)$, where g is a nonnegative function in $\mathcal{C}^1((0,\infty))$ and positive in $[a,b]$ such that $\lim_{r\to+\infty} \inf g(r) > 0$, Arcoya proved in [1], by using variational methods that problem (1.1) has at least one positive radial classical solution.

In this work, by using potential theory tools, we prove the existence of a unique positive classical solution of problem (1.1) for a larger class of functions q and we extend the result of Arcoya to the case $\sigma < 1$. Further, motivated by the results of [14], and by applying Karamata regular variation theory, we establish sharp estimates of the solution of problem (1.1).

To simplify our statements in this paper, we need some notation. We shall use $\mathcal{K}$ to denote the set of Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c \exp\left(\int_t^{\eta} \frac{z(s)}{s} ds\right),$$

for some $\eta > 0$, where $c > 0$ and z is a continuous function on $[0,\eta]$ with $z(0) = 0$.

Remark 1.1. It is obvious that $L \in \mathcal{K}$ if and only if L is a positive function in $\mathcal{C}^1((0,\eta])$ such that

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

Let $d := \operatorname{diam}(\Omega)$ and $\eta > d$. For $\lambda \leq 2$, $\sigma < 1$ and $L \in \mathcal{K}$ such that $\int_0^{\eta} t^{1-\lambda} L(t) dt < \infty$, we define the function $\Psi_{L,\lambda,\sigma}$ on $(0,d)$ by

$$\Psi_{L,\lambda,\sigma}(t) := \begin{cases} 1, & \text{if } \lambda < 1+\sigma, \\ \left(\int_t^{\eta} \frac{L(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 1+\sigma, \\ (L(t))^{\frac{1}{1-\sigma}}, & \text{if } 1+\sigma < \lambda < 2, \\ \left(\int_0^{t} \frac{L(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 2. \end{cases} \tag{1.2}$$

In what follows, we refer to the potential of a nonnegative measurable function f defined on Ω by

$$Vf(x) = \int_{\Omega} G(x,y) f(y) dy, \quad x \in \Omega,$$

where $G(x,y)$ is the Green's function of $(-\Delta)$ in Ω. We recall that if $f \in \mathcal{C}^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$, then $Vf \in \mathcal{C}^{2,\gamma}_{loc}(\Omega)$ and satisfies

$$-\Delta(Vf) = f \text{ in } \Omega. \tag{1.3}$$

In particular, if f is a radial positive function in $C(\Omega)$ such that

$$\int_a^b (b-r)(r-a)f(r)dr < \infty, \tag{1.4}$$

then the potential Vf is the unique positive classical solution of the following problem

$$\begin{cases} \frac{1}{r^{n-1}}(r^{n-1}u'(r))' = -f(r), & a < r < b, \\ u > 0 \text{ in } \Omega, & u(a) = u(b) = 0, \end{cases}$$

given by

$$Vf(r) = \frac{1}{(n-2)(a^{2-n}-b^{2-n})} \int_a^b t^{n-1} f(t)(\max(r,t)^{2-n} - b^{2-n})(a^{2-n} - \min(r,t)^{2-n})dt. \tag{1.5}$$

In the sequel, for two nonnegative functions f and g defined on a set S, the notation

$$f(x) \approx g(x), \quad x \in S,$$

means that there exists $c > 0$ such that

$$\frac{1}{c} f(x) \leq g(x) \leq cf(x) \quad \text{for all} \quad x \in S.$$

Now, let us introduce our hypothesis on the function q.

(H) q is a positive function in $\mathcal{C}^{\gamma}_{loc}(\Omega)$, $0 < \gamma < 1$, satisfying $x \in \Omega$,

$$q(x) \approx (|x|-a)^{-\lambda_1}(b-|x|)^{-\lambda_2} L_1(|x|-a)L_2(b-|x|),$$

where $\lambda_1, \lambda_2 \leq 2$ and $L_1, L_2 \in \mathcal{K}$ such that for $i \in \{1, 2\}$,

$$\int_0^\eta t^{1-\lambda_i} L_i(t)dt < \infty.$$

Remark 1.2. For $i \in \{1, 2\}$, we need to verify condition $\int_0^\eta t^{1-\lambda_i} L_i(t)dt < \infty$, in hypothesis (H), only if $\lambda_i = 2$. This is due to Lemma 2.1 below.

A typical example of a function satisfying (H) is given below.

Example 1.3. *Let q be the function defined in Ω by*

$$q(x) = (|x|-a)^{-\lambda_1}(b-|x|)^{-\lambda_2} \left(\log\left(\frac{3(b-a)}{|x|-a}\right) \log\left(\frac{3(b-a)}{b-|x|}\right) \right)^{-\mu},$$

where the real numbers λ_1, $\lambda_2 \leq 2$, and μ satisfy one of the following conditions:

(i) $\mu \in \mathbb{R}$ *if* $\lambda_1, \lambda_2 < 2$,
(ii) $\mu > 1$ *if* $\lambda_1 = 2$ *or* $\lambda_2 = 2$.

Then, if one of the above conditions is satisfied, the function q *satisfies* (H).

Now, we are ready to state our main result.

Theorem 1.4. *Assume* (H). *Then problem* (1.1) *has a unique positive classical solution satisfying for* $x \in \Omega$,

$$u(x) \approx \theta(x), \tag{1.6}$$

where θ *is the function defined on* Ω *by*

$$\theta(x) := (|x|-a)^{\min(1,\frac{2-\lambda_1}{1-\sigma})}(b-|x|)^{\min(1,\frac{2-\lambda_2}{1-\sigma})}\Psi_{L_1,\lambda_1,\sigma}(|x|-a)\Psi_{L_2,\lambda_2,\sigma}(b-|x|). \tag{1.7}$$

The proof of Theorem 1.4 is based on the sub-supersolution method and potential theory tools.

The paper is organized as follows. In Section 2, we state some already known results on functions in $\mathcal{K}$, useful for our study and we give estimates on some potential functions. Section 3 deals with the proof of Theorem 1.4. The last section is reserved for some applications. Throughout, the letter c will denote a generic positive constant which may vary from line to line.

2. PRELIMINARY RESULTS

2.1. TECHNICAL LEMMAS

In this section, we recapitulate some properties of functions belonging to the class $\mathcal{K}$ which need to be used in the paper. Applying Karamata's theorem, we get the following.

Lemma 2.1 ([16])**.** *Let* $\gamma \in \mathbb{R}$ *and* L *be a function in* $\mathcal{K}$ *defined on* $(0,\eta]$, $\eta > d$. *We have:*

(i) *If* $\gamma > -1$, *then* $\int_0^\eta s^\gamma L(s)ds$ *converges and*

$$\int_0^t s^\gamma L(s)ds \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

(ii) *If* $\gamma < -1$, *then* $\int_0^\eta s^\gamma L(s)ds$ *diverges and*

$$\int_t^\eta s^\gamma L(s)ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

Lemma 2.2 ([5,16])**.** *Let* $L_1, L_2 \in \mathcal{K}$ *be defined on* $(0,\eta]$ *for some* $\eta > d$, $p \in \mathbb{R}$ *and* $\varepsilon > 0$. *Then we have the following assertions:*

(i) $L_1L_2 \in \mathcal{K}$, $L_1^p \in \mathcal{K}$.
(ii) $\lim\limits_{t\to 0^+} t^\varepsilon L_1(t) = 0$.
(iii) $\lim\limits_{t\to 0^+} \dfrac{L_1(t)}{\int_t^\eta \frac{L_1(s)}{s}ds} = 0$.
(iv) *If* $\int\limits_0^\eta \frac{L_1(s)}{s}ds$ *converges, then* $\lim\limits_{t\to 0^+} \dfrac{L_1(t)}{\int_0^t \frac{L_1(s)}{s}ds} = 0$.

Remark 2.3. Let $L \in \mathcal{K}$ defined on $(0, \eta]$, $\eta > d$. Then using Remark 1.1 and Lemma 2.2 (iii), we deduce that

$$t \longmapsto \int\limits_t^\eta \frac{L(s)}{s}ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}ds$ converges, we obtain by Remark 1.1 and Lemma 2.2 (iv) that

$$t \longmapsto \int\limits_0^t \frac{L(s)}{s}ds \in \mathcal{K}.$$

Lemma 2.4. *Let* $\lambda \le 2$ *and* $L \in \mathcal{K}$ *defined on* $(0, \eta]$, $\eta > d$, *be such that* $\int_0^\eta t^{1-\lambda}L(t)dt < \infty$. *For* $t \in (0, \frac{b-a}{4})$, *we put*

$$I(t) = \int\limits_0^t s^{1-\lambda}L(s)ds \quad \text{and} \quad J(t) = t\left(1 + \int\limits_t^{\frac{b-a}{2}} s^{-\lambda}L(s)ds\right).$$

Then we have

$$I(t) \approx \begin{cases} t^{2-\lambda}L(t), & if\ \lambda < 2, \\ \int\limits_0^t \frac{L(s)}{s}ds, & if\ \lambda = 2, \end{cases}$$

and

$$J(t) \approx \begin{cases} t, & if\ \lambda < 1, \\ t\int\limits_t^{\frac{b-a}{2}} \frac{L(s)}{s}ds, & if\ \lambda = 1, \\ t^{2-\lambda}L(t), & if\ 1 < \lambda \le 2. \end{cases}$$

Proof. By using Lemma 2.1 (i), we have for $t \in (0, \frac{b-a}{4})$,

$$I(t) \approx \begin{cases} t^{2-\lambda}L(t), & if\ \lambda < 2, \\ \int\limits_0^t \frac{L(s)}{s}ds, & if\ \lambda = 2. \end{cases}$$

Next, to estimate J, we distinguish the following cases.

Case 1. If $\lambda < 1$, then applying Lemma 2.1 (i), we have $\int_0^\eta s^{-\lambda} L(s)ds < \infty$. Thus

$$J(t) \approx t \quad \text{for} \quad t \in \Big(0, \frac{b-a}{4}\Big).$$

Case 2. If $\lambda = 1$, then using the fact that $\int_0^\eta \frac{L(s)}{s} ds < \infty$, we get

$$1 + \int_t^{\frac{b-a}{2}} \frac{L(s)}{s} ds \approx \int_t^{\frac{b-a}{2}} \frac{L(s)}{s} ds \quad \text{for} \quad t \in \Big(0, \frac{b-a}{4}\Big).$$

This yields

$$J(t) \approx t \int_t^{\frac{b-a}{2}} \frac{L(s)}{s} ds \quad \text{for} \quad t \in \Big(0, \frac{b-a}{4}\Big).$$

Case 3. If $1 < \lambda \leq 2$, then by Lemma 2.1 (ii), we obtain that $\int_0^\eta s^{-\lambda} L(s)ds$ diverges and $\int_t^\eta s^{-\lambda} L(s)ds \approx t^{1-\lambda} L(t)$. So, applying Lemma 2.1 (i), we have

$$J(t) \approx t \int_t^{\frac{b-a}{2}} s^{-\lambda} L(s)ds \approx t^{2-\lambda} L(t) \quad \text{for} \quad t \in \Big(0, \frac{b-a}{4}\Big). \qquad \square$$

2.2. ASYMPTOTIC BEHAVIOR OF SOME POTENTIAL FUNCTIONS

In what follows, we are going to give estimates on the potential functions Vq and $V(q\theta^\sigma)$, where q is a function satisfying (H) and θ is the function given in (1.7).

Proposition 2.5. *Let q be a function satisfying (H). Then we have for $x \in \Omega$,*

$$Vq(x) \approx (|x| - a)^{\min(1,2-\lambda_1)} (b - |x|)^{\min(1,2-\lambda_2)} \Psi_{L_1,\lambda_1,0}(|x| - a) \Psi_{L_2,\lambda_2,0}(b - |x|).$$

Proof. Let q be a function satisfying (H). Then we have for $x \in \Omega$,

$$q(x) \approx (|x| - a)^{-\lambda_1} (b - |x|)^{-\lambda_2} L_1(|x| - a) L_2(b - |x|),$$

where $L_1, L_2 \in \mathcal{K}$, satisfying $\int_0^\eta t^{1-\lambda_1} L_1(t)dt < \infty$ and $\int_0^\eta t^{1-\lambda_2} L_2(t)dt < \infty$. Now, using the fact that the function

$$x \mapsto (|x| - a)^{-\lambda_1} (b - |x|)^{-\lambda_2} L_1(|x| - a) L_2(b - |x|)$$

is radial and satisfies the condition (1.4) we deduce by (1.5) that for $x \in \Omega$

$$Vq(x) \approx \int_a^b (\max(|x|, r)^{2-n} - b^{2-n})(a^{2-n} - \min(|x|, r)^{2-n}) \\ \times (r - a)^{-\lambda_1} (b - r)^{-\lambda_2} L_1(r - a) L_2(b - r)dr.$$

Put

$$h(t) := \int_a^b (\max(t,r)^{2-n} - b^{2-n})(a^{2-n} - \min(t,r)^{2-n}) \times (r-a)^{-\lambda_1}(b-r)^{-\lambda_2} L_1(r-a) L_2(b-r)dr, \quad t \in (a,b).$$

To prove the result, it sufficient to show that

$$h(|x|) \approx (|x|-a)^{\min(1,2-\lambda_1)}(b-|x|)^{\min(1,2-\lambda_2)} \times \Psi_{L_1,\lambda_1,0}(|x|-a)\Psi_{L_2,\lambda_2,0}(b-|x|) \quad \text{for} \quad x \in \Omega.$$

To reach our estimates, we distinguish the following cases.

Case 1. $a < |x| < a + \frac{b-a}{4}$. We have $|x|^{2-n} - b^{2-n} \approx 1$ and $a^{2-n} - |x|^{2-n} \approx |x| - a$. This implies that

$$h(|x|) \approx \int_a^{|x|} (a^{2-n} - r^{2-n})(r-a)^{-\lambda_1} L_1(r-a)(b-r)^{-\lambda_2} L_2(b-r)dr + (|x|-a)\left(\int_{|x|}^b (r-a)^{-\lambda_1} L_1(r-a)(r^{2-n} - b^{2-n})(b-r)^{-\lambda_2} L_2(b-r)dr\right).$$

Using that for $r \in (a,\, a + \frac{b-a}{2})$

$$a^{2-n} - r^{2-n} \approx r - a \quad \text{and} \quad (r^{2-n} - b^{2-n})(b-r)^{-\lambda_2} L_2(b-r) \approx 1,$$

we obtain that

$$\int_a^{|x|} (a^{2-n} - r^{2-n})(r-a)^{-\lambda_1} L_1(r-a)(b-r)^{-\lambda_2} L_2(b-r)dr \approx \int_a^{|x|} (r-a)^{1-\lambda_1} L_1(r-a)dr$$

and

$$\int_{|x|}^{a+\frac{b-a}{2}} (r-a)^{-\lambda_1} L_1(r-a)(r^{2-n} - b^{2-n})(b-r)^{-\lambda_2} L_2(b-r)dr \approx \int_{|x|}^{a+\frac{b-a}{2}} (r-a)^{-\lambda_1} L_1(r-a)dr.$$

Now, since for $r \in (a + \frac{b-a}{2}, b)$, we have

$$r^{2-n} - b^{2-n} \approx b - r \quad \text{and} \quad (r-a)^{-\lambda_1} L_1(r-a) \approx 1,$$

we get

$$\begin{aligned}&\int_{a+\frac{b-a}{2}}^{b}(r-a)^{-\lambda_1}L_1(r-a)(r^{2-n}-b^{2-n})(b-r)^{-\lambda_2}L_2(b-r)dr\\ &\approx \int_{a+\frac{b-a}{2}}^{b}(b-r)^{1-\lambda_2}L_2(b-r)dr.\end{aligned}$$

That is

$$\begin{aligned}h(|x|)\approx &\int_0^{|x|-a}s^{1-\lambda_1}L_1(s)ds\\ &+(|x|-a)\left(\int_{|x|-a}^{\frac{b-a}{2}}s^{-\lambda_1}L_1(s)ds+\int_0^{\frac{b-a}{2}}s^{1-\lambda_2}L_2(s)ds\right).\end{aligned}$$

Since $\int_0^{\frac{b-a}{2}} s^{1-\lambda_2}L_2(s)ds<\infty$, we get

$$\begin{aligned}h(|x|)&\approx \int_0^{|x|-a}s^{1-\lambda_1}L_1(s)ds+(|x|-a)\left(\int_{|x|-a}^{\frac{b-a}{2}}s^{-\lambda_1}L_1(s)ds+1\right)\\ &=I(|x|-a)+J(|x|-a),\end{aligned}$$

where I and J are the functions given in Lemma 2.4, by replacing L by L_1 and λ by λ_1. So, we reach

$$h(|x|)\approx\begin{cases}(|x|-a)^{2-\lambda_1}L_1(|x|-a)+(|x|-a), & \text{if } \lambda_1<1,\\ (|x|-a)\left(L_1(|x|-a)+\int_{|x|-a}^{\frac{b-a}{2}}\frac{L_1(s)}{s}ds\right), & \text{if } \lambda_1=1,\\ (|x|-a)^{2-\lambda_1}L_1(|x|-a), & \text{if } 1<\lambda_1<2,\\ \int_0^{|x|-a}\frac{L_1(s)}{s}ds+L_1(|x|-a), & \text{if } \lambda_1=2.\end{cases}$$

Using Lemma 2.2, we deduce that

$$h(|x|) \approx \begin{cases} |x| - a, & \text{if } \lambda_1 < 1, \\ (|x| - a) \displaystyle\int_{|x|-a}^{\eta} \frac{L_1(s)}{s} ds, & \text{if } \lambda_1 = 1, \\ (|x| - a)^{2-\lambda_1} L_1(|x| - a), & \text{if } 1 < \lambda_1 < 2, \\ \displaystyle\int_0^{|x|-a} \frac{L_1(s)}{s} ds, & \text{if } \lambda_1 = 2. \end{cases}$$

Hence, for $a < |x| < a + \frac{b-a}{4}$,

$$Vq(x) \approx (|x| - a)^{\min(1,2-\lambda_1)} \Psi_{L_1,\lambda_1,0}(|x| - a). \tag{2.1}$$

Case 2. $a + \frac{b-a}{4} \leq |x| \leq b - \frac{b-a}{4}$. Since the function h is continuous and positive in $[a + \frac{b-a}{4}, b - \frac{b-a}{4}]$, then we have

$$h(t) \approx 1 \quad \text{for} \quad t \in \Big[a + \frac{b-a}{4}, b - \frac{b-a}{4}\Big].$$

So, for $a + \frac{b-a}{4} \leq |x| \leq b - \frac{b-a}{4}$,

$$Vq(x) \approx 1. \tag{2.2}$$

On the other hand, we have for $a + \frac{b-a}{4} \leq |x| \leq b - \frac{b-a}{4}$,

$$(|x| - a)^{\min(1,2-\lambda_1)} (b - |x|)^{\min(1,2-\lambda_2)} \Psi_{L_1,\lambda_1,0}(|x| - a) \Psi_{L_2,\lambda_2,0}(b - |x|) \approx 1,$$

which gives the result.

Case 3. $b-\frac{b-a}{4}<|x|<b$. We have $|x|^{2-n}-b^{2-n}\approx b-|x|$ and $a^{2-n}-|x|^{2-n}\approx 1$. Then

$$h(|x|)\approx (b-|x|)\left(\int_a^{|x|}(a^{2-n}-r^{2-n})(r-a)^{-\lambda_1}L_1(r-a)(b-r)^{-\lambda_2}L_2(b-r)dr\right)$$

$$+\int_{|x|}^{b}(r-a)^{-\lambda_1}L_1(r-a)(r^{2-n}-b^{2-n})(b-r)^{-\lambda_2}L_2(b-r)dr.$$

Using that for $r\in(a,b-\frac{b-a}{4})$

$$a^{2-n}-r^{2-n}\approx r-a \quad \text{and} \quad (b-r)^{-\lambda_2}L_2(b-r)\approx 1,$$

we obtain

$$\int_a^{b-\frac{b-a}{4}}(a^{2-n}-r^{2-n})(r-a)^{-\lambda_1}L_1(r-a)(b-r)^{-\lambda_2}L_2(b-r)dr$$

$$\approx \int_a^{b-\frac{b-a}{4}}(r-a)^{1-\lambda_1}L_1(r-a)dr.$$

Now, since we have

$$(a^{2-n}-r^{2-n})(r-a)^{-\lambda_1}L_1(r-a)\approx 1 \quad \text{for} \quad r\in\left(b-\frac{b-a}{4},|x|\right),$$

we get

$$\int_{b-\frac{b-a}{4}}^{|x|}(a^{2-n}-r^{2-n})(r-a)^{-\lambda_1}L_1(r-a)(b-r)^{-\lambda_2}L_2(b-r)dr$$

$$\approx \int_{b-\frac{b-a}{4}}^{|x|}(b-r)^{-\lambda_2}L_2(b-r)dr.$$

Moreover, we have $r^{2-n}-b^{2-n}\approx b-r$ and $(r-a)^{-\lambda_1}L_1(r-a)\approx 1$ for $r\in(|x|,b)$. Then we reach

$$h(|x|)\approx (b-|x|)\left(\int_a^{b-\frac{b-a}{4}}(r-a)^{1-\lambda_1}L_1(r-a)dr+\int_{b-\frac{b-a}{4}}^{|x|}(b-r)^{-\lambda_2}L_2(b-r)dr\right)$$

$$+\int_{|x|}^{b}(b-r)^{1-\lambda_2}L_2(b-r)dr. \tag{2.3}$$

Hence,

$$\begin{aligned} h(|x|) \approx & \int_0^{b-|x|} s^{1-\lambda_2} L_2(s) ds \\ & + (b-|x|) \left(\int_0^{\frac{3(b-a)}{4}} s^{1-\lambda_1} L_1(s) ds + \int_{b-|x|}^{\frac{b-a}{4}} s^{-\lambda_2} L_2(s) ds \right). \end{aligned}$$

Since $\int_0^{\frac{3(b-a)}{4}} s^{1-\lambda_1} L_1(s) ds < \infty$, we deduce that

$$\begin{aligned} h(|x|) &\approx \int_0^{b-|x|} s^{1-\lambda_2} L_2(s) ds + (b-|x|) \left(1 + \int_{b-|x|}^{\frac{b-a}{2}} s^{-\lambda_2} L_2(s) ds \right) \\ &= I(b-|x|) + J(b-|x|), \end{aligned}$$

where I and J are given in Lemma 2.4, by replacing L by L_2 and λ by λ_2. So, we have

$$h(|x|) \approx \begin{cases} (b-|x|)^{2-\lambda_2} L_2(b-|x|) + (b-|x|), & \text{if } \lambda_2 < 1, \\ (b-|x|) \left(L_2(b-|x|) + \int_{b-|x|}^{\frac{b-a}{2}} \frac{L_2(s)}{s} ds \right), & \text{if } \lambda_2 = 1, \\ (b-|x|)^{2-\lambda_2} L_2(b-|x|), & \text{if } 1 < \lambda_2 < 2, \\ \int_0^{b-|x|} \frac{L_2(s)}{s} ds + L_2(b-|x|), & \text{if } \lambda_2 = 2. \end{cases}$$

Using Lemma 2.2, we get

$$h(|x|) \approx \begin{cases} b-|x|, & \text{if } \lambda_2 < 1, \\ (b-|x|) \int_{b-|x|}^{\eta} \frac{L_2(s)}{s} ds, & \text{if } \lambda_2 = 1, \\ (b-|x|)^{2-\lambda_2} L_2(b-|x|), & \text{if } 1 < \lambda_2 < 2, \\ \int_0^{b-|x|} \frac{L_2(s)}{s} ds, & \text{if } \lambda_2 = 2. \end{cases}$$

That is, for $b - \frac{b-a}{4} < |x| < b$, we obtain

$$Vq(x) \approx (b-|x|)^{\min(1,2-\lambda_2)} \Psi_{L_2,\lambda_2,0}(b-|x|). \tag{2.4}$$

Combining (2.1), (2.2) and (2.4), we conclude that for $x \in \Omega$,

$$Vq(x) \approx (|x|-a)^{\min(1,2-\lambda_1)}(b-|x|)^{\min(1,2-\lambda_2)}\Psi_{L_1,\lambda_1,0}(|x|-a)\Psi_{L_2,\lambda_2,0}(b-|x|).$$

This completes the proof. □

The following proposition plays a crucial role in the proof of our main result.

Proposition 2.6. *Let q be a function satisfying (H) and let θ be the function given by (1.7). Then for $x \in \Omega$, we have*

$$V(a\theta^\sigma)(x) \approx \theta(x).$$

Proof. Let $\lambda_1, \lambda_2 \leq 2$ and $L_1, L_2 \in \mathcal{K}$ satisfying $\int_0^\eta t^{1-\lambda_1}L_1(t)dt < \infty$ and $\int_0^\eta t^{1-\lambda_2}L_2(t)dt < \infty$, such that for $x \in \Omega$,

$$q(x) \approx (|x|-a)^{-\lambda_1}(b-|x|)^{-\lambda_2}L_1(|x|-a)L_2(b-|x|).$$

Putting $\mu_1 = \lambda_1 - \sigma\min(1, \frac{2-\lambda_1}{1-\sigma})$ and $\mu_2 = \lambda_2 - \sigma\min(1, \frac{2-\lambda_2}{1-\sigma})$. Then it is obvious to verify that $\mu_1, \mu_2 \leq 2$ and by using (1.7), we have for $x \in \Omega$,

$$\begin{aligned}&q(x)\theta^\sigma(x)\\ &\approx (|x|-a)^{-\mu_1}(b-|x|)^{-\mu_2}L_1(|x|-a)L_2(b-|x|)\Psi^\sigma_{L_1,\lambda_1,\sigma}(|x|-a)\Psi^\sigma_{L_2,\lambda_2,\sigma}(b-|x|)\\ &:= (|x|-a)^{-\mu_1}(b-|x|)^{-\mu_2}\tilde{L}_1(|x|-a)\tilde{L}_2(b-|x|).\end{aligned}$$

Using Remark 2.3, Lemmas 2.1 and 2.2, we deduce that $\tilde{L}_1, \tilde{L}_2 \in \mathcal{K}$ satisfying

$$\int_0^\eta t^{1-\mu_1}\tilde{L}_1(t)dt < \infty \quad \text{and} \quad \int_0^\eta t^{1-\mu_2}\tilde{L}_2(t)dt < \infty.$$

Hence, it follows by Proposition 2.5 that

$$V(q\theta^\sigma)(x) \approx (|x|-a)^{\min(1,2-\mu_1)}(b-|x|)^{\min(1,2-\mu_2)}\Psi_{\tilde{L}_1,\mu_1,0}(|x|-a)\Psi_{\tilde{L}_2,\mu_2,0}(b-|x|).$$

By calculus, we verify that, for $i \in \{1,2\}$, $\min(1, 2-\mu_i) = \min(1, \frac{2-\lambda_i}{1-\sigma})$ and $\Psi_{\tilde{L}_i,\mu_i,0} = \Psi_{L_i,\lambda_i,\sigma}$. So, we get

$$V(q\theta^\sigma)(x) \approx (|x|-a)^{\min(1,\frac{2-\lambda_1}{1-\sigma})}(b-|x|)^{\min(1,\frac{2-\lambda_2}{1-\sigma})}\Psi_{L_1,\lambda_1,\sigma}(|x|-a)\Psi_{L_2,\lambda_2,\sigma}(b-|x|).$$

This ends the proof. □

3. PROOF OF THEOREM 1.4

3.1. EXISTENCE AND ASYMPTOTIC BEHAVIOR

Let q be a function satisfying (H). The main idea is to find a subsolution and supersolution to the problem (1.1) of the form $cV(q\Phi^\sigma)$, where $c > 0$ and $\Phi(x) = (|x|-a)^{-\alpha}(b-|x|)^{-\beta}\tilde{L}_1(|x|-a)\tilde{L}_2(b-|x|)$, which will satisfy

$$V(q\Phi^\sigma) \approx \Phi. \tag{3.1}$$

The choice of the real α, β and the functions $\tilde{L}_1, \tilde{L}_2 \in \mathcal{K}$ are such that (3.1) is satisfied. By Proposition 2.6, the function θ satisfies (3.1). Hence, let $v := V(q\theta^\sigma)$ and let $M > 1$ such that

$$\frac{1}{M}\theta \leq v \leq M\theta, \tag{3.2}$$

that is, for $\sigma < 1$, we have

$$\frac{1}{M^{|\sigma|}}\theta^\sigma \leq v^\sigma \leq M^{|\sigma|}\theta^\sigma.$$

Put $c := M^{\frac{|\sigma|}{1-\sigma}}$. Then it follows from (1.3) that $\underline{u} = \frac{1}{c}v$ and $\bar{u} = cv$ are respectively a subsolution and a supersolution of problem (1.1).

Now, since $c \geq 1$, we get $\underline{u}(x) \leq \bar{u}(x)$ in Ω and thanks to the method of sub and supersolution, it follows that problem (1.1) has a classical solution u satisfying $\underline{u} \leq u \leq \bar{u}$ in Ω.

Finally, we deduce by using (3.2) that u satisfies (1.6). This completes the proof.

3.2. UNIQUENESS

If $\sigma < 0$, the uniqueness of the solution to problem (1.1) was established in recent papers (see for instance [2]). For the case $0 \leq \sigma < 1$, we aim to show that problem (1.1) has a unique positive classical solution in the cone

$$Y := \{u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega) : u(x) \approx \theta(x)\}.$$

Indeed, let u and v be two solutions of problem (1.1) in Y. Then there exists $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{u}{v} \leq M.$$

This implies that the set $J := \{t \in (1, +\infty) : \frac{1}{t}u \leq v \leq tu\}$ is not empty. Now, put $c := \inf J$, then we aim to show that $c = 1$. Suppose that $c > 1$, we obtain that

$$\begin{cases} -\Delta(v - c^{-\sigma}u) = q(x)(v^\sigma - c^{-\sigma}u^\sigma) \geq 0, \\ (v - c^{-\sigma}u)|_{\partial\Omega} = 0. \end{cases}$$

By the maximum principle, we deduce that $v \geq c^{-\sigma}u$. Using the same argument, we get $v \leq c^\sigma u$. This implies that $c^\sigma \in J$, but since $\sigma < 1$, we have $c^\sigma < c$. We reach a contradiction with the fact that $c = \inf J$. Hence, we have $c = 1$ and so $u = v$. This ends the proof.

4. APPLICATIONS

4.1. FIRST APPLICATION

Let $\eta > d$ and q be a positive function in $\mathcal{C}^\gamma_{loc}(\Omega)$, $0 < \gamma < 1$, satisfying for each $x \in \Omega$,

$$q(x) = (|x| - a)^{-2}(b - |x|)^{-\lambda}\left(\log\left(\frac{3(b-a)}{|x| - a}\right)\right)^{-2}\left(\log\left(\frac{3(b-a)}{b - |x|}\right)\right)^{-\mu},$$

where the real numbers $\lambda \leq 2$ and μ satisfy one of the following conditions:

(i) $\lambda < 2$ and $\mu \in \mathbb{R}$,
(ii) $\lambda = 2$ and $\mu > 1$.

Then, using Theorem 1.4, we deduce that problem (1.1) has a unique positive classical solution u satisfying for each $x \in \Omega$,

$$u(x) \approx \left(\log\left(\frac{3(b-a)}{|x|-a}\right)\right)^{\frac{-1}{1-\sigma}} \begin{cases} \left(\log(\frac{3(b-a)}{b-|x|})\right)^{\frac{1-\mu}{1-\sigma}}, & \text{if } \lambda = 2 \\ & \text{and } \mu > 1, \\ (b-|x|)^{\frac{2-\lambda}{1-\sigma}} \left(\log(\frac{3(b-a)}{b-|x|})\right)^{\frac{-\mu}{1-\sigma}}, & \text{if } 1+\sigma < \lambda < 2, \\ b-|x|, & \text{if } \lambda = 1+\sigma \\ & \text{and } \mu > 1, \\ (b-|x|) \left(\log(\log \frac{3(b-a)}{b-|x|})\right)^{\frac{1}{1-\sigma}}, & \text{if } \lambda = 1+\sigma \\ & \text{and } \mu = 1, \\ (b-|x|) \left(\log(\frac{3(b-a)}{b-|x|})\right)^{\frac{1-\mu}{1-\sigma}}, & \text{if } \lambda = 1+\sigma \\ & \text{and } \mu < 1, \\ b-|x|, & \text{if } \lambda < 1+\sigma. \end{cases}$$

4.2. SECOND APPLICATION

Let q be a function satisfying (H) and let $\sigma, \beta < 1$. We are interested in the following Dirichlet problem:

$$\begin{cases} -\Delta u + \frac{\beta}{u}|\nabla u|^2 = q(x)u^\sigma, \ x \in \Omega, \\ u > 0 \ \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

Put $v = u^{1-\beta}$. Then by calculus, we verify that v satisfies

$$\begin{cases} -\Delta v = (1-\beta)q(x)v^{\frac{\sigma-\beta}{1-\beta}}, \ x \in \Omega, \\ v > 0 \ \text{ in } \Omega, \quad v|_{\partial\Omega} = 0. \end{cases} \tag{4.2}$$

Applying Theorem 1.4 to problem (4.2), we obtain that there exists a unique positive classical solution v such that for each $x \in \Omega$,

$$v(x) \approx (|x|-a)^{\min(1,\frac{2-\lambda_1}{1-\mu})}(b-|x|)^{\min(1,\frac{2-\lambda_2}{1-\mu})}\Psi_{L_1,\lambda_1,\mu}(|x|-a)\Psi_{L_2,\lambda_2,\mu}(b-|x|),$$

where $\mu = \frac{\sigma-\beta}{1-\beta} < 1$.

Consequently, we deduce that problem (4.1) has a unique positive solution $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ satisfying for each $x \in \Omega$,

$$u(x) \approx (|x|-a)^{\min(\frac{1}{1-\beta},\frac{2-\lambda_1}{1-\sigma})}(b-|x|)^{\min(\frac{1}{1-\beta},\frac{2-\lambda_2}{1-\sigma})}\Psi_{L_1,\lambda_1,\sigma}(|x|-a)\Psi_{L_2,\lambda_2,\sigma}(b-|x|).$$

Acknowledgments

The authors are grateful to Professor Habib Mâagli and Professor Vicentiu Radulescu for their useful suggestions. Also, the authors want to express the sincere thanks to the anonymous referees and editors for their careful reading of the manuscript.

REFERENCES

[1] D. Arcoya, *Positive solutions for semilinear Dirichlet problems in an annulus*, J. Differential Equations **94** (1991), 217–227.

[2] S. Ben Othman, H. Mâagli, S. Masmoudi, M. Zribi, *Exat asymptotic behavior near the boundary to the solution for singular nonlinear Dirichlet problems*, Nonlinear Anal. **71** (2009), 4137–4150.

[3] F. Catrina, *Nonexistence of positive radial solutions for a problem with singular potential*, Adv. Nonlinear Anal. **3** (2014) 1, 1–13.

[4] A.B. Cavalheiro, *Existence and uniqueness of the solutions of some degenerate nonlinear elliptic equations*, Opuscula Math. **34** (2014) 1, 15–28.

[5] R. Chemmam, H. Mâagli, S. Masmoudi, M. Zribi, *Combined effects in nonlinear singular elliptic problems in a bounded domain*, Adv. Nonlinear Anal. **1** (2012), 301–318.

[6] R. Chemmam, A. Dhifli, H. Mâagli, *Asymptotic behavior of ground state solutions for semilinear and singular Dirichlet problem*, Electron. J. Differential Equations **88** (2011), 1–12.

[7] M.G. Crandall, P.H. Rabinowitz, L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), 193–222.

[8] X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **70** (1987), 69–92.

[9] M. Ghergu, V.D. Rădulescu, *On a class of sublinear singular elliptic problems with convection term*, J. Math. Anal. Appl. **311** (2005), 635–646.

[10] M. Ghergu, V.D. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Applications, Vol. 37, Oxford University Press, 2008, 320 pp.

[11] M. Ghergu, V.D. Rădulescu, *Nonlinear PDEs Mathematical Models in Biology, Chemistry and Population Genetics*, Springer Monographs in Mathematics, Springer Verlag, 2012.

[12] S. Gontara, H. Mâagli, S. Masmoudi, S. Turki, *Asymptotic behavior of positive solutions of a singular nonlinear Dirichlet problem*, J. Math. Anal. Appl. **369** (2010), 719–729.

[13] S. Jator, Z. Sinkala, *Uniqueness of positive radial solutions for* $\Delta u + f(u) = 0$ *in the annulus*, Int. J. Pure Appl. Math. **12** (2004), 23–31.

[14] H. Mâagli, *Asymptotic behavior of positive solutions of a semilinear Dirichlet problems*, Nonlinear Anal. **74** (2011), 2941–2947.

[15] H. Mâagli, S. Turki, Z. Zine El Abidine, *Asymptotic behavior of positive solutions of a semilinear Dirichlet problems outside the unit ball*, Electron. J. Differential Equations **95** (2013), 1–14.

[16] R. Seneta, *Regular Varying Functions*, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, Berlin, 1976.

[17] M. Tang, *Uniqueness of positive radial solutions for $\Delta u - u + up = 0$ on an annulus*, J. Differential Equations **189** (2003), 148–160.

[18] Z. Zhang, *The asymptotic behavior of the unique solution for the singular Lane-Emdem-Fowler equation*, J. Math. Anal. Appl. **312** (2005), 33–43.

[19] H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **109** (1994), 1–7.

Safa Dridi
dridi.safa87@yahoo.fr

Campus Universitaire
Faculté des Sciences de Tunis
Département de Mathématiques
2092 Tunis, Tunisia

Bilel Khamessi
bilel.khamassi@yahoo.com

Campus Universitaire
Faculté des Sciences de Tunis
Département de Mathématiques
2092 Tunis, Tunisia

Received: February 18, 2014.
Revised: May 5, 2014.
Accepted: May 5, 2014.