THE ARCHIVES OF TRANSPORT

VOL. XXV-XXVI NO 1-2 2013

Mathematical Model for Vibrations Analysis of The Tram Wheelset

Andrzej Grzyb*
Karol Bryk**
Marek Dudzik***

Received March 2013

Abstract

This paper presents a mathematical model for the analysis of the 105Na tram trolley small vibrations. In the paper there is also presented a physical model of the analysed tram trolley, according to which the mathematical model was created. The leads were made using the methods of analytical mechanics. For the model, the Lagrange II type equations were determined in matrices form. Using the model it is possible to create simulations of the 105Na tram trolley motion.

1. Introduction

In railway vehicle's dynamic modeling the Lagrange II type equations are widely used. These equations allow obtaining a set of differential equations, which describes the physical model without internal reactions occurring between the objects of the model. To describe the physical models using the Lagrange equations, it is necessary to introduce the concept of generalized coordinates. The generalized coordinates are the least numerous set of variables, which clearly describes the position of an element of the physical model. These variables' number is defined as a number of degrees of freedom of the analyzed model. Therefore, geometrical coordinates of each object forming part of the analyzed model can be represented by generalized coordinates and written as:

*Tadeusz Kosciuszko Cracow University of Technology, Instytut Pojazdów Szynowych, Jana Pawła II 37 Street, 31-864, Cracow, Poland
**IPS TABOR, Warszawska 181 Street, 61-055, Poznan, Poland
***Tadeusz Kosciuszko Cracow University of Technology, Katedra Trakcji i Sterowania Ruchem, Warszawska 24 Street, 31-155, Cracow, Poland

$$\begin{aligned} x_1 &= x_1(q_1,\ldots,q_m,t) \\ &\vdots \\ x_n &= x_n(q_1,\ldots,q_m,t) \end{aligned} \tag{1.1}$$

where

$x_1,\ldots,x_n$ – geometrical coordinates of each object forming part of the analyzed physical model

n – number of geometrical coordinates of the physical model

$q_1,\ldots,q_m$ – generalized coordinates of the physical model

t – time

For the physical model where generalized coordinates were defined the Lagrange II type equations can be written as:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_1}\right)-\frac{\partial E}{\partial q_1}=-\frac{\partial U}{\partial q_1}-\frac{\partial D}{\partial q_1}+Q_1 \\ &\vdots \\ &\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_m}\right)-\frac{\partial E}{\partial q_m}=-\frac{\partial U}{\partial q_m}-\frac{\partial D}{\partial q_m}+Q_m \end{aligned} \tag{1.2}$$

where

E – kinetic energy of the analyzed model described in generalized coordinates

U – potential energy of the analyzed model described in generalized coordinates

D – dissipation function of the analyzed model's energy described in generalized coordinates

m – number of generalized coordinates

$Q_1,\ldots,Q_m$ – external generalized forces described as [5]:

$$\begin{aligned} Q_1 &= P_1\frac{\partial x_1}{\partial q_1}+\ldots+P_n\frac{\partial x_n}{\partial q_1} \\ &\vdots \\ Q_m &= P_1\frac{\partial x_1}{\partial q_m}+\ldots+P_n\frac{\partial x_n}{\partial q_m} \end{aligned} \tag{1.3}$$

where

$P_1,\ldots,P_n$ – external forces related to movements or external torques related to rotation angle for geometrical coordinates: $x_1,\ldots,x_n$.

For the purpose of this study the set of equations (1.2) of the 105Na tram trolley was written assuming linearity and stationarity of the analyzed physical objects. These equations were transformed into a set of equations described by generalized coordinates and time (1.4) and (1.5), stored using quadratic forms (1.6) [5].

$$U = U\left[x_1\left(q_1,\dots,q_m,t\right),\dots,x_n\left(q_1,\dots,q_s,t\right)\right] \tag{1.4}$$

$$E = E\left[x_1\left(\dot{q}_1,\dots,\dot{q}_m,q_1,\dots,q_m,t\right),\dots,x_n\left(\dot{q}_1,\dots,\dot{q}_m,q_1,\dots,q_m,t\right)\right] \tag{1.5}$$

$$\begin{aligned} E &= \frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\cdot\mathbf{E}\cdot\dot{\mathbf{x}} \\ D &= \frac{1}{2}\dot{\mathbf{x}}^{\mathbf{T}}\cdot\mathbf{D}\cdot\dot{\mathbf{x}} \\ U &= \frac{1}{2}\mathbf{x}^{\mathbf{T}}\cdot\mathbf{U}\cdot\mathbf{x} \end{aligned} \tag{1.6}$$

where

$\mathbf{E}, \mathbf{D}, \mathbf{U}$ – quadratic form matrices of: kinetic energy, dissipation function and potential energy.
$\mathbf{x}, \dot{\mathbf{x}}$ – geometrical and velocity coordinates vectors described as (1.7)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} \tag{1.7}$$

For the stationary, linear set of equations of geometric coordinates and their velocities, their relationship to general coordinates is presented in (1.8).

$$\mathbf{x} = \mathbf{W}\cdot\mathbf{x_q}, \quad \dot{\mathbf{x}} = \mathbf{W}\cdot\dot{\mathbf{x}}_\mathbf{q} \tag{1.8}$$

where

$\mathbf{x_q} = \left(q_1,\dots,q_n\right)^{\mathrm{T}}$ – general coordinates vector (1.9)

$\mathbf{W}$ – constraints matrix,

$$\mathbf{W}^{\mathrm{T}} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial q_m} & \cdots & \frac{\partial x_n}{\partial q_m} \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial q_1} & \cdots & \frac{\partial \dot{x}_n}{\partial q_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \dot{x}_1}{\partial q_m} & \cdots & \frac{\partial \dot{x}_n}{\partial q_m} \end{bmatrix} \tag{1.10}$$

When the matrices in equation (1.2) are invariant in time, the Lagrange II type equations can be written as:

$$\mathbf{W}^{\mathrm{T}} \cdot \mathbf{E}_{\nabla} \cdot \mathbf{W} \cdot \ddot{\mathbf{x}}_{\mathbf{q}} = -\mathbf{W}^{\mathrm{T}} \cdot \mathbf{U}_{\nabla} \cdot \mathbf{W} \cdot \mathbf{x}_{\mathbf{q}} - \mathbf{W}^{\mathrm{T}} \cdot \mathbf{D}_{\nabla} \cdot \mathbf{W} \cdot \dot{\mathbf{x}}_{\mathbf{q}} + \mathbf{W} \cdot \mathbf{P} \tag{1.11}$$

where

$\mathbf{P}$ – external forces vector

$$\mathbf{P} = \left[P_1, \ldots, P_n\right]^{\mathrm{T}} \tag{1.12}$$

$\mathbf{E}_{\nabla}, \mathbf{D}_{\nabla}, \mathbf{U}_{\nabla}$ – gradient matrices of kinetic energy, dissipation function and potential energy

$$\nabla_{(\mathbf{x})} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}, \quad \nabla_{(\dot{\mathbf{x}})} = \begin{bmatrix} \frac{\partial}{\partial \dot{x}_1} \\ \vdots \\ \frac{\partial}{\partial \dot{x}_n} \end{bmatrix} \tag{1.13}$$

$$\begin{aligned} \nabla_{(\dot{\mathbf{x}})}\mathbf{E} &= \mathbf{E}_{\nabla} \cdot \mathbf{W} \cdot \dot{\mathbf{x}} \\ \nabla_{(\dot{\mathbf{x}})}\mathbf{D} &= \mathbf{D}_{\nabla} \cdot \mathbf{W} \cdot \dot{\mathbf{x}} \\ \nabla_{(\mathbf{x})}\mathbf{U} &= \mathbf{U}_{\nabla} \cdot \mathbf{W} \cdot \dot{\mathbf{x}} \end{aligned} \tag{1.14}$$

2. Description of the Analyzed Tram Trolley, its Kinematic Analysis and Physical Model

The analyzed tram trolley is a part of 105Na tram's power transmission system. It was being produced in 1979 – 1992 by Konstal industries in Chorzów. Nowadays it is the most numerously exploited tram trolley in Poland.

The characteristic feature of the described tram trolley is a frame consisting of two parts joint–connected.

Figure 2.1 presents functional model of the analyzed tram trolley. In this model zero degree suspension was omitted.

To describe the motion of each of the objects in the model there were used two sets of coordinates (figure 2.2). The first set of coordinates (with the superscript " ' "), which is rigidly connected to an object and its coordinate origin is situated in the object's centre of inertia. The second set of coordinates (without the superscript " ' "), moves with the undisturbed movement of the vehicle. When the object moves only with the undisturbed movement, both sets of coordinates are the same. The position of any point of the object in the second set for small range of rotation angles changes is described by approximate relations (2.1):

$$
\begin{aligned}
x &= x_C + x' - \phi_z \cdot y' + \phi_y \cdot z' \\
y &= y_C + \phi_z \cdot x' + y' - \phi_x \cdot z' \\
z &= z_C - \phi_y \cdot x' + \phi_x \cdot y' + z'
\end{aligned}
\tag{2.1}
$$

In the analyzed tram trolley the frame is constructed from two parts, which are connected by joints A and B (figure 2.1). These joints limit mutual movement of the two parts of the frame. The frame's system has 7 degrees of freedom. As the generalized coordinates we can choose six coordinates describing the position of the right-side frame in 3D coordinate system and rotation angle between the frame's halves (the halves can rotate relatively to each other only around the straight line connecting the joints A and B). Since both parts of the frame, considered as independent objects, have 12 degrees of freedom, and after combining them by bonds the number of degrees of freedom decreased to 7. Therefore five bond equations were written.

In order to determine these equations there were set A and B points locations in the second set of coordinates C_{iP}, x_{ip} , y_{iP}, z_{ip}.

$$
\begin{aligned}
x_{\mathrm{iP}}^{\mathrm{A}} &= x_{\mathrm{iP}} + k_1 - \phi_{ziP} \cdot d_1 - \phi_{yiP} \cdot h \\
y_{\mathrm{iP}}^{\mathrm{A}} &= y_{\mathrm{iP}} + \phi_{\mathrm{ziP}} \cdot k_1 + d_1 + \phi_{\mathrm{xiP}} \cdot h \\
z_{\mathrm{iP}}^{\mathrm{A}} &= z_{\mathrm{iP}} - \phi_{\mathrm{yiP}} \cdot k_1 + \phi_{\mathrm{xiP}} \cdot d_1 - h \\
x_{\mathrm{iP}}^{\mathrm{B}} &= x_{\mathrm{iP}} - k_2 + \phi_{\mathrm{ziP}} \cdot d_2 - \phi_{\mathrm{yiP}} \cdot h \\
y_{\mathrm{iP}}^{\mathrm{B}} &= y_{\mathrm{iP}} - \phi_{\mathrm{ziP}} \cdot k_2 - d_2 + \phi_{\mathrm{xiP}} \cdot h \\
z_{\mathrm{iP}}^{\mathrm{B}} &= z_{\mathrm{iP}} + \phi_{\mathrm{yiP}} \cdot k_2 - \phi_{\mathrm{xiP}} \cdot d_2 - h
\end{aligned}
\tag{2.2}
$$

The same points in the set of coordinates related to the left–side frame are described:

$$
\begin{aligned}
x_{\mathrm{iL}}^{\mathrm{A}} &= x_{\mathrm{iL}} + k_2 - \phi_{\mathrm{ziL}} \cdot d_2 - \phi_{\mathrm{yi|L}} \cdot h \\
y_{\mathrm{iL}}^{\mathrm{A}} &= y_{\mathrm{iL}} + \phi_{\mathrm{ziL}} \cdot k_2 + d_2 + \phi_{\mathrm{xiL}} \cdot h \\
z_{\mathrm{iL}}^{\mathrm{A}} &= z_{\mathrm{iL}} - \phi_{\mathrm{yiL}} \cdot k_2 + \phi_{\mathrm{xiL}} \cdot d_2 - h \\
x_{\mathrm{iL}}^{\mathrm{B}} &= x_{\mathrm{iL}} - k_1 + \phi_{\mathrm{ziL}} \cdot d_1 - \phi_{\mathrm{yi|L}} \cdot h \\
y_{\mathrm{iL}}^{\mathrm{B}} &= y_{\mathrm{iL}} - \phi_{\mathrm{ziL}} \cdot k_1 + d_1 + \phi_{\mathrm{xiL}} \cdot h \\
z_{\mathrm{iL}}^{\mathrm{B}} &= z_{\mathrm{iL}} + \phi_{\mathrm{yiL}} \cdot k_1 - \phi_{\mathrm{xiL}} \cdot d_1 - h
\end{aligned}
\tag{2.3}
$$

The location of a point in the set of coordinates C_{iP}, x_{ip} , y_{iP}, z_{ip}, can be described by use of the location in the set of coordinates C_{iL}, x_{iL} , y_{iL}, z_{iL}, using their relation (these sets of coordinates are mutually shifted – figure 3.2):

$$
\begin{aligned}
x_{iP} &= x_{iL} + k_1 - k_2 \\
y_{iP} &= y_{iL} - d_2 + d_1 \\
z_{iP} &= z_{iP}
\end{aligned}
\tag{2.4}
$$

Since both parts of the frame have together seven degrees of freedom, in the obtained relations there are seven independent variables: $\mathrm{x_{iP}}$, $\mathrm{y_{iP}}$, $\mathrm{z_{iP}}$, φ_{xiP}, φ_{yiP}, φ_{ziP}, φ_{yiL}.

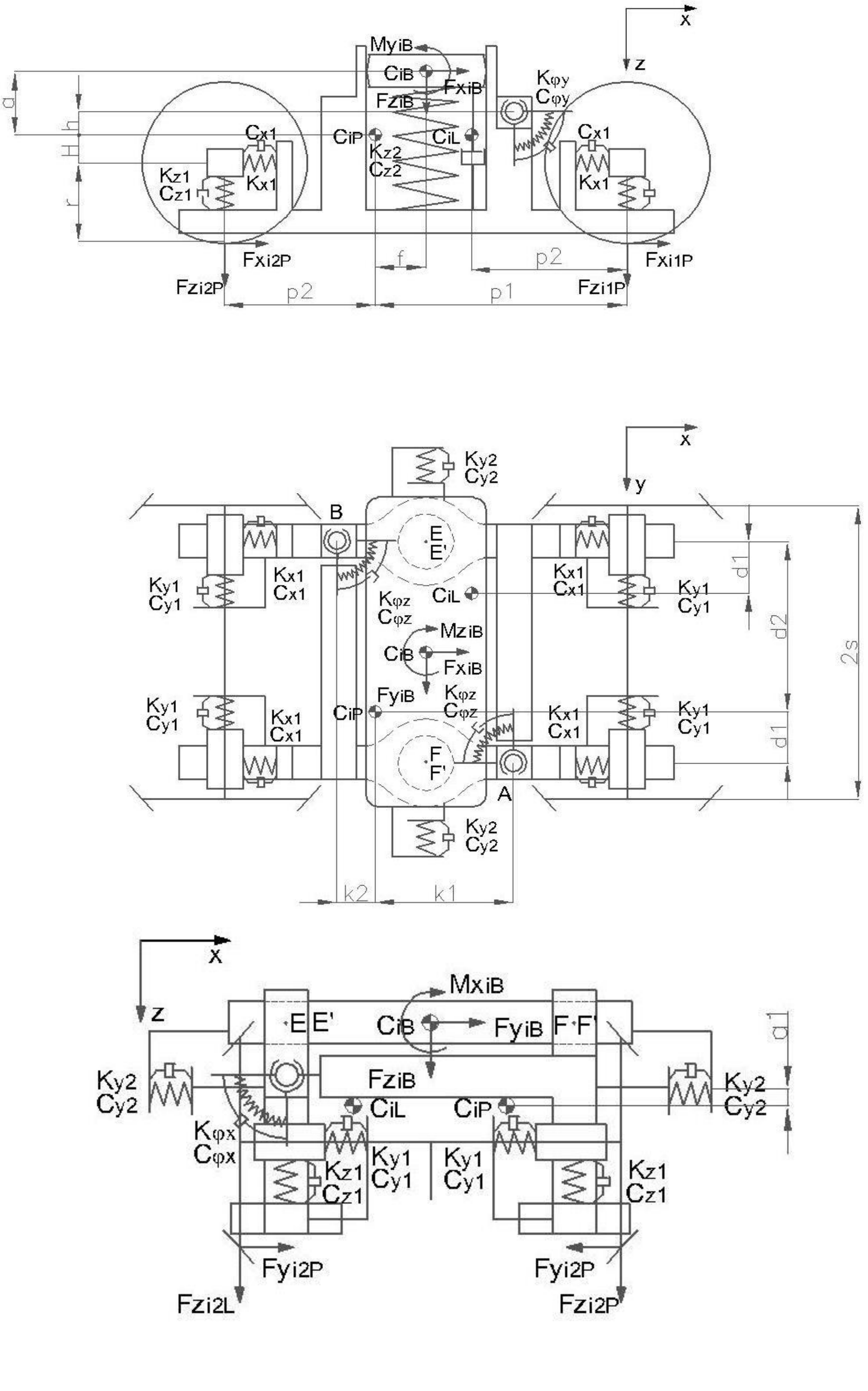

Fig. 2.1. 105Na tram trolley's model

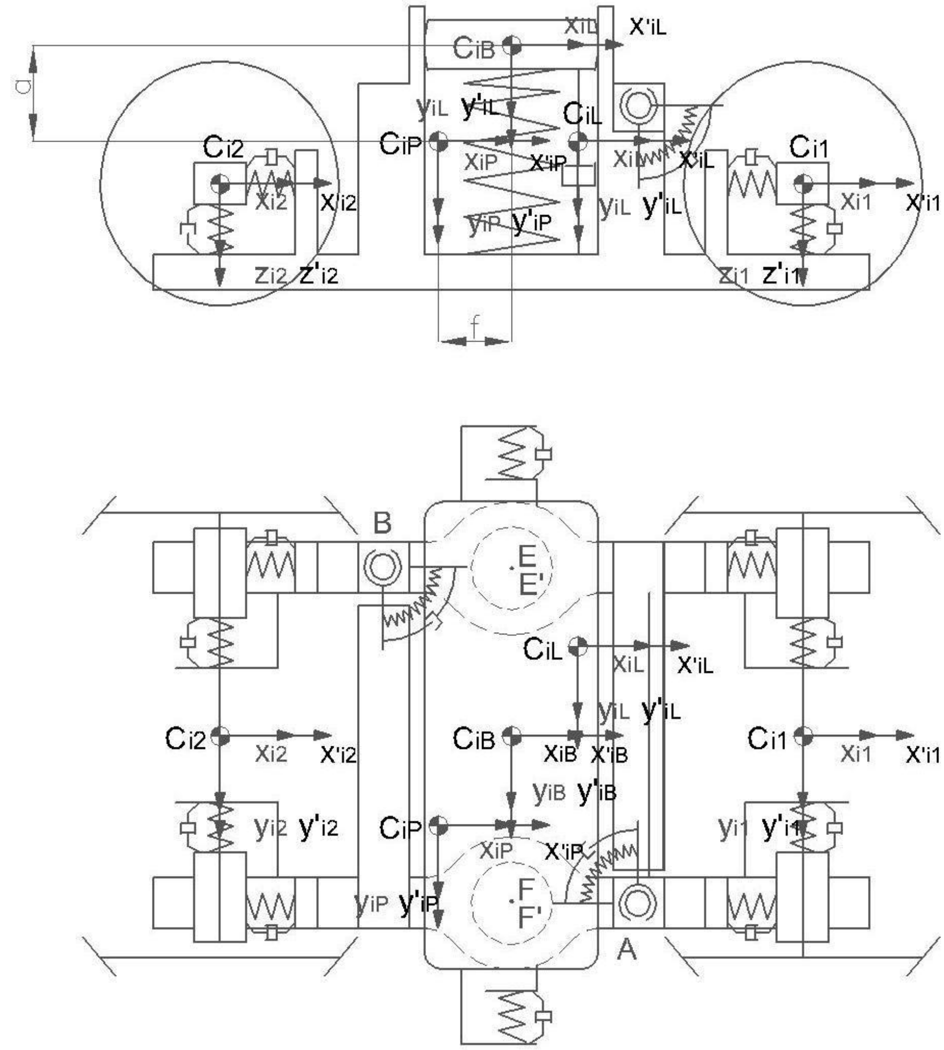

Fig. 2.2. Sets of coordinates used in the model

In the considered model there are 30 geometric coordinates (the number of rigid blocks multiplied by 6) and 23 general coordinates (the number of geometrical coordinates minus number of independent bonds).

$$\mathbf{x} = \mathbf{W} \cdot \mathbf{x_q} \tag{2.5}$$

where:

$\mathbf{W}$ – bonds matrix
$\mathbf{x}$ – geometrical coordinates vector

$$\mathbf{x} = \left[x_{iP}, y_{iP}, z_{iP}, \phi_{xiP}, \phi_{yiP}, \phi_{ziP}, x_{iL}, y_{iL}, z_{iL}, \phi_{xiL}, \phi_{yiL}, \phi_{ziL}, x_{iB}, y_{iB}, z_{iB}, \phi_{xiB}, \phi_{yiB}, \phi_{ziB}, x_{i1}, y_{i1}, z_{i1}, \phi_{xi1}, \phi_{zi1}, \phi_{zi1}, x_{i2}, y_{i2}, z_{i2}, \phi_{xi2}, \phi_{yi2}, \phi_{zi2}\right]^{T} \tag{2.6}$$

$\mathbf{x_q}$ – generalized (independent) coordinates vector

$$\mathbf{x_q} = \left[x_{iP}, y_{iP}, z_{iP}, \phi_{xiP}, \phi_{yiP}, \phi_{ziP}, \phi_{yiL}, y_{iB}, z_{iB}, \phi_{xiB}, \phi_{yiB}, x_{i1}, y_{i1}, z_{i1}, \phi_{xi1}, \phi_{zi1}, \phi_{zi1}, x_{i2}, y_{i2}, z_{i2}, \phi_{xi2}, \phi_{yi2}, \phi_{zi2}\right]^{T} \tag{2.7}$$

3. The Movement Equation

For the linear and stationary model the Lagrange II type equations of tram trolley can be written as:

$$\mathbf{W^T} \cdot \mathsf{E}_{\nabla} \cdot \mathbf{W} \cdot \ddot{\mathbf{x}}_{\mathbf{q}} = -\mathbf{W^T} \cdot \mathsf{D}_{\nabla} \cdot \mathbf{W} \cdot \dot{\mathbf{x}}_{\mathbf{q}} + \mathbf{W^T} \cdot \mathsf{U}_{\nabla} \cdot \mathbf{W} \cdot \mathbf{x_q} + \mathbf{W^T} \cdot \mathbf{F} \tag{3.1}$$

where
$\mathbf{x_q}$ – generalized (independent) coordinates vector,
$\mathbf{W}$ – bonds matrix
E_{∇} – energy gradient matrix, determined by the relation:

$$\nabla_{(\dot{x})}\mathsf{E} = \mathsf{E}_{\nabla} \cdot \dot{\mathbf{x}} \tag{3.2}$$

D_{∇} – energy dissipation gradient matrix:

$$\nabla_{(\dot{x})}\mathsf{D} = \mathsf{D}_{\nabla} \cdot \dot{\mathbf{x}} \tag{3.3}$$

U_{∇} – potential energy gradient matrix:

$$\nabla \mathsf{U}_{(x)} = \mathsf{U}_{\nabla} \cdot \mathbf{x} \tag{3.4}$$

∇ – gradient operator – In the analyzed case it refers to differentiation by geometrical coordinates or geometrical velocities:

$$\nabla_{(\mathbf{x})} = \left[\frac{\partial}{\partial x_1} \quad \cdots \quad \frac{\partial}{\partial x_i} \right]^{\mathrm{T}} \tag{3.5}$$

$\mathbf{F}$ – external forces vector

After the calculations the matrices $\mathbf{E}_\nabla$ and $\mathbf{U}_\nabla$ take the (3.7) and (3.8) forms.

The matrix $\mathbf{D}_\nabla$ has similar form to the matrix $\mathbf{U}_\nabla$, however it has attenuation factors instead of elasticity coefficients.

In each of the matrices, to the elements of the i–row or the j–column there were assigned:

$$\mathbf{i},\,\mathbf{j} = \left[x_{iP}, y_{iP}, z_{iP}, \varphi_{xiP}, \varphi_{yiP}, \varphi_{ziP}, x_{iL}, y_{iL}, z_{iL}, \varphi_{xiL}, \varphi_{yiL}, \varphi_{ziL}, x_{iB}, y_{iB}, z_{iB}, \varphi_{xiB}, \varphi_{yiB}, \varphi_{ziB}, x_{i1}, y_{i1}, z_{i1}, \varphi_{xi1}, \varphi_{yi1}, \varphi_{zi1}, x_{i2}, y_{i2}, z_{i2}, \varphi_{xi2}, \varphi_{yi2}, \varphi_{ziP} \right]$$

$$\mathbf{W} = \left(\begin{array}{*{23}{c}}
1 \\
 & 1 \\
 & & 1 \\
 & & & 1 \\
 & & & & 1 \\
 & & & & & 1 \\
1 & & & & -h & d_2-d_1 & h \\
 & 1 & & & h*k/d & k_1-k_2 & h*k/d \\
 & & 1 & d_1-d_2 & -k_3 & & -k_4 \\
 & & & 1 & -k/d & & k/d \\
 & & & & & & 1 \\
 & & & & & 1 \\
1 & & & & (h+a)/2 & (d_2-d_1)/2 & (h-a)/2 \\
 & & & & & & & 1 \\
 & & & & & & & & 1 \\
 & & & & & & & & & 1 \\
 & & & & & & & & & & 1 \\
 & & & & (a-h)/d & 1 & -(a-h)/d \\
 & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & & & & & & & & & 1
\end{array} \right) \tag{3.6}$$

$$\mathbf{E}_\nabla = \left(\mathrm{E}_{i,j} \right) \tag{3.7}$$

The non–zero elements of the matrix $\mathbf{E}_\nabla$ are:

$E_{1,1} = E_{2,2} = E_{3,3} = m_{iP}$

$E_{4,4} = J_{xiP}$

$E_{4,5} = E_{5,4} = -2J_{xyiP}$

$E_{4,6} = E_{6,4} = -2J_{zxiP}$

$E_{5,5} = J_{yiP}$

$E_{5,6} = E_{6,5} = -2J_{yziP}$

$E_{6,6} = J_{ziP}$

$E_{7,7} = E_{8,8} = E_{9,9} = m_{iL}$

$E_{10,10} = J_{xiL}$

$E_{10,11} = E_{11,10} = -2J_{xyiL}$

$E_{10,12} = E_{12,10} = -2J_{zxiL}$

$E_{11,11} = J_{yiL}$

$E_{11,12} = -2J_{yziL}$

$E_{12,12} = J_{ziL}$

$E_{13,13} = E_{14,14} = E_{15,15} = m_{iB}$

$E_{16,16} = J_{xiB}$

$E_{17,17} = J_{yiB}$

$E_{18,18} = J_{ziB}$

$E_{19,19} = m_{iB}$

$E_{20,20} = E_{21,21} = m_{i1}$

$E_{22,22} = J_{xi1}$

$E_{23,23} = J_{yi1}$

$E_{24,24} = J_{ziB}$

$E_{25,25} = E_{26,26} = E_{27,27} = m_{i1}$

$E_{28,28} = J_{xi1}$

$E_{29,29} = J_{yi1}$

$E_{30,30} = J_{zi1}$

$$\mathbf{U}_{\nabla} = \left(u_{i,j}\right) \tag{3.8}$$

The non–zero elements of the matrix $\mathbf{U}_{\nabla}$ are:

$u_{1,1} = 2k_{x1}$

$u_{1,5} = u_{5,1} = 2k_{x1} \cdot H$

$u_{1,6} = u_{6,1} = -2k_{x1} \cdot d_1$

$u_{1,19} = u_{19,1} = -k_{x1}$

$u_{1,24} = u_{24,1} = -k_{x1} \frac{d}{2}$

$u_{1,25} = u_{25,1} = -k_{x1}$

$u_{1,30} = u_{30,1} = -k_{x1} \frac{d}{2}$

$u_{2,2} = 2k_{y1} + k_{y2}$

$u_{2,4} = u_{4,2} = -2k_{y1}H + k_{y2}a_1$

$u_{2,6} = u_{6,2} = k_{y1}(p_1 - p_2) + k_{y2}f$

$u_{2,14} = u_{14,2} = -k_{y2}$

$u_{2,16} = u_{16,2} = k_{y2}(a - a_1)$

$u_{2,20} = u_{20,2} = -k_{y1}$

$u_{2,30} = u_{30,2} = -k_{y1}$

$u_{3,3} = 2k_{z1} + k_{z2}$

$u_{3,4} = u_{4,3} = 2k_{z1}d_1 + k_{z2}d_1$

$u_{3,5} = u_{5,3} = k_{z1}(p_2 - p_1) - k_{z2}f$

$u_{3,15} = u_{15,3} = -k_{z2}$

$u_{3,16} = u_{16,3} = -k_{z2} \frac{d}{2}$

$u_{3,21} = u_{21,3} = k_{z1}$

$u_{3,22} = u_{22,3} = -k_{z1} \frac{d}{2}$

$u_{3,27} = u_{27,3} = k_{z1}$

$u_{3,28} = u_{28,3} = -k_{z1} \frac{d}{2}$

$u_{4,4} = 2k_{z1}d_1^2 + 2_{y1}H^2 + k_{z2}d_1^2$
$+k_{y2}a_1^2 + 2k_{\phi x}$

$u_{4,5} = u_{5,4} = k_{z1}d_1(p_2 - p_1)$
$-k_{z2}d_1f$

$u_{4,6} = u_{6,4} = -k_{y1}H(p_1 - p_2)$
$+k_{y2}a_1f$

$u_{4,10} = u_{10,4} = -2k_{\phi x}$

$u_{4,14} = u_{14,4} = -k_{y2}a_1$

$u_{4,15} = u_{15,4} = -k_{z2}d_1$

$u_{4,16} = u_{16,4} = -k_{z2}d_1 \frac{d}{2}$
$+k_{y2}a_1(a - a_1)$

$u_{4,20} = u_{20,4} = -k_{u1}H$

$u_{4,21} = u_{21,4} = -k_{z1}d_1$

$u_{4,22} = u_{22,4} - k_{z1}d_1 \frac{d}{2}$

$u_{4,26} = u_{26,4} = -k_{u1}H$

$u_{4,27} = u_{27,4} = -k_{z1}d_1$

$u_{4,28} = u_{28,4} - k_{z1}d_1 \frac{d}{2}$

$u_{5,5} = k_{z1}(p_1^2 + p_2^2) + 2k_{x1}H^2$
$+2k_{z2}f^2 + 2k_{\phi y}$

$u_{5,6} = u_{6,5} = -2k_{x1} \cdot H \cdot d_1$

$u_{5,11} = u_{11,5} = -2k_{\phi y}$

$u_{5,15} = u_{15,5} = k_{z2}f$

$u_{5,16} = u_{16,5} = k_{z2}f \frac{d}{2}$

$u_{5,19} = u_{19,5} = -k_{x1}H$

$u_{5,21} = u_{21,5} = k_{z1}p_1$

$u_{5,22} = u_{22,5} = k_{z1}p_1 \frac{d}{2}$

$u_{5,24} = u_{24,5} = k_{x1}H \frac{d}{2}$

$u_{5,25} = u_{25,5} = -k_{x1}H$

$u_{5,27} = u_{27,5} = -k_{z1}p_2$

$u_{5,28} = u_{28,5} = -k_{z1}p_2 \frac{d}{2}$

$u_{5,30} = u_{30,5} = k_{x1}H \frac{d}{2}$

$u_{6,6} = 2k_{x1}d_1^2 + k_{y1}(p_1^2 + p_2^2)$
$+k_{y2}f^2 + 2k_{\phi z}$

$u_{6,12} = u_{12,6} = -2k_{\phi z}$

$u_{6,14} = u_{14,6} = -k_{y2}f$

$u_{6,16} = u_{16,6} = k_{y2}f(a - a_1)$

$u_{6,19} = u_{19,6} = k_{x1}d_1$

$u_{6,20} = u_{20,6} = -k_{y1}p_1$

$u_{6,24} = u_{24,6} = -k_{x1}d_1 \frac{d}{2}$

$u_{6,25} = u_{25,6} = k_{x1}d_1$

$u_{6,26} = u_{26,6} = k_{y1}p_2$

$u_{6,30} = u_{30,6} = -k_{x1}d_1 \frac{d}{2}$

$u_{7,7} = 2k_{x1}$

$u_{7,11} = u_{11,7} = 2k_{x1}H$

$u_{7,12} = u_{12,7} = 2k_{x1}d_1$

$u_{7,19} = u_{19,7} = -k_{x1}$

$u_{7,24} = u_{24,7} = -k_{x1} \frac{d}{2}$

$u_{7,25} = u_{25,7} = -k_{x1}$

$u_{7,30} = u_{30,7} = -k_{x1} \frac{d}{2}$

$u_{8,8} = 2k_{y1} + k_{y2}$

$u_{8,10} = u_{10,8} = -2k_{y1} \cdot H + k_{y2} \cdot a_1$

$u_{8,12} = u_{12,8} = k_{y1}(-p_1 + p_2) - k_{y2}f$

$u_{8,14} = u_{14,8} = -k_{y2}$

$u_{8,16} = u_{16,8} = k_{y2}(a - a_1)$

$u_{8,20} = u_{20,8} = -k_{y1}$

$u_{8,30} = u_{30,8} = -k_{y1}$

$u_{9,9} = 2k_{z1} + k_{z2}$

$u_{9,10} = u_{10,9} = -2k_{z1}d_1 - k_{z2}d_1$

$u_{9,11} = u_{11,9} = k_{z1}(-p_2 + p_1) + k_{z2}f$

$u_{9,15} = u_{15,9} = -k_{z2}$

$u_{9,16} = u_{16,9} = k_{z2} \frac{d}{2}$

$u_{9,21} = u_{21,9} = -k_{z1}$

$u_{9,22} = u_{22,9} = k_{z1} \frac{d}{2}$

$u_{9,27}=u_{27,9}=-k_{z1}$	$u_{11,15}=u_{15,11}=-k_{z2}f$	$u_{12,25}=u_{25,12}=-k_{x1}d_1$
$u_{9,28}=u_{28,9}=k_{z1}\frac{d}{2}$	$u_{11,16}=u_{16,11}=k_{z2}f\frac{d}{2}$	$u_{12,26}=u_{26,12}=k_{y1}p_2$
$u_{10,10}=2k_{z1}d_1^2+2_{y1}H^2+k_{z2}d_1^2$	$u_{11,19}=u_{19,11}=-k_{x1}H$	$u_{12,30}=u_{30,12}=-k_{x1}d_1\frac{d}{2}$
$+k_{y2}a_1^2+2k_{\phi x}$	$u_{11,21}=u_{21,11}=k_{z1}p_2$	$u_{14,14}=2k_{y2}$
$u_{10,11}=u_{11,10}=k_{z1}d_1(p_2-p_1)-k_{z2}d_1f$	$u_{11,22}=u_{22,11}=-k_{z1}p_2\frac{d}{2}$	$u_{14,16}=u_{16,14}=-2k_{y2}(a-a_1)$
$u_{10,12}=u_{12,10}=-k_{y1}H(p_2-p_1)-k_{y2}a_1f$	$u_{11,24}=u_{24,11}=-k_{x1}H\frac{d}{2}$	$u_{15,15}=2k_{z2}$
$u_{10,14}=u_{14,10}=-k_{y2}a_1$	$u_{11,25}=u_{25,11}=-k_{x1}H$	$u_{16,16}=k_{z2}\frac{d^2}{2}+2k_{y2}(a-a_1)$
$u_{10,15}=u_{15,10}=k_{z2}d_1$	$u_{11,27}=u_{27,11}=-k_{z1}p_1$	$u_{19,19}=2k_{x1}$
$u_{10,16}=u_{16,10}=-k_{z2}d_1\frac{d}{2}+k_{y2}a_1(a-a_1)$	$u_{11,28}=u_{28,5}=k_{z1}p_1\frac{d}{2}$	$u_{20,20}=2k_{y1}$
$u_{10,20}=u_{20,10}=k_{y1}H$	$u_{11,30}=u_{30,11}=-k_{x1}H\frac{d}{2}$	$u_{21,21}=2k_{z1}$
$u_{10,21}=u_{21,10}=k_{z1}d_1$	$u_{12,12}=2k_{x1}d_1^2+k_{y1}(p_1^2+p_2^2)$	$u_{22,22}=k_{z1}\frac{d^2}{2}$
$u_{10,22}=u_{22,10}-k_{z1}d_1\frac{d}{2}$	$+k_{y2}f^2+2k_{\phi z}$	$u_{24,24}=k_{x1}\frac{d^2}{2}$
$u_{10,26}=u_{26,10}=k_{y1}H$	$u_{12,16}=u_{16,12}=-k_{y2}f(a-a_1)$	$u_{25,25}=2k_{x1}$
$u_{10,27}=u_{27,10}=k_{z1}d_1$	$u_{12,19}=u_{19,12}=-k_{x1}d_1$	$u_{26,26}=2k_{y1}$
$u_{10,28}=u_{28,10}-k_{z1}d_1\frac{d}{2}$	$u_{12,20}=u_{20,12}=-k_{y1}p_1$	$u_{27,27}=2k_{z1}$
$u_{11,11}=k_{z1}(p_1^2+p_2^2)+2k_{x1}H^2$	$u_{12,24}=u_{24,12}=-k_{x1}d_1\frac{d}{2}$	$u_{28,28}=k_{z1}\frac{d^2}{2}$
$+2k_{z2}f^2+2k_{\phi y}$		$u_{30,30}=k_{x1}\frac{d^2}{2}$
$u_{11,12}=u_{12,11}=2k_{x1}Hd_1$		

where

$$k=k_1+k_2 \qquad d=d_1+d_2 \tag{3.9}$$

$$-k_3=k_2-d_1\frac{k_1+k_2}{d_1+d_2} \qquad -k_4=-k_1+d_1\frac{k_1+k_2}{d_1+d_2} \tag{3.10}$$

$$\mathbf{F}=\big[0,\ 0,\ 0,\ 0,\ 0,\ 0,0,\ 0,\ 0,\ 0,\ 0,\ 0,F_{xiB},F_{yiB},F_{ziB},M_{xiB},M_{yiB},M_{ziB},(F_{xi1P}+F_{xi1L}),(-F_{yi1P}+F_{yi1L}),$$
$$(-F_{zi1P}-F_{zi1L}),(-F_{zi1P}s+F_{zi1L}s+F_{yi1P}r-F_{yi1L}r),(F_{xi1P}r+F_{xi1L}r),(-F_{xi1P}s+F_{xi1L}s),(F_{xi2P}+F_{xi2L}),(-F_{yi2P}+F_{yi2L}),$$
$$(-F_{zi2P}-F_{zi2L}),(-F_{zi2P}s+F_{zi2L}s+F_{yi2P}r-F_{yi2L}r),(F_{xi2P}r+F_{xi2L}r),(-F_{xi2P}s+F_{xi2L}s)\big]^T \tag{3.11}$$

The information on kinematic excitations, which may be the basis to complete the elements of the F matrix, can be used on the basis of the materials contained in the paper [2].

4. Summary

In the paper there was presented the mathematical model of 105Na tram trolley using the Lagrange II type equations' matrix form. For this purpose there were determined bonds occurring in the analyzed model. There were also set kinematic energy, potential energy and energy dissipation function equations, which served to create wanted mathematical model's equations. These equations after connecting with box's or torsion pivot's equations and with equations modeling the wheel-rail contact forces can be written after transforming to the set of differential equations form and then can be used to the analysis of small vibrations and stabilization of the vehicle's movement in mathematical programs such as Matlab or Mathematica.

References

1. Gąsowski W., Marciniak Z.: Construction and models of trolleys and suspension systems for wagons and locomotives for high-speed rail. Poznan University of Technology Publishing House, Poznan 1993.
2. Grzyb A., Czauderna T.: Research and analysis of trams' kinematic vibration excitations. TTS Rail Transport Technology, no 9, 64–68, 2009.
3. Leitner R.: Overview of higher mathematics for students. Scientific and Technical Publishing House, Warsaw 1995.
4. Leyko J.: General Mechanics. PWN Scientific Publishing House, Warsaw 1980.
5. Sobczyk T.J.: Methodical aspects of induction machines' mathematical modeling. Scientific and Technical Publishing House, Warsaw 2004.