ON THE HYPERREFLEXIVITY OF SUBSPACES
OF TOEPLITZ OPERATORS ON REGIONS
IN THE COMPLEX PLANE

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Abstract. The results of hyperreflexivity or 2-hyperreflexivity for subspaces of Toeplitz operators on the Hardy spaces on Jordan regions or upper half-plane are given.

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1. INTRODUCTION

The existence of an invariant subspace for the bounded operator on the Hilbert space (equally the algebra generated by the operator) can be equivalently put as the existence of a rank one operator in the preannihilator of the algebra generated by the operator. The reflexivity of an algebra of operators (or more generally a subspace of operators) means that there are so many rank one operators in the preannihilator of the algebra (or subspace) of operators that they determine the algebra itself (the subspace itself). The hyperreflexivity (much stronger property than reflexivity) of an algebra of operators (or more generally a subspace) means that the usual distance from any operator to the algebra (or to the subspace) can be controlled by the distance given by rank one operators in the preannihilator of the algebra (or subspace). Changing rank one operators to rank k operators we obtain definition of k-reflexivity and k-hyperreflexivity, respectively. The precise definitions are given in Section 2.

The hyperreflexivity of the algebra of analytic Toeplitz operators on the Hardy space on the unit disc in the complex plane was shown in [7]. The subspace of all Toeplitz operators on this Hardy space is not reflexive (it is transitive [3]), but it is 2-reflexive [3] and even 2-hyperreflexive [12]. There was also proved that every weak* closed subspace of all Toeplitz operators on this Hardy space is 2-hyperreflexive. The purpose of this note is to move this properties to the Toeplitz operators on the Hardy

spaces on the Jordan regions in the complex plane or the upper half-plane. Namely we will prove that the algebra of analytic Toeplitz operators on the Hardy spaces on Jordan regions in the complex plane or on the upper half-plane is hyperreflexive, the subspace of all Toeplitz operators on these Hardy spaces is 2-hyperreflexive and we will get 2-hyperreflexivity of any weak* closed subspace of all Toeplitz operators on these Hardy spaces.

2. DEFINITIONS AND PRELIMINARIES

If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators and by $\tau c(\mathcal{H})$ we denote the set of *trace class* operators on $\mathcal{H}$. Duality between $\tau c(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ is given by

$$\langle A, t \rangle := \operatorname{tr}(At) \text{ for } A \in \mathcal{B}(\mathcal{H}) \text{ and } t \in \tau c(\mathcal{H}).$$

The trace norm in $\tau c(\mathcal{H})$ will be denoted by $\| \cdot \|_1$. Recall that

$$\|t\|_1 = \sup\{|\operatorname{tr}(At)| : A \in \mathcal{B}(\mathcal{H}), \|A\| \leqslant 1\} \quad \text{for } t \in \tau c(\mathcal{H}). \tag{2.1}$$

If $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, then by $\mathcal{S}_\perp$ we denote the *preannihilator* of $\mathcal{S}$ and if $\mathcal{M} \subset \tau c(\mathcal{H})$, then by $\mathcal{M}^\perp$ we denote the *annihilator* of $\mathcal{M}$. We will write the set of operators of rank at most k as $F_k(\mathcal{H})$.

Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ be a subspace and $A \in \mathcal{B}(\mathcal{H})$ be an operator. By $d(A, \mathcal{S})$, we will denote the usual distance from A to the subspace $\mathcal{S}$, i.e. $d(A, \mathcal{S}) := \inf\{\|A - S\| : S \in \mathcal{S}\}$. If $\mathcal{S}$ is weak* closed, then the distance $d(A, \mathcal{S})$ can be calculated by trace class operators, i.e.

$$d(A, \mathcal{S}) = \sup\{|\operatorname{tr}(At)| : t \in \mathcal{S}_\perp, \ \|t\|_1 \leqslant 1\}. \tag{2.2}$$

Set

$$\alpha_k(A, \mathcal{S}) := \sup\{|\operatorname{tr}(At)| : t \in \mathcal{S}_\perp \cap F_k(\mathcal{H}), \ \|t\|_1 \leqslant 1\}. \tag{2.3}$$

We now recall after [2] and [12] the definition of k-reflexivity and k-hyperreflexivity. A subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is called *k-reflexive* if $\mathcal{S} = (\mathcal{S}_\perp \cap F_k(\mathcal{H}))^\perp$. We call a subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ *k-hyperreflexive* if there is a constant $c > 0$ such that for all $A \in \mathcal{B}(\mathcal{H})$, we have

$$d(A, \mathcal{S}) \leqslant c\, \alpha_k(A, \mathcal{S}). \tag{2.4}$$

The smallest constant c satisfying (2.4) is called the *k-hyperreflexive constant* of $\mathcal{S}$ and denoted by $\kappa_k(\mathcal{S})$. It is known that every weak* closed k-hyperreflexive subspace is k-reflexive. If $k = 1$ then the letter k will be omitted and the definition above coincide with the definition of reflexivity which was introduced for algebras in [21] and extended for subspaces of operators in [15] and with the definition of hyperreflexivity which was introduced for algebras in [1] and extended for subspaces of operators in [14].

We will require the following lemma.

Lemma 2.1. *Let* $\mathcal{H}$, $\mathcal{K}$ *be Hilbert spaces. If opeator* $U : \mathcal{H} \to \mathcal{K}$ *is an isometric isomorphism, then:*

(a) *the operator* $\widetilde{U}$, *defined by* $\widetilde{U}(A) := UAU^{-1}$ *for* $A \in \mathcal{B}(\mathcal{H})$, *is an isometric isomorphism and weak* homeomorphism from* $\mathcal{B}(\mathcal{H})$ *onto* $\mathcal{B}(\mathcal{K})$,
(b) $U\mathcal{S}_\perp U^{-1} = (U\mathcal{S}U^{-1})_\perp$ *for* $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$,
(c) $UF_k(\mathcal{H})U^{-1} = F_k(\mathcal{K})$.

Proof. By [4, Exercise 2, p.61], we have that the operator $\widetilde{U}$ is an isometric isomorphism of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$. Moreover, it is easy to verify that

$$U\tau c(\mathcal{H})U^{-1} = \tau c(\mathcal{K}) \tag{2.5}$$

and

$$\text{tr}(At) = \text{tr}(UAtU^{-1}) \quad \text{for} \quad A \in \mathcal{B}(\mathcal{H}), t \in \tau c(\mathcal{H}). \tag{2.6}$$

Hence, it follows that $\widetilde{U}$ and $\widetilde{U}^{-1}$ are weak* continuous, so the proof of the condition (a) is complete. Condition (b) is a consequence of (a). As for (c), it follows by (a) that the rank one operator $g \otimes h$ on $\mathcal{H}$ satisfies

$$U(g \otimes h)U^{-1} = Ug \otimes Uh,$$

which implies (c). $\square$

The following lemma will play a crucial role in the proofs of the paper.

Lemma 2.2. *Let* $\mathcal{H}$, $\mathcal{K}$ *be Hilbert spaces and let* $U : \mathcal{H} \to \mathcal{K}$ *be an isometric isomorphism. If* $\mathcal{S}$ *is a weak* closed subspace of* $\mathcal{B}(\mathcal{H})$, *then:*

(a) $d(A, \mathcal{S}) = d(UAU^{-1}, U\mathcal{S}U^{-1})$ *for* $A \in \mathcal{B}(\mathcal{H})$,
(b) $\alpha_k(A, \mathcal{S}) = \alpha_k(UAU^{-1}, U\mathcal{S}U^{-1})$ *for* $A \in \mathcal{B}(\mathcal{H})$,
(c) *the subspace* $\mathcal{S}$ *is* k-hyperreflexive with constant c if and only if $U\mathcal{S}U^{-1}$ is k-hyperreflexive with constant c.

Proof. To see (a) and (b) note first that $\|t\|_1 = \|UtU^{-1}\|_1$ for $t \in \tau c(\mathcal{H})$ by (2.1) and (2.6). Hence, by (2.2), (2.5), (2.6) and Lemma 2.1, for any $A \in \mathcal{B}(\mathcal{H})$, we have

$$d(A, \mathcal{S}) = \sup\{|\text{tr}(At)| : t \in \mathcal{S}_\perp, \ \|t\|_1 \leqslant 1\} =$$
$$= \sup\{|\text{tr}(UAtU^{-1})| : UtU^{-1} \in U\mathcal{S}_\perp U^{-1}, \ \|UtU^{-1}\|_1 \leqslant 1\} =$$
$$= \sup\{|\text{tr}(UAU^{-1}UtU^{-1})| : UtU^{-1} \in (U\mathcal{S}U^{-1})_\perp, \ \|UtU^{-1}\|_1 \leqslant 1\} =$$
$$= d(UAU^{-1}, U\mathcal{S}U^{-1}).$$

So, the proof of (a) is complete. By similar arguments, we get (b). Finally, combining (a) and (b) we obtain (c). $\square$

3. k-HYPERREFLEXIVITY OF TOEPLITZ OPERATORS ON JORDAN REGIONS

Let $\Omega \subset \mathbb{C}$ be a Jordan region – a simply connected domain which boundary is an analytic Jordan curve. Let $a \in \Omega$ be an arbitrary point. Set $L^2(\partial\Omega) := L^2(\partial\Omega, \omega_a)$ and $L^\infty(\partial\Omega) := L^\infty(\partial\Omega, \omega_a)$, where ω_a is the harmonic measure on $\partial\Omega$ for the point a. Dependence on a is suppressed since all harmonic measures are boundedly mutually absolutely continuous (see for instance [10, Theorem 1.6.1]).

Definition 3.1. The *Hardy space* $H^2(\Omega)$ is the set of functions F analytic on Ω such that $|F|^2$ has a harmonic majorant. By $H^\infty(\Omega)$ we denote the space of all bounded analytic functions on Ω.

If Ω is the unit disc $\mathbb{D}$ and $a = 0$, then the harmonic measure ω_0 becomes the Lebesgue measure m on the unit circle $\mathbb{T}$ and the definition above of $H^2(\mathbb{D})$ is equivalent to the classical definition of the Hardy space on the unit disc. It is well known (see for instance [20]) that the space $H^2(\Omega)$ ($H^\infty(\Omega)$, respectively) can be regarded as a closed subspace of $L^2(\partial\Omega)$ (weak* closed subspace of $L^\infty(\partial\Omega)$, respectively). We use $P_{H^2(\Omega)}$ to denote the orthogonal projection of $L^2(\partial\Omega)$ onto $H^2(\Omega)$. The standard citations for Hardy spaces on Jordan regions are [9, 10, 20].

Definition 3.2. For each $\Phi \in L^\infty(\partial\Omega)$, the *Toeplitz operator* on $H^2(\Omega)$ with the symbol Φ is the operator T_Φ defined by

$$T_\Phi F := P_{H^2(\Omega)}(\Phi F), \quad F \in H^2(\Omega).$$

If $\Phi \in H^\infty(\Omega)$, then T_Φ is called an *analytic Toeplitz operator*.

By $\mathcal{T}(\Omega)$ we denote the space of all Toeplitz operators and by $\mathcal{A}(\Omega)$ the algebra of all analytic Toeplitz operators on $H^2(\Omega)$.

In [21] it was shown that the algebra $\mathcal{A}(\mathbb{D})$ is reflexive. Every weak* closed subspace of $\mathcal{A}(\mathbb{D})$ is reflexive (see [5]). In [3] it was proved that $\mathcal{T}(\mathbb{D})$ is not reflexive but it is 2-reflexive. Moreover, full characterization of reflexive subspaces of Toeplitz operators was given. In [16,17] similar results was proved for the Toeplitz operators on the Hardy space on the upper half-plane and on Jordan regions respectively.

Now we recall some key results about hyperreflexivity of Toeplitz operators on the unit disc.

Theorem 3.3.

(1) ([7, 12]) *The algebra $\mathcal{A}(\mathbb{D})$ is hyperreflexive and $\kappa(\mathcal{A}(\mathbb{D})) < 13$.*
(2) ([12, 18]) *The space $\mathcal{T}(\mathbb{D})$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}(\mathbb{D})) \leqslant 2$.*
(3) ([12]) *Every weak* closed subspace of $\mathcal{T}(\mathbb{D})$ is 2-hyperreflexive with constant at most 5.*

We will show that the theorem above can be generalized to Toeplitz operators on the Hardy space over any Jordan region.

Theorem 3.4. *Let Ω be a Jordan region. Then:*

(1) *The algebra $\mathcal{A}(\Omega)$ is hyperreflexive and $\kappa(\mathcal{A}(\Omega)) < 13$.*
(2) *The space $\mathcal{T}(\Omega)$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}(\Omega)) \leqslant 2$.*
(3) *Every weak* closed subspace of $\mathcal{T}(\Omega)$ is 2-hyperreflexive with constant at most 5.*

Proof. Let $a \in \Omega$ and $\gamma \colon \Omega \to \mathbb{D}$ be the conformal mapping ($\gamma(a) = 0$ and $\gamma'(a) > 0$) such that $\gamma(\partial\Omega) = \mathbb{T}$. Due to [20] the operator U_2 defined by $U_2 f = f \circ \gamma$, $f \in H^2(\mathbb{D})$ is an isometric isomorphism between spaces $H^2(\mathbb{D})$ and $H^2(\Omega)$. By Lemma 2.1, it follows that $\widetilde{U}_2 \colon \mathcal{B}(H^2(\mathbb{D})) \to \mathcal{B}(H^2(\Omega))$, $\widetilde{U}_2(A) := U_2 A U_2^{-1}$ is a weak* homeomorphism. Moreover, $\widetilde{U}_2(\mathcal{T}(\mathbb{D})) = \mathcal{T}(\Omega)$ and $\widetilde{U}_2(\mathcal{A}(\mathbb{D})) = \mathcal{A}(\Omega)$ (see [17, Theorem 5.3]). Hence the theorem is an immediate consequence of Lemma 2.2 and Theorem 3.3. $\square$

4. k-HYPERREFLEXIVITY OF TOEPLITZ OPERATORS ON THE UPPER HALF-PLANE

We will denote by $L^p(\mathbb{R})$ the L^p spaces of complex functions with the usual Lebesgue measure on $\mathbb{R}$. Let $\mathbb{C}_+$ denote the upper half-plane.

Definition 4.1. The *Hardy space $H^2(\mathbb{C}_+)$* is the space of all analytic functions $F \colon \mathbb{C}_+ \to \mathbb{C}$ such that $\sup_{y>0} \left(\int_{\mathbb{R}} |F(x+iy)|^2 \, dx \right)^{1/2} < \infty$. By $H^\infty(\mathbb{C}_+)$ we denote the space of all bounded analytic functions on $\mathbb{C}_+$.

Recall that the space $H^2(\mathbb{C}_+)$ ($H^\infty(\mathbb{C}_+)$, respectively) can be identified with a corresponding closed subspace of $L^2(\mathbb{R})$ (weak* closed subspace of $L^\infty(\mathbb{R})$, respectively). The standard work on Hardy spaces on the upper half-plane is [9, 11, 13, 19].

Let $P_{H^2(\mathbb{C}_+)}$ be the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Similarly as before we introduce the definition of Toeplitz operators on the upper half-plane.

Definition 4.2. For each $\Phi \in L^\infty(\mathbb{R})$, the *Toeplitz operator* on $H^2(\mathbb{C}_+)$ with symbol Φ is the operator T_Φ defined by

$$T_\Phi F = P_{H^2(\mathbb{C}_+)}(\Phi F), \quad F \in H^2(\mathbb{C}_+).$$

If $\Phi \in H^\infty(\mathbb{C}_+)$, then T_Φ is called an *analytic Toeplitz operator*.

We write $\mathcal{T}(\mathbb{C}_+)$ for the space of all Toeplitz operators and $\mathcal{A}(\mathbb{C}_+)$ for the algebra of all analytic Toeplitz operators on $H^2(\mathbb{C}_+)$.

The next theorem is a generalization of Theorem 3.3 on the upper half-plane.

Theorem 4.3.

(1) *The algebra $\mathcal{A}(\mathbb{C}_+)$ is hyperreflexive and $\kappa(\mathcal{A}(\mathbb{C}_+)) < 13$.*
(2) *The space $\mathcal{T}(\mathbb{C}_+)$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}(\mathbb{C}_+)) \leqslant 2$.*
(3) *Every weak* closed subspace of $\mathcal{T}(\mathbb{C}_+)$ is 2-hyperreflexive with constant at most 5.*

Proof. Let $\gamma(z) = \frac{z-i}{z+i}$, $z \in \mathbb{C}_+$ be the usual conformal mapping of the upper half-plane onto the unit disc which takes $\mathbb{R}$ onto $\mathbb{T} \setminus \{1\}$. We know that the operator U_2 defined by $(U_2 f)(z) := \frac{1}{\sqrt{\pi}} \frac{1}{z+i} f(\gamma(z))$, where $f \in H^2(\mathbb{D})$, $z \in \mathbb{C}_+$ is an isometric

isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{C}_+)$ (see for instance [19]). Lemma 2.1 now gives that $\widetilde{U}_2 : \mathcal{B}(H^2(\mathbb{D})) \to \mathcal{B}(H^2(\mathbb{C}_+))$, $\widetilde{U}_2(A) := U_2 A U_2^{-1}$ is a weak* homeomorphism. Furthermore, we have that $\widetilde{U}_2(\mathcal{T}(\mathbb{D})) = \mathcal{T}(\mathbb{C}_+)$ and $\widetilde{U}_2(\mathcal{A}(\mathbb{D})) = \mathcal{A}(\mathbb{C}_+)$ (see [16, Theorem 4.4]). Therfore, the proof of the theorem follows clearly from Lemma 2.2 and Theorem 3.3. $\qquad\square$

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