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TRANSFER MATRICES WITH POSITIVE COEFFICIENTS OF DESCRIPTOR LINEAR SYSTEMS

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Abstract – Transfer matrices with positive coefficients of descriptor positive linear continuous-time systems are addressed. Two methods of checking of the positivity of descriptor linear systems are proposed. It is shown that if the positive descriptor system is asymptotically stable then all coefficients of its transfer matrix are positive.

Key words – descriptor, linear, continuous-time, system, positivity, transfer matrix

1 INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1], [4], [7].

Positive linear systems with different fractional orders have been addressed in [8], [9], [13]. Stability of standard and positive systems has been investigated in [5], [12] and of fractional systems in [3], [6]. Descriptor positive systems have been analyzed in [2], [10], [11], [12].

In this paper the positivity of descriptor linear continuoustime systems and its transfer matrices with positive coefficients are addressed. The paper is organized as follows. In section 2 some preliminaries concerning standard positive linear continuous-time and discrete-time systems are recalled. Two methods of checking of the positivity of descriptor linear continuous-time and discrete-time systems are proposed in section 3. In section 4 it is shown that if the positive descriptor system is asymptotically stable then its transfer matrix has positive coefficients. Concluding remarks are given in section 5.

The following notation will be used: $\mathfrak{R}$ - the set of real numbers, $\mathfrak{R}^{n\times m}$- the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n\times m}$- the set of $n \times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n\times 1}$, M_n - the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n \times m$ identity matrix.

2 STANDARD POSITIVE LINEAR SYSTEMS

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$, $y(t) \in \mathfrak{R}^p$ are the state, input and output vectors and $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$, $C \in \mathfrak{R}^{p\times n}$, $D \in \mathfrak{R}^{p\times m}$.

Definition 1. [7] The system (1) is called (internally) positive if $x(t) \in \mathfrak{R}_+^n$ and $y(t) \in \mathfrak{R}_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \mathfrak{R}_+^n$ and all inputs $u(t) \in \mathfrak{R}_+^m$, $t \geq 0$.

Theorem 1. [7] The system (1) is positive if and only if

$$A \in M_n, B \in \mathfrak{R}_+^{n\times m}, C \in \mathfrak{R}_+^{p\times n}, D \in \mathfrak{R}_+^{p\times m}. \tag{2}$$

The transfer matrix of the system (1) is given by

$$T(s) = C[I_n s - A]^{-1} B + D. \tag{3}$$

Definition 2. [7] The positive system (1) is called asymptotically stable and the matrix $A \in M_n$ is Hurwitz if

$$\lim_{t\to\infty} x(t) = 0, \text{ for } u(t) = 0. \tag{4}$$

Theorem 2. [7] The positive system (1) is asymptotically stable (the matrix A is Hurwitz) if and only if one of the following equivalent conditions is satisfied:
All coefficients of the polynommial

$$\mathrm{Det}[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{5}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n-1$.

There exists a strictly positive vector $\lambda = [\,\lambda_1 \ldots \lambda_n\,]^T, \lambda_k > 0, k = 1, \ldots, n$ such that

$$A\lambda < 0. \tag{6}$$

Now let us consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, i \in Z_+ = 0, 1, \dots, \tag{7a}$$

$$y_i = Cx_i + Du_i, \tag{7b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 3. [7] The system (7) is called (internally) positive if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 3. [7] The system (7) is positive if and only if

$$A \in \Re_+^{n\times n}, B \in \Re_+^{n\times m}, C \in \Re_+^{p\times n} D \in \Re_+^{p\times m}. \tag{8}$$

Definition 4. [7] The positive system (7) is called asymptotically stable and the matrix $A \in \Re_+^{n\times n}$ is Schur if

$$\lim_{i\to\infty} x_i = 0, \text{ for } u_i = 0. \tag{9}$$

Theorem 4. [7] The positive system (7) is asymptotically stable (the matrix A is Schur) if and only if one of the following equivalent conditions is satisfied:

All coefficients of the polynomial

$$\det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 \tag{10}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \dots, n-1$.

There exists a strictly positive vector $\lambda = [\ \lambda_1 \dots \lambda_n\ ]^T, \lambda_k > 0, k = 1, \dots, n$ such that

$$A\lambda < \lambda. \tag{11}$$

3 DESCRIPTOR POSITIVE LINEAR SYSTEMS

Consider the descriptor continuous-time linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{12a}$$
$$y(t) = Cx(t), \tag{12b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$. It is assumed that the pencil (E, A) of (12a) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbb{C} \tag{13}$$

(the field of complex numbers)

Definition 5. The descriptor system (12) is called (internally) positive if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p, t \geq 0$ for every consistent nonnegative initial conditions $x(0) \in \Re_+^n$ and all inputs $u^{(k)}(t) = \frac{d^k u(t)}{dt^k} \in \Re_+^m, t \geq 0, k = 0, 1, \dots, q$ and q is the index of E.

The transfer matrix of the system (12) is given by

$$T(s) = C[Es - A]^{-1}B \in \Re^{p\times m}(s), \tag{14}$$

where $\Re^{p\times m}(s)$ is the set of $p \times m$ rational matrices in s. The transfer matrix (14) can be always decomposed into the strictly proper transfer matrix

$$T_{sp}(s) = C_1\left[I_{n_1}s - A_1\right]^{-1}B_1 \tag{15a}$$

and the polynomial matrix

$$P(s) = D_0 + D_1 s + \dots + D_q s^q \in \Re^{p\times m}[s], \tag{15b}$$

where $\Re^{p\times m}[s]$ is the set of $p \times m$ polynomial matrices in s.

Theorem 5. The descriptor system (12) is positive if and only if

$$A_1 \in M_{n_1}, B_1 \in \Re_+^{n_1\times m}, C_1 \in \Re_+^{p\times n_1} \tag{16a}$$

and

$$D_k \in \Re_+^{p\times m} \text{ for } k = 0, 1, \dots, q. \tag{16b}$$

Proof. By Theorem 1 the system with (15a) is positive if and only if the conditions (16a) are satisfied. Taking into account that $T(s) = T_{sp} + P(s)$ we conclude that the descriptor system (12) is positive if and only if the conditions (16a) and (16b) are satisfied.

Example 1. Consider the descriptor system (12) with the transfer matrix

$$T(s) = \begin{bmatrix} \frac{s^3 + 7s^2 + 16s + 11}{s^2 + 4s + 3} & \frac{2s^3 + 15s^2 + 36s + 25}{s^2 + 4s + 3} \end{bmatrix}, \tag{17}$$

The transfer matrix decomposed into the strictly proper part

$$T_{sp}(s) = \begin{bmatrix} \frac{s+2}{s^2+4s+3} & \frac{2s+4}{s^2+4s+3} \end{bmatrix} \tag{18}$$

and the polynomial part

$$P(s) = D_0 + D_1 s = [s + 3 \quad 2s + 7]. \tag{19}$$

From (18) we have

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, C_1 = [1 \quad 0]. \tag{20}$$

The matrices (20) satisfy the condition (16a) and

$$D_0 = [3 \quad 7], D_1 = [1 \quad 2]. \tag{21}$$

Therefore, by Theorem 5 the descriptor system (12) with (17) is positive.

Now let us consider the descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \qquad i \in Z_+ \tag{22a}$$
$$y_i = Cx_i, \tag{22b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times m}$. It is assumed that the pencil (E, A) of (22) is regular, i.e.

$$\det[Ez - A] \neq 0 \text{ for some } z \in \mathbb{C} \tag{23}$$

Definition 6. The descriptor system (22) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for every consistent nonnegative initial conditions $x_0 \in \Re_+^n$ and all $u_i \in \Re_+^m$, $i \in Z_+$.
The transfer matrix of the system (22) is given by

$$T(z) = C[Ez - A]^{-1}B \in \Re^{p\times m}(z) \tag{24}$$

and can be decomposed into the strictly proper part

$$T_{sp}(z) = C_1[I_{n1}z - A_1]^{-1}B_1 \tag{25a}$$

and the polynomial part

$$P(z) = D_0 + D_1 z + \dots + D_q z^q \in \Re^{p\times m}[z]. \tag{25b}$$

Theorem 6. The descriptor system (22) is positive if and only if

$$A_1 \in \Re_+^{n\times n}, B_1 \in \Re_+^{n_1\times m}, \qquad C_1 \in \Re_+^{p\times n_1} \tag{26a}$$

and

$$D_k \in \Re_+^{p\times n_1} for\ k = 0, 1, \dots, q. \tag{26b}$$

Proof. Proof is similar to the proof of Theorem 5.
It is assumed that in (12) and (22):

- the singular matrix E has only $n_1 < n$ linearly independent columns,
- the pencil (E,A) is regular, i.e. (13) or (23) is satisfied.

In this case there exist nonsingular matrices $P \in \Re^{n\times n}$ and $Q \in \Re^{n\times n}$ monomial (in each row and in each column only one entry is positive and the remaining entries are zero) such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix},$$
$$n = n_1 + n_2 \tag{27}$$

where $N \in \Re^{n_2 \times n_2}$ is the nilpotent matrix such that $N^{\mu} = 0$, $N^{\mu-1} \neq 0$, μ is the nilpotency index, $A_1 \in \Re^{n_1 \times n_1}$ and n_1 = degdet[Es – A].

Premultiplying the equation (12a) by the matrix $P \in \Re^{n \times n}$ and defining new state vector

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Q^{-1}x(t),\ x_1(t) \in \Re^{n_1},\ x_2(t) \in \Re^{n_2} \tag{28}$$

we obtain

$$\dot{x}_1(t) = A_1x_1(t) + B_1u(t), \tag{29a}$$
$$N\dot{x}_2(t) = x_2(t) + B_2u(t), \tag{29b}$$

where $A_1 \in \Re^{n_1 \times n_1}$, $B_1 \in \Re^{n_1 \times m}$, $B_2 \in \Re^{n_2 \times m}$ and

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB.$$

Note that if $Q \in \Re_+^{n \times n}$ is monomial then $Q^{-1} \in \Re_+^{n \times n}$ and $x_1(t) \in \Re_+^{n_1}$ and $x_2(t) \in \Re_+^{n_2}$ for $t \geq 0$ if $x(t) \in \Re_+^n$, $t \geq 0$. Defining $CQ = [C_1 \quad C_2]$, $C_1 \in \Re_+^{p \times n_1}$, $C_2 \in \Re_+^{p \times n_2}$ for any $C \in \Re_+^{p \times n}$ from (12b)

$$y(t) = C_1x_1(t) + C_2x_2(t), \tag{30}$$

it is easy to verify that

$$\begin{aligned} T(s) &= C[Es - A]^{-1}B = CQ[P(Es - A)Q]^{-1}PB \\ &= [C_1 \quad C_2]\begin{bmatrix} I_{n_1}s - A_1 & 0 \\ 0 & Ns - I_{n_2} \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= C_1\left[I_{n_1}s - A_1\right]^{-1}B_1 \\ &\quad -C_2\left[I_{n_2} + Ns + \cdots + N^{\mu-1}s^{\mu-1}\right]B_2. \end{aligned} \tag{31}$$

From (31) we have the following theorem.

Theorem 7. The descriptor continuous-time system (12) is positive if and only if

$$A_1 \in M_{n_1}, B_1 \in \Re_+^{n_1 \times m}, -B_2 \in \Re_+^{n_2 \times m},$$
$$C_1 \in \Re_+^{p \times n_1}, C_2 \in \Re_+^{p \times n_2}. \tag{32}$$

Example 2. Consider the descriptor system (12) with matrixes

$$E=\begin{bmatrix}0 & 0.25 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0.5 & 0 & 0 & 0\end{bmatrix}, A=\begin{bmatrix}0 & 0 & 0.5 & 0\\ 0 & 0.5 & 0 & 0\\ 0.5 & 0 & 0 & -2\\ -1 & 0 & 0 & 1\end{bmatrix}$$
$$B=\begin{bmatrix}0 & -0.5\\ -1 & -2\\ 1 & 0\\ 0 & 2\end{bmatrix}, C=[0 \quad 1.5 \quad 1 \quad 1]. \tag{33}$$

The pencil of the system is regular since

$$det[Es-A]=\begin{vmatrix}0 & 0.25s & -0.5 & 0\\ 0 & -0.5 & 0 & 0\\ -0.5 & 0 & 0 & s+2\\ 0.5s+1 & 0 & 0 & -1\end{vmatrix}$$
$$=0.125s^2+0.5s+0.375\neq 0. \tag{34}$$

In this case the matrices P and Q have the forms

$$P=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}, Q=\begin{bmatrix}0 & 2 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \tag{35}$$

and

$$PEQ=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix},$$
$$PAQ=\begin{bmatrix}-2 & 1 & 0 & 0\\ 1 & -2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$
$$PB=\begin{bmatrix}1 & 0\\ 0 & 2\\ 0 & -1\\ -1 & -2\end{bmatrix}, \qquad CQ=[1 \quad 0 \quad 1 \quad 3]. \tag{36}$$

The equations (29) take the forms

$$\dot{x}_1(t)=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}x_1(t)+\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}u(t),$$
$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\dot{x}_2(t)=x_2(t)+\begin{bmatrix}0 & -1\\ -1 & -2\end{bmatrix}u(t),$$
$$y(t)=[1 \quad 0]x_1(t)+[1 \quad 3]x_2(t). \tag{37}$$

By Theorem 7 the descriptor system (37) is positive since

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \in M_2, \qquad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in \Re_+^{2\times 2},$$
$$-B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \in \Re_+^{2\times 2}, \qquad C_1 = [1 \quad 0] \in \Re_+^{1\times 2},$$
$$C_2 = [1 \quad 3] \in \Re_+^{1\times 2}\,. \tag{38}$$

Repeating the considerations for the descriptor discrete-time system (22) we obtain the following theorem.

Theorem 8. The descriptor discrete time system (22) is positive if and only if

$$A_1 \in \Re_+^{n_1\times n_1}, B_1 \in \Re_+^{n_1\times m}, -B_2 \in \Re_+^{n_2\times m},$$
$$C_1 \in \Re_+^{p\times n_1}, C_2 \in \Re_+^{p\times n_2} \tag{39}$$

4 TRANSFER MATRICES OF POSITIVE DESCRIPTOR LINEAR SYSTEMS

Consider the standard positive continuous-time linear system (1).

Theorem 9. If the matrix $A \in M_n$ is Hurwitz and $B \in \Re_+^{n\times m}, C \in \Re_+^{p\times n}, D \in \Re_+^{p\times m}$ of the linear positive system (1), then all coefficients of the transfer matrix (3) are positive.

Proof. First by induction with respect to n we shall show that the matrix

$$[I_n s - A]^{-1} \in \Re^{n\times n}(s) \tag{40}$$

has positive coefficients. The hypothesis is true for n = 1 since

$$[s+a]^{-1} = \frac{1}{s+a} \tag{41}$$

and for n = 2

$$[I_2 s - A_2]^{-1} = \begin{bmatrix} s + a_{11} & -a_{12} \\ -a_{21} & s + a_{22} \end{bmatrix}^{-1}$$
$$= \frac{1}{s^2 + a_1 s + a_0} \begin{bmatrix} s + a_{22} & a_{12} \\ a_{21} & s + a_{11} \end{bmatrix}, \tag{42}$$

where $a_1 = a_{11} + a_{22} \geq 0, a_0 = a_{11}a_{22} - a_{12}a_{21} \geq 0$.

Assuming that the hypothesis is valid for n − 1 (the matrix $[I_{n-1}s - A_{n-1}]^{-1}$) we shall show that it is also true for n (the matrix $[I_n s - A_n]^{-1}$). It is easy to check that the inverse matrix of the matrix

$$[I_n s - A_n] = \begin{bmatrix} I_{n-1}s - A_{n-1} & u_n \\ u_n & s + a_{nn} \end{bmatrix},$$

$$u_n = -\begin{bmatrix} a_{1n} \\ \vdots \\ a_{n-1,n} \end{bmatrix}, u_n = -[a_{n1} \quad \cdots \quad a_{n,n-1}] \tag{43}$$

has the form

$$[I_n s - A_n]^{-1} = \begin{bmatrix} \bar{A}_{n-1}^{-1} + \dfrac{\bar{A}_{n-1}^{-1} u_n v_n \bar{A}_{n-1}^{-1}}{a_n} & -\dfrac{\bar{A}_{n-1}^{-1} u_n}{a_n} \\ -\dfrac{v_n \bar{A}_{n-1}^{-1}}{a_n} & \dfrac{1}{a_n} \end{bmatrix}, \tag{44}$$

where $\bar{A}_{n-1} = [I_{n-1}s - A_{n-1}]$ and

$$a_n = (s + a_{nn}) - v_n [I_{n-1}s - A_{n-1}]^{-1} u_n. \tag{45}$$

By assumption the matrix $[I_{n-1}s - A_{n-1}]^{-1}$ has all positive coefficients and the rational function (45) has positive coefficients. Taking into account that u_n and v_n have non-negative entries, we conclude that $-\frac{[I_{n-1}s - A_{n-1}]^{-1} u_n}{a_n}$ and $\frac{v_n [I_{n-1}s - A_{n-1}]^{-1}}{a_n}$ are column and row rational vectors with positive coefficients. By the same arguments the matrix

$$\frac{[I_{n-1}s - A_{n-1}]^{-1} u_n v_n [I_{n-1}s - A_{n-1}]^{-1}}{a_n} \tag{46}$$

has also all rational entries in s with positive coefficients and the matrix (43) has positive coefficients. Therefore, if $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ and $D \in \Re_+^{p\times m}$ then all coefficients of the transfer matrix (3) are positive.

Now Theorem 9 will be extended to the positive descriptor linear systems (12).

Theorem 10. If the positive descriptor system (12) is asymptotically stable, then all coefficients of its transfer matrix are positive.

Proof. By Theorem 5 the descriptor system (12) is positive if the conditions (16) are satisfied. If conditions (16a) are satisfied, then by Theorem 9 the transfer matrix of the strictly proper part has positive coefficients. Note that by Theorem 5 if the descriptor system is positive then the conditions (16b) are satisfied and the transfer matrix of the positive descriptor system (12) has positive coefficients.

Example 3. (Continuation of Example 2) The transfer matrix of the positive descriptor linear system (12) with (17) has the form (18) and its coefficients are positive. Note that the transfer matrix (19) of its strictly proper part has also positive coefficients.

Theorem 11. If the matrix $A \in \Re_+^{n\times n}$ of the descriptor discrete-time system (22) is a Schur matrix then the matrix

$$[I_n(z+1) - A]^{-1} \in \Re^{n\times n}(z) \tag{47}$$

has positive coefficients.

Proof. By Theorem 4 if $A \in \Re_+^{n\times n}$ of the discrete-time system is a Schur matrix then the matrix $A - I_n \in M_n$ is the Hurwitz matrix. The remaining elements of the proof are similar as in the proof of Theorem 9.

Example 4. The matrix

$$A = \begin{bmatrix} 0.6 & 0 & 0.2 \\ 0.1 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0.5 \end{bmatrix} \tag{48}$$

of the discrete-time linear system is asymptotically stable (Schur) since the polynomial

$$\begin{aligned} det[I_3(z+1) - A] &= \begin{vmatrix} z+0.4 & 0 & -0.2 \\ -0.1 & z+0.6 & -0.2 \\ -0.2 & -0.1 & z+0.5 \end{vmatrix} \\ &= z^3 + 1.5z^2 + 0.68z + 0.082 \end{aligned} \tag{49}$$

positive coefficients. The inverse matrix (47) for (48) has the form

$$\begin{aligned} det[I_3(z+1) - A]^{-1} &= \begin{bmatrix} z+0.4 & 0 & -0.2 \\ -0.1 & z+0.6 & -0.2 \\ -0.2 & -0.1 & z+0.5 \end{bmatrix}^{-1} \\ &= \frac{1}{z^3 + 1.5z^2 + 0.68z + 0.082} \times \\ &\begin{bmatrix} z^2 + 1.1z + 0.28 & 0.02 & 0.2z + 0.12 \\ 0.1z + 0.01 & z^2 + 0.9z + 0.16 & 0.2z + 0.06 \\ 0.2z + 0.11 & 0.1z + 0.04 & z^2 + z + 0.24 \end{bmatrix}. \end{aligned} \tag{50}$$

that all coefficients of the matrix (50) are positive. The transfer function of the positive asymptotically stable system with (48) and

$$\mathrm{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathrm{C} = [1 \quad 0 \quad 2] \tag{51}$$

has the form

$$\mathrm{T(z)} = \mathrm{C[I_n(z) - A]^{-1}B} = \frac{\mathrm{z^2 - 0.5z + 0.5}}{\mathrm{z^3 - 1.5z^2 + 0.68z - 0.094}}. \tag{52}$$

Note that (52) has some negative coefficients. Therefore, Theorem 9 is not true for positive discrete-time linear systems.

5 CONCLUDING REMARKS

Transfer matrices with positive coefficients of descriptor positive linear continuous-time and discrete-time systems have been addressed. Two methods for checking of the positivity of descriptor linear systems have been proposed. It has been shown that if the positive descriptor linear system is asymptotically stable then its transfer matrix has positive coefficients (Theorem 9). The considerations have been illustrated by numerical examples.

The considerations can be easily extended to fractional descriptor linear continuous-time and discrete-time systems.

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