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PHENOMENOLOGICAL MODELING OF MECHANICAL PROPERTIES OF METAL FOAM

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In this paper, constitutive relations for metal foam under complex load are introduced using a phenomenological concept. On the basis of stress-strain and transversal-longitudinal strain dependencies in uniaxial tests, a form of the function for strain energy density is specified, and further, the criterion for loss of stability due to the occurrence of plastic flow is defined. The theoretical investigations are illustrated by a short numerical example.

Key words: metal foam, strain energy density function, phenomenological method of modeling, energy conservation principle

1. Introduction

Modeling of mechanical properties can give an answer to the question about conditions of material damage. This phenomenon was taken into account by Petryk (1991) and Perzyna (2005). We have here a fundamental description of material properties with the use of displacement, strain and stress tensors for continuous materials. Recently, metal foams are materials which have gained wide use. In many papers, attention was focused on good properties of foams, such as: low weight, energy absorption and non-flammability. Based on experimental results, a model of the porous material was investigated by Gibson and Ashby (1999). The impact of technological process of manufacturing or microstructural parameters on varied properties of the foam was investigated by Koza *et al.* (2002). A large number of scientific and research works are devoted to engineering employment of foams. Hutchinson and He (2000) took into account buckling of cylindrical sandwich shells with metal foam cores. Mahjoob and Vafai (2008) discussed an influence of microstructural parameters of a metal foam such as porosity and density on heat transfer. Unfortunately, most of available publications do not provide information about physical mechanisms of metal foams destruction. For this reason, an energy method based on a phenomenological approach is promising, which involves construction of a strain energy density function only on the basis of relations between stress and strain in longitudinal and transversal directions, derived from the uniaxial tension test of the material sample, without any knowledge of the internal structure of the material. The material is interpreted here as a black box. The input signal is a loading program, and the output signal is the deformation state. The identification of a manner of processing of this signal is equivalent to model construction of mechanical properties of material.

Application of the energy conservation principle to the description and identification of stable or unstable states of aluminum was presented by Wegner and Kurpisz (2009). In the paper presented now, the essential problem is the use of experimental characteristics from uniaxial tension tests to construct the strain energy function for an aluminium foam.

2. Uniaxial tension tests and their approximations for the aluminium foam

To obtain basic information about physical properties of the aluminium foam, uniaxial tension and compression tests were made (Grant No. 0807/B/T02/2010/38 supported by the Ministry of Science and Higher Education in Poland). In the beginning, a cubic sample with length of the edge 50 mm was subjected to a static uniaxial tension test and then a compression test with simultaneous measurement of the longitudinal force and longitudinal and transversal strains. The plots obtained in the tension test are presented in Fig. 1.

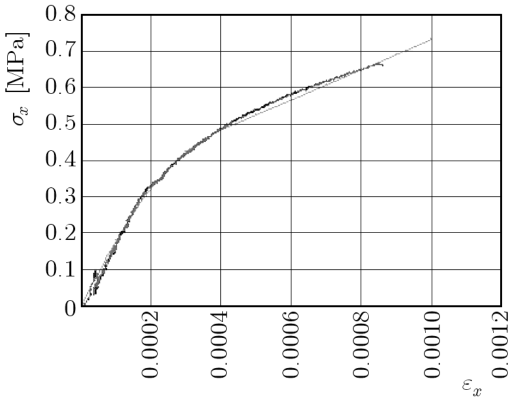

Fig. 1. Experimental relation between stress and strain for uniaxial tension (black line) and its approximation (Eq. (2.1), gray line)

The approximation of experimental results, $\sigma_A(\varepsilon_x)$ [MPa], can be expressed as

$$\sigma_A(\varepsilon_x) = \begin{cases} -1963261\varepsilon_x^2 + 1999\varepsilon_x & \text{for} \quad 0 \leqslant \varepsilon_x \leqslant 0.00038 \\ 414(\varepsilon_x - 0.00038) + 0.476 & \text{for} \quad 0.00038 < \varepsilon_x < 0.001 \end{cases} \tag{2.1}$$

Hence, the longitudinal stiffness coefficient, $\widetilde{E}_A(\varepsilon_x)$ [MPa], takes the form

$$\widetilde{E}_A(\varepsilon_x) = \frac{\sigma_A(\varepsilon_x)}{\varepsilon_x} = \begin{cases} -1963261\varepsilon_x + 1999 & \text{for} \quad 0 \leqslant \varepsilon_x \leqslant 0.00038 \\ 414 + \dfrac{0.31868}{\varepsilon_x} & \text{for} \quad 0.00038 < \varepsilon_x < 0.001 \end{cases} \tag{2.2}$$

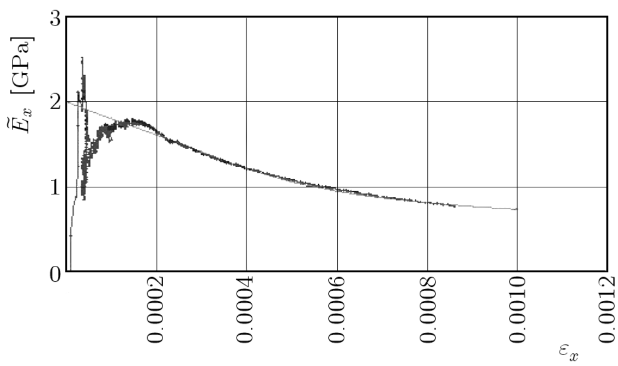

Fig. 2. Stiffness coefficient (black line) and its approximation (gray line)

The characteristics obtained for the uniaxial compression test with cyclic unloading are presented in Figs. 3-5.

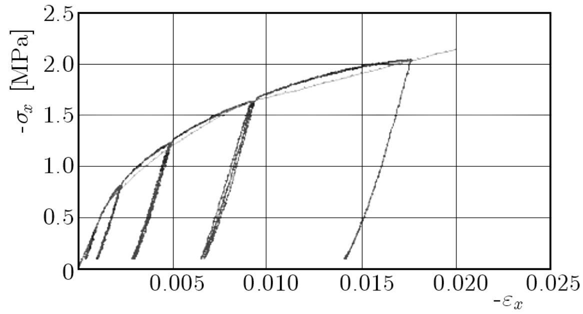

Fig. 3. Experimental relation between stress and strain (black line) and its approximation (gray line) in the compression test with cyclic unloading

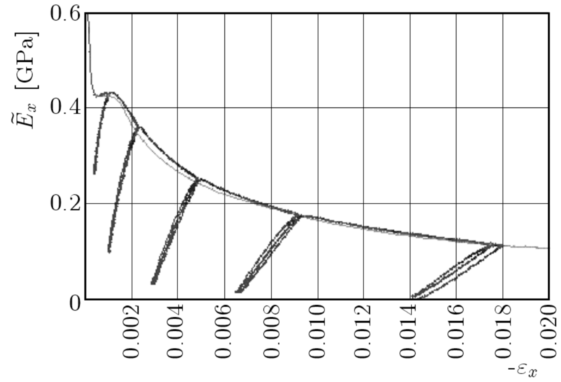

Fig. 4. Stiffness coefficient (black line) and its approximation (gray line) in the compression test with cyclic unloading

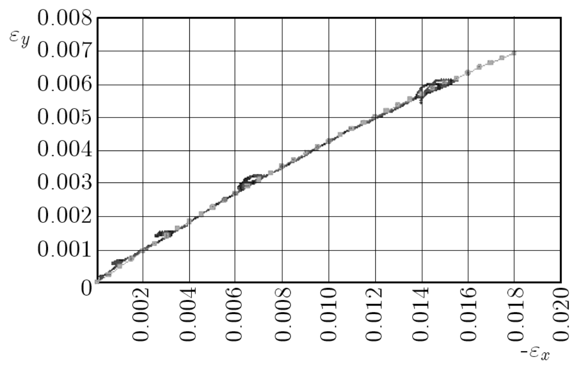

Fig. 5. Experimental relation between transversal and longitudinal strain (black line) and its approximation (gray line) in the compression test with cyclic unloading

The approximation of stress-strain $(\sigma_A(\varepsilon_x)$ [MPa]$)$ relation for uniaxial compression can be written in the form

$$\sigma_A(\varepsilon_x) = \begin{cases} 426\varepsilon_x & \text{for} \quad -0.00137 \leqslant \varepsilon_x \leqslant 0 \\ -\sqrt{-306\varepsilon_x - 0.079} & \text{for} \quad -0.0088 \leqslant \varepsilon_x \leqslant -0.00137 \\ 47.3(\varepsilon_x + 0.0088) - 1.62 & \text{for} \quad -0.015 < \varepsilon_x < 0.0088 \end{cases} \tag{2.3}$$

Hence, the approximation of the stiffness coefficient, $\widetilde{E}_A(\varepsilon_x)$ [MPa], can be expressed as

$$\widetilde{E}_A(\varepsilon_x) = \begin{cases} 426 & \text{for} \quad -0.00137 \leqslant \varepsilon_x \leqslant 0 \\ \dfrac{\sqrt{-306\varepsilon_x - 0.079}}{-\varepsilon_x} & \text{for} \quad -0.0088 \leqslant \varepsilon_x \leqslant -0.00137 \\ 47.3\Big(1 + \dfrac{0.0088}{\varepsilon_x}\Big) - \dfrac{1.62}{\varepsilon_x} & \text{for} \quad -0.015 < \varepsilon_x < 0.0088 \end{cases} \tag{2.4}$$

The transversal strain can be approximated by the function

$$\varepsilon_y(\varepsilon_x) = -5.3\varepsilon_x^2 - 0.48\varepsilon_x \qquad \text{where} \qquad -0.018 \leqslant \varepsilon_x \leqslant 0 \tag{2.5}$$

or using the coefficient of transversal strain

$$\widetilde{\nu}(\varepsilon_x) = -\frac{\varepsilon_y}{\varepsilon_x} = 5.3\varepsilon_x + 0.48 \qquad \text{where} \qquad -0.018 \leqslant \varepsilon_x \leqslant 0 \tag{2.6}$$

3. Geometrical interpretation of the deformation process

The deformation process of a metal foam can be presented by the use of geometrical interpretation. Because the foam can be regarded as an isotropic material (for the mean distance of pores considerably smallest than the dimension of a material piece and the same in all directions), the coefficients of stiffness and transversal strain are the same in different directions. However, the coefficients of stiffness and transversal strain do not have constant values, like the elastic modulus and Poisson ratio in most linear elastic and isotropic material models described in literature. That is why the relations between strain and stress components undergo complication. Let us take the space of main components of the deformation state. Each deformation process is represented by the curve whose points are determined by the current deformation state.

Every displacement along the deformation path C needs a work. Because the $(\varepsilon_1 \times \varepsilon_2 \times \varepsilon_3)$-space is potential, the work is not dependent on the shape of the path C. The relations between the strain and stress components take the form

$$\begin{aligned} \varepsilon_1 &= \frac{\sigma_1}{E(\varepsilon_1)} - \nu(\varepsilon_2)\frac{\sigma_2}{E(\varepsilon_2)} - \nu(\varepsilon_3)\frac{\sigma_3}{E(\varepsilon_3)} \\ \varepsilon_2 &= -\nu(\varepsilon_1)\frac{\sigma_1}{E(\varepsilon_1)} + \frac{\sigma_2}{E(\varepsilon_2)} - \nu(\varepsilon_3)\frac{\sigma_3}{E(\varepsilon_3)} \\ \varepsilon_3 &= -\nu(\varepsilon_1)\frac{\sigma_1}{E(\varepsilon_1)} - \nu(\varepsilon_2)\frac{\sigma_2}{E(\varepsilon_2)} + \frac{\sigma_3}{E(\varepsilon_3)} \end{aligned} \tag{3.1}$$

and after a series of transformations, we have the following relations

$$\sigma_i(\varepsilon_1, \varepsilon_2, \varepsilon_3) = E(\varepsilon_i)\frac{A_i(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{B_i(\varepsilon_1, \varepsilon_2, \varepsilon_3)} \qquad i = 1, 2, 3 \tag{3.2}$$

where

$$\begin{aligned} A_1(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \nu(\varepsilon_2)[1 + \nu(\varepsilon_3)](\varepsilon_1 - \varepsilon_2) - [1 + \nu(\varepsilon_2)][\varepsilon_1 + \varepsilon_3\nu(\varepsilon_3)] \\ B_1(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \nu(\varepsilon_2)[1 + \nu(\varepsilon_1)][1 + \nu(\varepsilon_3)] - [1 + \nu(\varepsilon_2)][1 - \nu(\varepsilon_1)\nu(\varepsilon_3)] \\ A_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \nu(\varepsilon_3)[1 + \nu(\varepsilon_1)](\varepsilon_2 - \varepsilon_3) - [1 + \nu(\varepsilon_3)][\varepsilon_2 + \varepsilon_1\nu(\varepsilon_2)] \\ B_2(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \nu(\varepsilon_3)[1 + \nu(\varepsilon_2)][1 + \nu(\varepsilon_1)] - [1 + \nu(\varepsilon_3)][1 - \nu(\varepsilon_1)\nu(\varepsilon_2)] \\ A_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \nu(\varepsilon_1)[1 + \nu(\varepsilon_2)](\varepsilon_3 - \varepsilon_1) - [1 + \nu(\varepsilon_1)][\varepsilon_3 + \varepsilon_2\nu(\varepsilon_2)] \\ B_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \nu(\varepsilon_1)[1 + \nu(\varepsilon_3)][1 + \nu(\varepsilon_2)] - [1 + \nu(\varepsilon_1)][1 - \nu(\varepsilon_2)\nu(\varepsilon_3)] \end{aligned} \tag{3.3}$$

and $\nu(\varepsilon_i)$ is the approximation of the transversal strain coefficient.

Let us assume that the deformation path C can be written as

$$C: \begin{cases} \varepsilon_1 = \varepsilon_1^K t \\ \varepsilon_2 = \varepsilon_2^K t \\ \varepsilon_3 = \varepsilon_3^K t \end{cases} \tag{3.4}$$

where $t \in \langle 0,1 \rangle$ and ε_i^K, $i = 1,2,3$ are the end components of the deformation state. Putting (3.4) to (3.2) and (3.3), we receive

$$\sigma_i(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) = E(\varepsilon_i^K t) \frac{A_i(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t)}{B_i(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t)} \qquad i = 1,2,3 \tag{3.5}$$

where

$$\begin{aligned} A_1(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) &= \nu(\varepsilon_2^K t)[1+\nu(\varepsilon_3^K t)](\varepsilon_1^K t - \varepsilon_2^K t) - [1+\nu(\varepsilon_2^K t)][\varepsilon_1^K t + \varepsilon_3^K t \nu(\varepsilon_3^K t)] \\ B_1(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) &= \nu(\varepsilon_2^K t)[1+\nu(\varepsilon_1^K t)][1+\nu(\varepsilon_3^K t)] - [1+\nu(\varepsilon_2^K t)][1-\nu(\varepsilon_1^K t)\nu(\varepsilon_3^K t)] \\ A_2(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) &= \nu(\varepsilon_3^K t)[1+\nu(\varepsilon_1^K t)](\varepsilon_2^K t - \varepsilon_3^K t) - [1+\nu(\varepsilon_3^K t)][\varepsilon_2^K t + \varepsilon_1^K t \nu(\varepsilon_2^K t)] \\ B_2(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) &= \nu(\varepsilon_3^K t)[1+\nu(\varepsilon_2^K t)][1+\nu(\varepsilon_1^K t)] - [1+\nu(\varepsilon_3^K t)][1-\nu(\varepsilon_1^K t)\nu(\varepsilon_2^K t)] \\ A_3(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) &= \nu(\varepsilon_1^K t)[1+\nu(\varepsilon_2^K t)](\varepsilon_3^K t - \varepsilon_1^K t) - [1+\nu(\varepsilon_1^K t)][\varepsilon_3^K t + \varepsilon_2^K t \nu(\varepsilon_2^K t)] \\ B_3(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t) &= \nu(\varepsilon_1^K t)[1+\nu(\varepsilon_3^K t)][1+\nu(\varepsilon_2^K t)] - [1+\nu(\varepsilon_1^K t)][1-\nu(\varepsilon_2^K t)\nu(\varepsilon_3^K t)] \end{aligned} \tag{3.6}$$

Expressions (3.5) assign the stress state $(\sigma_1, \sigma_2, \sigma_3)$ to an optional indirect deformation state $(\varepsilon_1^K t, \varepsilon_2^K t, \varepsilon_3^K t)$.

4. Strain energy density function

Let us assume that we want to appoint the work of change of the deformation state from P_1 to P_2 along path C (see Fig. 6). Because the discussed points are very close to each other,

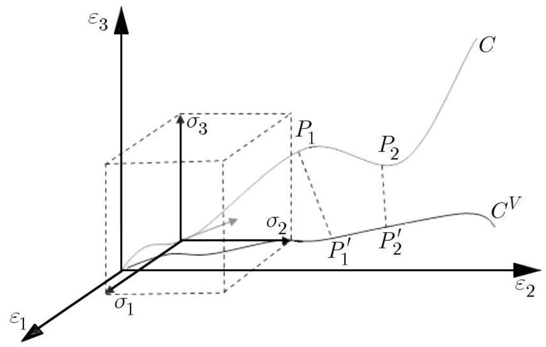

Fig. 6. A concept of the material deformation process due to its volumetric part

so the generalized loads, realizing this slip, assume a constant value. If we accept that current dimensions of the elementary cube in the deformation state P_1 are $l_1 \times l_2 \times l_3$, and respectively in P_2 – $(l_1 + \Delta l_1) \times (l_2 + \Delta l_2) \times (l_3 + \Delta l_3)$, then the increment of work can be written as

$$\Delta W^{P_1 P_2} = F_1 \Delta l_1 + F_2 \Delta l_2 + F_3 \Delta l_3 \tag{4.1}$$

After dividing (4.1) by the volume in the initial state V_0, we have

$$\frac{\Delta L^{P_1 P_2}}{V_0} = \frac{F_1}{l_2^0 l_3^0} \frac{\Delta l_1}{l_1^0} + \frac{F_2}{l_1^0 l_3^0} \frac{\Delta l_2}{l_2^0} + \frac{F_3}{l_1^0 l_2^0} \frac{\Delta l_3}{l_3^0} = \sum_{i=1}^{3} \sigma_i \frac{\Delta l_i}{l_i^0} \tag{4.2}$$

Hence, the increment of work density caused by the increment of deformation state components $\Delta\varepsilon = [\Delta\varepsilon_1, \Delta\varepsilon_2, \Delta\varepsilon_3]$ takes the form

$$\frac{\Delta L^{P_1 P_2}}{V_0} = \sum_{i=1}^{3} \sigma_i \Delta\varepsilon_i \tag{4.3}$$

Hence, on the base of definition of the contour integral, we can write

$$\frac{\Delta L}{V_0} = \int_C \sigma_1 \, d\varepsilon_1 + \int_C \sigma_2 \, d\varepsilon_2 + \int_C \sigma_3 \, d\varepsilon_3 \tag{4.4}$$

and finally, by using the energy conservation principle, we have

$$W = \int_0^1 \Big(\sum_{i=1}^{3} \sigma_i(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K, t)\varepsilon_i^K \Big) \, dt = \widehat{f}(\varepsilon_1^K, \varepsilon_2^K, \varepsilon_3^K) \tag{4.5}$$

where W is the strain energy density function. The analytical form of W can be indicated by substituting (3.5) to (4.5).

4.1. Extraction of volumetric part of energy

A pure volume deformation process takes place if the material is under hydrostatic pressure. In this case, (3.1) takes the form

$$\begin{aligned}
\varepsilon_1^V &= \frac{ks}{E(\varepsilon_1^V)} - \nu(\varepsilon_2^V)\frac{ks}{E(\varepsilon_2^V)} - \nu(\varepsilon_3^V)\frac{ks}{E(\varepsilon_3^V)} \\
\varepsilon_2^V &= -\nu(\varepsilon_1^V)\frac{ks}{E(\varepsilon_1^V)} + \frac{ks}{E(\varepsilon_2^V)} - \nu(\varepsilon_3^V)\frac{ks}{E(\varepsilon_3^V)} \\
\varepsilon_3^V &= -\nu(\varepsilon_1^V)\frac{ks}{E(\varepsilon_1^V)} - \nu(\varepsilon_2^V)\frac{ks}{E(\varepsilon_2^V)} + \frac{ks}{E(\varepsilon_3^V)}
\end{aligned} \tag{4.6}$$

where $s \in \langle 0, 1 \rangle$ is dimesionless quantity and $k < 0$. The solution to the system of equations (4.6) is parametric and gives a pure volumetric deformations path

$$C^V : \begin{cases} \varepsilon_1^V = \varphi_1(s) \\ \varepsilon_2^V = \varphi_2(s) \\ \varepsilon_3^V = \varphi_3(s) \end{cases} \tag{4.7}$$

so the pure volumetric deformation energy can be written in the form

$$W^V = \sum_{i=1}^{3} \int_{C^V} \sigma_i^V \, d\varepsilon_i^V = \int_{C^V} \Big(\sum_{i=1}^{3} ks\varphi_i'(s) \Big) \, ds \tag{4.8}$$

Received relation (4.8) can not be directly used to extract the volumetric part of energy. It is necessary to find a relation between the deformation states P_1 and P_1' (see Fig. 6). On the base of (3.4) and (4.7), we have

$$\frac{\Delta V}{V} = \prod_{i=1}^{3} [1 + \varepsilon_i(t)] - 1 = a(t) - 1 \tag{4.9}$$

and on the other hand

$$\frac{\Delta V}{V}=\prod_{i=1}^{3}[1+\varphi_i(s)]-1=b(s)-1 \tag{4.10}$$

hence

$$\prod_{i=1}^{3}[1+\varphi_i(s)]=\prod_{i=1}^{3}(1+\varepsilon_i^K t) \tag{4.11}$$

and

$$s=h(\varepsilon_1^K,\varepsilon_2^K,\varepsilon_3^K,t)=b^{-1}(a(\varepsilon_1^K,\varepsilon_2^K,\varepsilon_3^K,t)) \tag{4.12}$$

On the base of (4.12), the volumetric part of energy can be written as

$$\begin{aligned}W^V&=\sum_{i=1}^{3}\int\limits_{C^V}\sigma_i\,d\varepsilon_i^V\\&=\int\limits_0^1\Big[\sum_{i=1}^{3}\sigma_i(\varepsilon_1^K,\varepsilon_2^K,\varepsilon_3^K,t)\varphi_i'(h(\varepsilon_1^K,\varepsilon_2^K,\varepsilon_3^K,t))\Big]h'(\varepsilon_1^K,\varepsilon_2^K,\varepsilon_3^K,t)\,dt\end{aligned} \tag{4.13}$$

so, it is a function of the deformation state $\varepsilon=(\varepsilon_1^K,\varepsilon_2^K,\varepsilon_3^K)$.

5. Stability assumptions

Based on the Jaunzemis criterion of material stability (1967) and the strength hypothesis presented by Wegner (2000a,b, 2005, 2009), the damage of material begins in the place, where, in geometrical interpretation, the state of loss of convexity of the strain energy density function appears.

The stability regions are there where the strain energy density function is convex. So, the stability criterion takes the form

$$\delta^2 W=\sum_{i=1}^{3}\sum_{j=3}^{3}\frac{\partial^2 W}{\partial\varepsilon_i\partial\varepsilon_j}\delta\varepsilon_i\,\delta\varepsilon_j>0 \tag{5.1}$$

and on the base of the Sylvester theorem about quadratic form, we have

$$\begin{array}{lll}\det\mathbf{M}>0 & & \\ \det\mathbf{M}_1>0 & \det\mathbf{M}_2>0 & \det\mathbf{M}_3>0\\ \det\mathbf{M}_{1,2}>0 & \det\mathbf{M}_{1,3}>0 & \det\mathbf{M}_{2,3}>0\end{array} \tag{5.2}$$

where

$$\mathbf{M}=\begin{bmatrix}\dfrac{\partial^2 W}{\partial\varepsilon_1^2} & \dfrac{\partial^2 W}{\partial\varepsilon_1\partial\varepsilon_2} & \dfrac{\partial^2 W}{\partial\varepsilon_1\partial\varepsilon_3}\\ \dfrac{\partial^2 W}{\partial\varepsilon_1\partial\varepsilon_2} & \dfrac{\partial^2 W}{\partial\varepsilon_2^2} & \dfrac{\partial^2 W}{\partial\varepsilon_2\partial\varepsilon_3}\\ \dfrac{\partial^2 W}{\partial\varepsilon_1\partial\varepsilon_3} & \dfrac{\partial^2 W}{\partial\varepsilon_2\partial\varepsilon_3} & \dfrac{\partial^2 W}{\partial\varepsilon_3^2}\end{bmatrix}$$

$$\mathbf{M}_1 = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \varepsilon_2^2} & \dfrac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3} \\ \dfrac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3} & \dfrac{\partial^2 W}{\partial \varepsilon_3^2} \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \varepsilon_1^2} & \dfrac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3} \\ \dfrac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3} & \dfrac{\partial^2 W}{\partial \varepsilon_3^2} \end{bmatrix} \qquad \mathbf{M}_3 = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \varepsilon_1^2} & \dfrac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2} \\ \dfrac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2} & \dfrac{\partial^2 W}{\partial \varepsilon_2^2} \end{bmatrix}$$

$$\mathbf{M}_{2,3} = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \varepsilon_1^2} \end{bmatrix} \qquad \mathbf{M}_{3,1} = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \varepsilon_2^2} \end{bmatrix} \qquad \mathbf{M}_{2,3} = \begin{bmatrix} \dfrac{\partial^2 W}{\partial \varepsilon_3^2} \end{bmatrix}$$

If we take into consideration damage of a material due to plastic flow, then the increment of volume change is equal to zero

$$\begin{aligned} &\delta\Big(\frac{\Delta V}{V}\Big) = \delta((1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1) = 0 \\ &\delta\Big(\frac{\Delta V}{V}\Big) = (1+\varepsilon_2+\varepsilon_3)\delta\varepsilon_1 + (1+\varepsilon_1+\varepsilon_3)\delta\varepsilon_2 + (1+\varepsilon_1+\varepsilon_2)\delta\varepsilon_3 = 0 \end{aligned} \tag{5.3}$$

and (5.1) can be written in the form

$$\begin{aligned} &A(\varepsilon_1,\varepsilon_2,\varepsilon_3) + B(\varepsilon_1,\varepsilon_2,\varepsilon_3)[(1+\varepsilon_2+\varepsilon_3)\delta\varepsilon_1 + (1+\varepsilon_1+\varepsilon_3)\delta\varepsilon_2 + (1+\varepsilon_1+\varepsilon_2)\delta\varepsilon_3]^2 \\ &\quad - \Big[(1+\varepsilon_2+\varepsilon_3)^2 B(\varepsilon_1,\varepsilon_2,\varepsilon_3) - \frac{\partial^2 W}{\partial \varepsilon_1^2}\Big]\delta\varepsilon_1^2 - \Big[(1+\varepsilon_1+\varepsilon_3)^2 B(\varepsilon_1,\varepsilon_2,\varepsilon_3) - \frac{\partial^2 W}{\partial \varepsilon_2^2}\Big]\delta\varepsilon_2^2 \\ &\quad - \Big[(1+\varepsilon_1+\varepsilon_2)^2 B(\varepsilon_1,\varepsilon_2,\varepsilon_3) - \frac{\partial^2 W}{\partial \varepsilon_3^2}\Big]\delta\varepsilon_3^2 \\ &\quad - 2\Big[(1+\varepsilon_2+\varepsilon_3)(1+\varepsilon_1+\varepsilon_3)B(\varepsilon_1,\varepsilon_2,\varepsilon_3) - \frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2}\Big]\delta\varepsilon_1\,\delta\varepsilon_2 \\ &\quad - 2\Big[(1+\varepsilon_2+\varepsilon_3)(1+\varepsilon_1+\varepsilon_2)B(\varepsilon_1,\varepsilon_2,\varepsilon_3) - \frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3}\Big]\delta\varepsilon_1\,\delta\varepsilon_3 \\ &\quad - 2\Big[(1+\varepsilon_1+\varepsilon_3)(1+\varepsilon_1+\varepsilon_2)B(\varepsilon_1,\varepsilon_2,\varepsilon_3) - \frac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3}\Big]\delta\varepsilon_2\,\delta\varepsilon_3 = 0 \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} A(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \frac{\partial^2 W}{\partial \varepsilon_1^2}\delta\varepsilon_1^2 + \frac{\partial^2 W}{\partial \varepsilon_2^2}\delta\varepsilon_2^2 + \frac{\partial^2 W}{\partial \varepsilon_3^2}\delta\varepsilon_3^2 + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2}\delta\varepsilon_1\,\delta\varepsilon_2 + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3}\delta\varepsilon_1\,\delta\varepsilon_3 \\ &\quad +2\frac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3}\delta\varepsilon_2\,\delta\varepsilon_3 \\ B(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= \frac{\partial^2 W}{\partial \varepsilon_1^2} + \frac{\partial^2 W}{\partial \varepsilon_2^2} + \frac{\partial^2 W}{\partial \varepsilon_3^2} + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2} + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3} + 2\frac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3} \end{aligned}$$

which, after a series of transformations, gives

$$\begin{aligned} \delta^2 W &= \Big[(1+\varepsilon_2+\varepsilon_3)^2 B(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{\partial^2 W}{\partial \varepsilon_1^2}\Big]\delta\varepsilon_1^2 + \Big[(1+\varepsilon_1+\varepsilon_3)^2 B(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{\partial^2 W}{\partial \varepsilon_2^2}\Big]\delta\varepsilon_2^2 \\ &\quad + \Big[(1+\varepsilon_1+\varepsilon_2)^2 B(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{\partial^2 W}{\partial \varepsilon_3^2}\Big]\delta\varepsilon_3^2 \\ &\quad + 2\Big[(1+\varepsilon_2+\varepsilon_3)(1+\varepsilon_1+\varepsilon_3)B(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2}\Big]\delta\varepsilon_1\,\delta\varepsilon_2 \\ &\quad + 2\Big[(1+\varepsilon_2+\varepsilon_3)(1+\varepsilon_1+\varepsilon_2)B(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3}\Big]\delta\varepsilon_1\,\delta\varepsilon_3 \\ &\quad + 2\Big[(1+\varepsilon_1+\varepsilon_3)(1+\varepsilon_1+\varepsilon_2)B(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3}\Big]\delta\varepsilon_2\,\delta\varepsilon_3 \end{aligned} \tag{5.5}$$

Therefore, on the base of the Sylvester theorem, we can write

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \leqslant 0$$

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \leqslant 0 \qquad \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} \leqslant 0 \qquad \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} \leqslant 0 \tag{5.6}$$

$$\begin{vmatrix} x_{11} \end{vmatrix} \leqslant 0 \qquad \begin{vmatrix} x_{22} \end{vmatrix} \leqslant 0 \qquad \begin{vmatrix} x_{33} \end{vmatrix} \leqslant 0$$

where for $i,j = 1,2,3$

$$\begin{aligned} x_{ii} &= (1+\Theta-\varepsilon_i)^2\Big(\frac{\partial^2 W}{\partial \varepsilon_1^2} + \frac{\partial^2 W}{\partial \varepsilon_2^2} + \frac{\partial^2 W}{\partial \varepsilon_3^2} + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2} + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3} + 2\frac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3}\Big) + \frac{\partial^2 W}{\partial \varepsilon_i^2} \\ x_{ij} &= x_{ji} = (1+\Theta-\varepsilon_i)(1+\Theta-\varepsilon_j)\Big(\frac{\partial^2 W}{\partial \varepsilon_1^2} + \frac{\partial^2 W}{\partial \varepsilon_2^2} + \frac{\partial^2 W}{\partial \varepsilon_3^2} + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_2} \\ &\quad + 2\frac{\partial^2 W}{\partial \varepsilon_1 \partial \varepsilon_3} + 2\frac{\partial^2 W}{\partial \varepsilon_2 \partial \varepsilon_3}\Big) + \frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} \end{aligned} \tag{5.7}$$

The plastic flow appears if at least one of inequalities (5.6) is satisfied.

6. Example

Let us take that nonlinear porous material characteristics can be written in form (2.4) and (2.6).

After a series of transformations according to (3.2)-(3.5), the strain energy density function (4.5) for three axial pressures takes the form

$$W(\varepsilon_1,\varepsilon_2,\varepsilon_3) = 426\int_0^1 \Big(\sum_{i=1}^{3} \varepsilon_i \frac{A_i(\varepsilon_1,\varepsilon_2,\varepsilon_3,t)}{B_i(\varepsilon_1,\varepsilon_2,\varepsilon_3,t)}\Big)\,dt \tag{6.1}$$

and, after a series of transformations, we get

$$\begin{aligned} W(\varepsilon_1,\varepsilon_2,\varepsilon_3) &= -\frac{328.02(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2) + 604.92(\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3)}{7.54(\varepsilon_1+\varepsilon_2+\varepsilon_3)} \\ &\quad \cdot \Big[1 + \frac{0.876}{7.54(\varepsilon_1+\varepsilon_2+\varepsilon_3)}\ln\Big(1 - \frac{7.54(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{0.876}\Big)\Big] \end{aligned} \tag{6.2}$$

The strain energy density function for the subspace $\Omega = \varepsilon_1 \times \varepsilon_2 \times 0$ is presented in Fig. 7.

The solution to equations (4.6) gives the relation

$$\varepsilon_1^V = \varepsilon_2^V = \varepsilon_3^V = -\frac{0.04s}{426 - 10.6s} \tag{6.3}$$

Basing on (4.12), the relation between s and t takes the form

$$s = \frac{426\Big(1 - \sqrt[3]{(1+\varepsilon_1 t)(1+\varepsilon_2 t)(1+\varepsilon_3 t)}\Big)}{10.64 - 10.6\sqrt[3]{(1+\varepsilon_1 t)(1+\varepsilon_2 t)(1+\varepsilon_3 t)}} \tag{6.4}$$

and finally, according to (4.13), the volumetric part of energy can be written as

$$W^V(\varepsilon_1,\varepsilon_2,\varepsilon_3) = -41.32(\varepsilon_1+\varepsilon_2+\varepsilon_3)\Big[1 + \frac{0.876}{7.54(\varepsilon_1+\varepsilon_2+\varepsilon_3)}\ln\Big(1 - \frac{7.54(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{0.876}\Big)\Big] \tag{6.5}$$

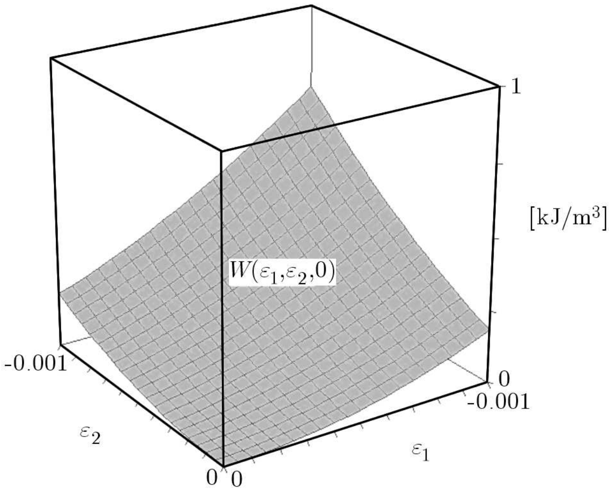

Fig. 7. Strain energy density function

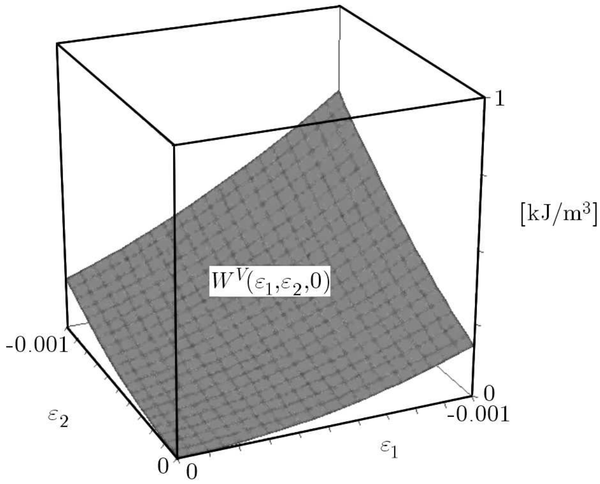

Fig. 8. Volumetric part of the strain energy density function

The plot of the volumetric part of the strain energy density function is shown in Fig. 9.

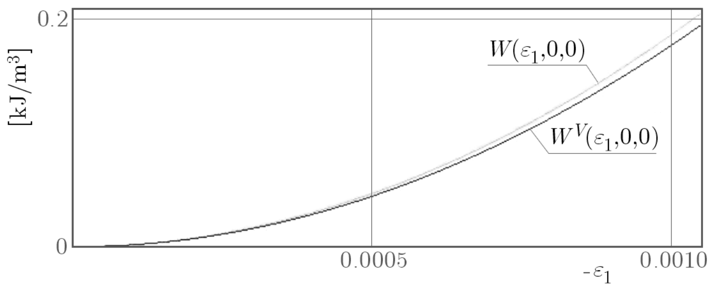

Fig. 9. Plot of the strain energy density function and its volumetric part for $\varepsilon_2 = \varepsilon_3 = 0$

7. Conclusions

The main conclusions are:

- Direct application of the known Hook law for description of the relation between stress and strain in metal foams is impossible due to nonlinear elastic properties of the material.

- The strain energy density function can be used for description of properties of nonlinear elastic porous materials.
- Material characteristics have a significant influence on the shape of plot of the energy density function and, in consequence, on regions of stability.
- The strain energy density function is symmetric for an isotropic metal foam.
- The volumetric part of the strain energy density function can be properly extracted by assuming an appropriate interpretation for finding an adequate equivalent of the hydrostatic pressure.
- From the point of view of energy, the volumetric deformation is the main form of deformations, and the deformation of the material is stable due to numerical investigations of inequalities (5.2) for the elastic range $\varepsilon_i \in \langle -0.001, 0 \rangle$.

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Modelowanie metodą fenomenologiczną mechanicznych właściwości pian metalowych

Streszczenie

W niniejszej pracy, bazując na fenomenologicznym sposobie podejścia, wyprowadzono związki konstytutywne piany metalowej poddanej złożonemu stanowi obciążenia. Na podstawie zależności naprężenia oraz odkształcenia poprzecznego od odkształcenia podłużnego uzyskanych w jednoosiowych próbach, wyznaczono wzór opisujący funkcję gęstości energii wewnętrznej, a następnie sformułowano analityczną postać kryteriów zniszczenia materiału ze względu na wystąpienie plastycznego płynięcia. Rozważania teoretyczne zilustrowano prostym przykładem numerycznym.

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