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Pure torsion problem in tensor notation

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Abstract: The paper examines the application of the tensor calculus to the classic problem of the pure torsion of prismatic rods. The introduction contains a short description of the reference frames, base vectors, contravariant and covariant vector coordinates when applying the Einstein summation convention. Torsion formulas were derived according to Coulomb's and Saint-Venant's theories, while, as a link between the theories, so-called Navier's error was discussed. Groups of the elasticity theory equations were used.

Keywords: pure torsion, tensor calculus, covariant/contravariant basses, vector components.

1. Introduction

The tensor notation used here is in accordance with the notation applied in the publications [1-2]. The description is known as index notation, although there are some variants to it. Prof. Jan Rychlewski consequently used the dyadic notation during his lectures on continuum mechanics at the Institute of Fundamental Technical Problems in Warsaw at the end of 1970s. In the monograph [3], vectors and tensors are written as boldface letters without indices. In this case, an additional notation with regard to components is necessary.

There are numerous monographs where tensors are used, albeit without the application of a uniform and consistent method. Let us recall here Sokolikoff's repeatedly reprinted monograph [4], for instance.

The Einstein summation convention (1916) concerns the geometrical/physical objects with upper and lower indices where there are various pairs of identical indices and where one is upper and the other one – lower. This implies summation over the paired indices over the whole range of their values.

In Fig. 1.a-b, the same vector $\vec{u}$ is drawn as an oriented segment in two different dimensional coordinate systems. Although the base vectors are singular, they are different. In Fig. 1.a and Fig.1.b, the coordinate lines are the same, however, the projections of the $\vec{u}$ vector differ significantly. In the case of Fig. 1.a, the parallel projection takes place, and in the case of Fig. 1.b, the orthogonal projection is applied. As a result, base vectors are denoted as $\vec{g}^m$ and $\vec{g}_n$ respectively, where $\vec{g}^m$ creates the contravariant base and $\vec{g}_n$ stands for the covariant base. By virtue of summation convention, respective presentations of the $\vec{u}$ vector are possible.

$$\vec{u} = u_m \vec{g}^m \text{ and } \vec{u} = u^n \vec{g}_n \text{ , where } m, n = 1, 2. \tag{i}$$

The vector coefficients create sets u_m and u^n adequately to their bases.

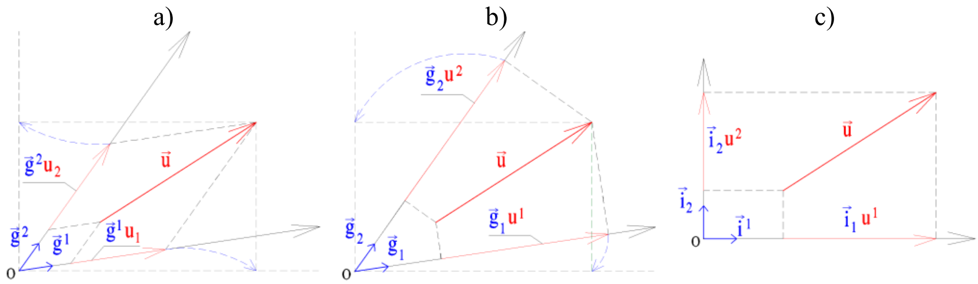

Fig 1. Base vectors as well as the contravariant and covariant vectors' components in two dimensions a) covariant vector components b) contravariant vector components c) Cartesian coordinate system

In Fig 1., the transformation of skew coordinates into a Cartesian system is also suggested. Such transformation reveals that in the case of the Cartesian coordinate system there is no distinction between the contravariant and covariant bases and, as a consequence, the summation convention plays only the role of an "alternator".

The skew coordinate systems were investigated by Kaliski, [5].

The metric tensor includes the information on the coordinate system and is obtained by the scalar product of base vectors

$$g_{mn} = \vec{g}_m \cdot \vec{g}_n\,,\ \delta_{mn} = \vec{i}_m \cdot \vec{i}_n = \delta^{mn} = \vec{i}^m \cdot \vec{i}^n = \delta^n_m = \vec{i}_m \cdot \vec{i}^n\,, \tag{ii}$$

Additionally,

$$g^n_m = \delta^n_m\,. \tag{iii}$$

Where g_{mn} is the metric tensor of an arbitrary coordinate system with the base vectors g_m. δ^n_m is a metric tensor for a Cartesian coordinate system, also known as Kronecker's delta. Here, the m, n indices can run over 1, 2, 3, for instance. The metric tensor allows to increase or decrease an index which leads to associated tensors in the following way

$$A^n_m g_{nk} = A_{mk}\,. \tag{iv}$$

The $\vec{u}$ vector coefficients can be found as a scalar product of the u and base vectors, according to the following formula:

$$u_k = \vec{u} \cdot \vec{g}_k\,. \tag{v}$$

In the linear theory of elasticity there are the following groups of problems:
- deformation, strain tensor ε^n_m,
- constitutive relations, stress tensor σ^j_i,
- equilibrium equation,
- boundary conditions or initial-boundary conditions,
- compatibility relation.

This group set will be used below. The pure torsion problem of a bar belongs to the classical approaches in mechanics. It can be found in [4], [6-7]. One of the latest publications on the theory of elasticity is a monograph [8]. In the field of the continuum mechanics, the book [10] can be worth to mention.

2. Coulomb's theory

We assume:

I) the summation convention and tensor calculus in three-dimensional space will be implemented,

II) the theory of small deformations and the principle of stiffness apply

$$\varepsilon_{ij}^{(sum.)}=\frac{1}{2}\left(\nabla_i u_j+\nabla_j u_i \pm \nabla_i u_m \nabla_j u^m\right)\rightarrow \varepsilon_{ij}^{(sum.)}=\frac{1}{2}\left(\nabla_i u_j+\nabla_j u_i\right)=\varepsilon_{ji}^{(sum.)}, \qquad (1)$$

where ∇_i stands for covariant derivative and in Cartesian coordinate system becomes to ordinary derivative $\dfrac{\partial}{\partial x_i}$,

$$\varepsilon_{ij}^{(sum.)}=\varepsilon_{ij}+\varepsilon_{ij}^{(init.)},\ \varepsilon_{ij}^{(init.)}=0, \qquad (1.1)$$

III) material is isotropic and homogenous, Hooke's law is valid –

$$\sigma_{ij}=2\mu\varepsilon_{ij}+\lambda J_1 g_{ij},\ J_1=\varepsilon_{ij}g^{ij}, \qquad (2)$$

where J_1 is the first invariant of the strain tensor, $J_1=\varepsilon_{ij}g^{ij}=\varepsilon_i^i$,

IV) the pure torsion of straight rod of a constant circular section is considered, (Fig.2.), in Cartesian coordinate system we have

$\vec{T}=T_m\vec{i}^{\,m}=\vec{0}$, (resultant internal forces vector) (2.3.1)

$\vec{M}=M_k\vec{i}^{\,k}=M_1\vec{i}^{\,1}+M_2\vec{i}^{\,2}+M_3\vec{i}^{\,3}=M_3\vec{i}^{\,3}=\vec{\mathfrak{M}}$, (resultant vectors of internal moments) (3.2)

where $\mathfrak{M}=const.\neq 0$; (3.3)

V) x_1, x_2 are the principal axes in the sense of inertial moments,

VI) the Coulomb’s assumption is assumed – in case of pure torsion the cross-sections of the rod turn each other like infinitely rigid discs,

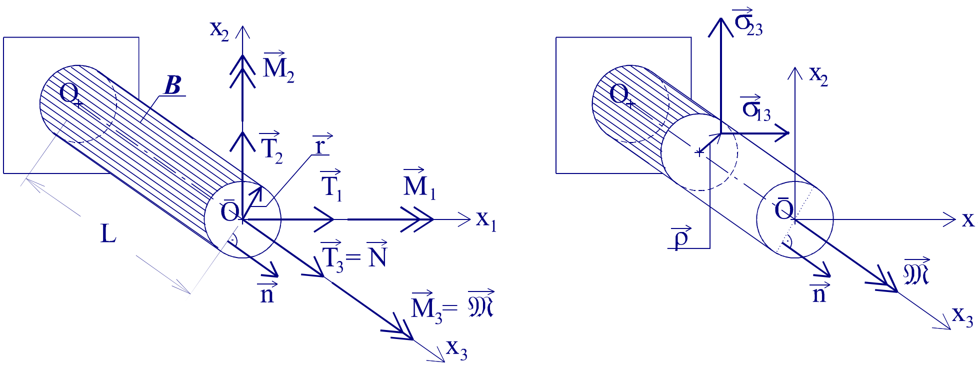

Fig.2. Cartesian coordinate system, internal force vectors and stress vectors

VII) the Cauchy's relation is valid

$$\vec{t} = \sigma_{ij}\, n^i\, \vec{g}^j\,, \tag{4}$$

the stress vector $\vec{t}$ is related to stress tensor σ_{ij} by cutting the body (point) with plane which is oriented by outward normal vector $\vec{n}$ at any chosen body point; in detail

$$\vec{n} = n^m\, \vec{g}_m\,,\ |\vec{n}| = 1\,, \tag{5}$$

VIII) statics is analysed.

3. Deformation – displacements - geometric relations

Lets repeat again - in case of Cartesian coordinates, for tensors of valence one or two, there are no distinction between covariant, contravariant or shifted indices –

$$\vec{i}_m = \vec{i}^{\,m}\,,\ u_m = u^m\,,\ x_m = x^m\,, \varepsilon_{mn} = \varepsilon^{mn} = \varepsilon^n_m\,, \tag{6}$$

where m =1, 2, 3, (three dimensions).

In initial state, for an arbitrary point P of a rod cross-section, in polar description, its position is uniquely defined by position vector $\vec{\rho}_P$. After torsion, in the actual configuration, the point P rotates to the position P' and is depicted be vector $\vec{\rho}_{P'}$. In analysed case, the difference between actual and initial configurations (P' and P) depicts the deformation, (Fig. 3.).

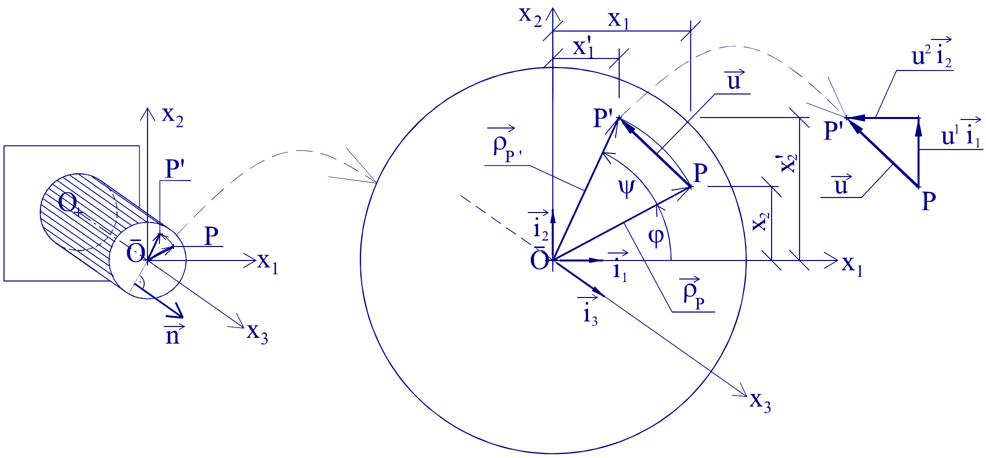

Fig. 3. Deformation

Assumption VI) implies that

$$u_3 = 0\,, \tag{7}$$

it results the only two (of three) non zero displacement vector components

$$\vec{u} = u_1\vec{i}^{\,1} + u_2\vec{i}^{\,2} + 0\vec{i}^{\,3} = u_{\overline{m}}\vec{i}^{\,\overline{m}}\,,\ \overline{m} = 1{,}2. \tag{7.1}$$

and a circular trajectory of movement

$$\left|\vec{\rho}_P\right| = \left|\vec{\rho}_{P'}\right| = \left|\vec{\rho}\right| = \rho \tag{8}$$

and

$$\vec{\rho}_P + \vec{u} = \vec{\rho}_{P'} \,. \tag{9}$$

Where:

$$\vec{\rho}_P = x_{\overline{m}} \vec{i}^{\,\overline{m}} = x_1 \vec{i}^{\,1} + x_2 \vec{i}^{\,2} \,, \;\; \vec{\rho}_{P'} = x'_{\overline{m}} \vec{i}^{\,\overline{m}} = x'_1 \vec{i}^{\,1} + x'_2 \vec{i}^{\,2} \tag{10}$$

or

$$\vec{u} = \vec{\rho}_{P'} - \vec{\rho}_P \Rightarrow u_{\overline{m}} = x'_{\overline{m}} - x_{\overline{m}} \,. \tag{11}$$

Additionally the polar coordinate system is introduced. By virtue of II) assumption, the rotational angle of a disc $\psi = \psi(x_3)$ is a small one, i.e.

$$\psi \to 0 \,, \tag{12}$$

this implies as follows

$$\begin{cases} \lim\limits_{\psi \to \pm 0} \cos\psi = f_1(\psi) = 1 \\ \lim\limits_{\psi \to \pm 0} \sin\psi = f_2(\psi) = \psi \end{cases} . \tag{13}$$

On the basis of (8) the Cartesian coordinates of P i P’ can be written in the form

$$\begin{cases} x_1 = \rho \cos\varphi \\ x_2 = \rho \sin\varphi \end{cases} , \tag{14.1}$$

and

$$\begin{cases} x'_1 = \rho \cos(\varphi + \psi) \Rightarrow \lim\limits_{\psi \to \pm 0} x'_1 = \rho\,(\cos\varphi - \psi\,\sin\varphi) = x_1 - \psi\, x_2 \\ x'_2 = \rho \sin(\varphi + \psi) \Rightarrow \lim\limits_{\psi \to \pm 0} x'_2 = \rho\,(\sin\varphi + \psi\,\cos\varphi) = x_2 + \psi\, x_1 \end{cases} . \tag{14.2}$$

Using (11), (14.1-14.2), the non zeros components of displacement vector $\vec{u}$ are as follows

$$\begin{cases} u_1 = -\psi\, x_2 \\ u_2 = \psi\, x_1 \end{cases} . \tag{15}$$

4. Strain tensor components

Having known displacement vector (7), (15) we can use (1) to determine the components of strain tensor in Cartesian coordinate system

$$\varepsilon_{ij} = \varepsilon_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right) \xrightarrow{Cart.} \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\left(u_{j},_{i} + u_{i},_{j}\right), \tag{16}$$

adequately we obtain

$$\varepsilon_{11} = u_1,_1 = 0\,,\ \varepsilon_{22} = u_2,_2 = 0\,,\ \varepsilon_{33} = u_3,_3 = 0\,, \tag{17.1-3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(u_2,_1 + u_1,_2\right) = \frac{1}{2}\left(\psi - \psi\right) = 0\,,$$

$$\varepsilon_{13} = \frac{1}{2}\left(u_3,_1 + u_1,_3\right) = \frac{1}{2}u_1,_3 = -\frac{1}{2}\frac{d\psi}{dx_3}x_2 = -\frac{1}{2}\omega x_2\,, \tag{17.4-6}$$

$$\varepsilon_{23} = \frac{1}{2}\left(u_3,_2 + u_2,_3\right) = \frac{1}{2}u_2,_3 = \frac{1}{2}\frac{d\psi}{dx_3}x_1 = \frac{1}{2}\omega x_1\,.$$

In (17) the above notation is introduced

$$\psi\left(x_3\right) = \int_0^{x_3} \omega\left(\xi\right)d\xi \tag{18.1}$$

where $\omega\left(x_3\right) = \dfrac{d\psi}{dx_3}$. (18.2)

The strain tensor can be presented in the matrix form

$$\varepsilon_{ij} = \begin{bmatrix} 0 & 0 & -\dfrac{\omega x_2}{2} \\ 0 & 0 & \dfrac{\omega x_1}{2} \\ -\dfrac{\omega x_2}{2} & \dfrac{\omega x_1}{2} & 0 \end{bmatrix}. \tag{18.3}$$

Hence, the first invariant of ε_{ij} (dilatation) becomes as follows

$$J_1 = \varepsilon_{ij}g^{ij} \xrightarrow{Cart.} \varepsilon_{ij}\delta^{ij} = \varepsilon_1^1 + \varepsilon_2^2 + \varepsilon_3^3 = 0\,. \tag{19}$$

5. Constitutive relation – stresses

By means of (2) which is valid for isotropic and homogenous material we can find the set of stress tensor components. Applying (19) we get

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda J_1 g_{ij} = 2G\varepsilon_{ij}\,. \tag{20}$$

Where G is the Kirchhoff modulus. The non zeros components are in only two cases

$\sigma_{13} = -G\omega\, x_2$ and $\sigma_{23} = G\omega\, x_1$. (21)

In matrix notation one arrives to

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & -G\omega x_2 \\ 0 & 0 & G\omega x_1 \\ -G\omega x_2 & G\omega x_1 & 0 \end{bmatrix} = \sigma_{ji} \tag{21.1}$$

6. Equilibrium state

In the general case, the state of equilibrium expresses two conditions of resetting the resultant vectors of forces and moments acting on/in the body respectively. In the problem of pure torsion, the symmetry of stress tensor (21.1) leads to the first condition of the form

$$\vec{\mathfrak{M}} = \int_A \vec{\rho} \times \vec{\mathrm{t}}\; dA = \int_A \in^{abc} \rho_a \mathrm{t_b} \vec{g}_c \, dA \xrightarrow{Cart.} \int_A e^{abc} x_a \mathrm{t}_b \vec{i}_c \, dA = \mathfrak{M}\,\vec{i}_3 \,. \tag{22}$$

The rod cross-section is characterized by normal vector $\vec{n}$ which Cartesian components (Fig. 1.) are reduced as follows

$$\vec{n} = n_k \vec{i}^{\,k} = 0\vec{i}^{\,1} + 0\vec{i}^{\,2} + 1\vec{i}^{\,3}, \; \vec{n}: \quad n_k = n^k = [0,0,1]. \tag{23}$$

Using (21) we can write the components of a stress vector (4) in the cross-section

$$\vec{\mathrm{t}} = \sigma_{bk}\, n^k \, \vec{g}^{\,b} \xrightarrow{Cart.} \sigma_{bk}\, n^k \, \vec{i}^{\,b} = \mathrm{t}_b \vec{i}^{\,b} \,, \tag{24.1}$$

$$t_b = \sigma_{bk} n^k = \sigma_{b1} n^1 + \sigma_{b2} n^2 + \sigma_{b3} n^3 = \sigma_{b3} 1 = \sigma_{b3} \,. \tag{24.2}$$

On the basis of assumption IV) and appropriately (19), (22) do (24) we obtain

$$\mathfrak{M} = \int_A (x_1 \sigma_{23} - x_2 \sigma_{13}) dA = G\omega \int_A \left[(x_1)^2 + (x_2)^2 \right] \; dA = G\omega \, J_o \,, \tag{25}$$

which is usually written in the form

$$\omega = \frac{\mathfrak{M}}{G J_o} \,. \tag{26}$$

By means of – IV), (3.1), (26) we can write

$$\mathfrak{M} = const. \rightarrow \qquad \omega = const., \qquad \psi = \omega x_3 \tag{26.1}$$

$G J_o$ – is torsion stiffness against rotation. Here J_o is a centrifugal moment of inertia.

Now, applying (15), (17), (21), (26) and (26.1), sequentially components of displacement vector as well as strain and stress tensors can be rewritten as follows

$$u_1 = -\frac{\mathfrak{M}}{G J_o} x_2 x_3 \,, \; u_2 = \frac{\mathfrak{M}}{G J_o} x_1 x_3 \,, \tag{27}$$

$$\varepsilon_{13} = -\frac{\mathfrak{M}}{2 G J_o} x_2 \,, \; \varepsilon_{23} = \frac{\mathfrak{M}}{2 G J_o} \, x_1 \,, \tag{28}$$

$$\sigma_{13} = -\frac{\mathfrak{M}}{J_o} \, x_2 \,, \; \sigma_{23} = \frac{\mathfrak{M}}{J_o} \, x_1 \,. \tag{29}$$

Let us find extremes of shearing stresses in the rod cross-section. Now, the stress vector is written as a sum of normal and shearing stresses, see (21), (24) and Fig. 2, then we arrive at

$$\vec{t} = \vec{\sigma} + \vec{\tau} = t_b \vec{i}^{\,b} = \sigma_{b3} \vec{i}^{\,b} = \sigma_{13} \vec{i}^{\,1} + \sigma_{23} \vec{i}^{\,2} = \vec{\tau}\ . \tag{30}$$

The modulus of $\vec{\tau}$ can be calculated as

$$\tau = |\vec{\tau}| = \sqrt{\vec{\tau} \cdot \vec{\tau}} = \sqrt{\sigma_{a3} \vec{i}^{\,a} \cdot \sigma_{b3} \vec{i}^{\,b}} = \sqrt{\sigma_{a3}\, \sigma_{b3} \delta^{ab}} = \sqrt{(\sigma_{13})^2 + (\sigma_{23})^2} =$$

$$= G\omega \sqrt{(x_1)^2 + (x_2)^2} = G\omega R = \frac{\mathfrak{M}}{J_o}\, \rho\,. \tag{31}$$

Let us analyse vectors $\vec{\tau}$ and $\vec{\rho}$

$$\vec{\tau} \cdot \vec{\rho} = \sigma_{a3} \vec{i}^{\,a} \cdot x_b \vec{i}^{\,b} = \sigma_{a3}\, x_b\, \delta^{ab} = \sigma_{13}\, x_1 + \sigma_{23}\, x_2 = (-G\omega x_2) x_1 + (G\omega x_1) x_2 = 0\,, \tag{32}$$

or in other way (Fig. 4.) –

$$\vec{\tau} \cdot \vec{\rho} = 0 \Leftrightarrow \vec{\tau} \perp \vec{\rho}\ ; \tag{33}$$

$\tau_{extr.}$ occurs at the cross-section circuit, and its vector is tangential to the circuit.

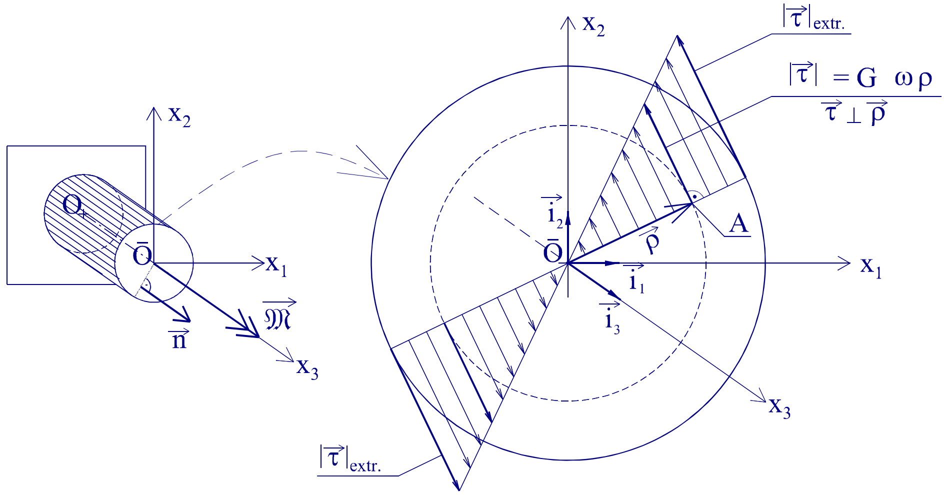

Fig. 4. Shearing stress vector

7. Navier's error

Without success, the Coulomb's theory for cross-sections of any shape was used by Claude-Louis Navier. We will show that assumption VI) is valid only for bars with a circular cross-section.

Consider the equilibrium in the case of pure torsion. The edge surface is free of any load. On this cross-section border surface we select an arbitrary point B, Fig. 5.

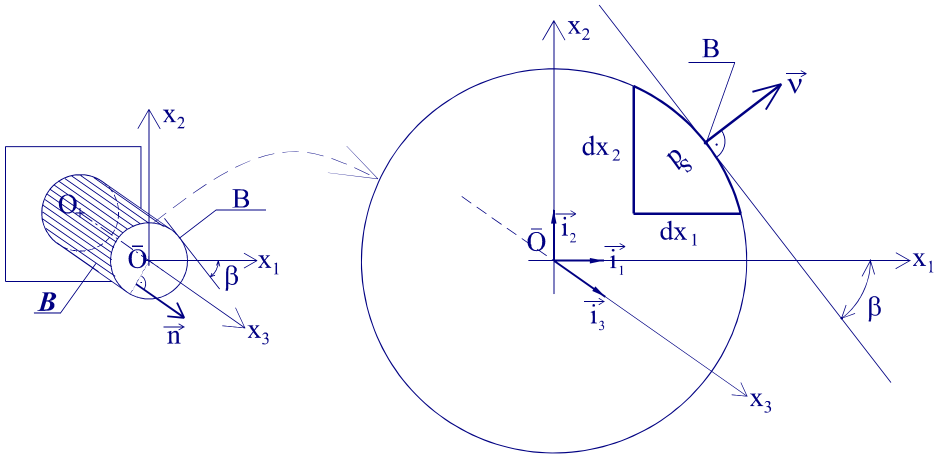

Fig. 5. Vector $\vec{n}$ and geometrical relationships on the side of the rod

We analyze the state of stress in the infinitesimal vicinity of the point B on the distance ds, Fig. 5. The tangent line to the B side is inclined at an angle β to the axis x_1, this results as follows

$$\sin\beta = \frac{dx_2}{ds}, \ \cos\beta = -\frac{dx_1}{ds}. \tag{34}$$

In the Cartesian system coordinates, the normal unit vector to the side surface at the point B has following components

$$|\vec{n}| = 1, \ n_1 = n^1 = 1\sin\beta = \frac{dx_2}{ds}, \ n_2 = n^2 = 1\cos\beta = -\frac{dx_1}{ds}, \ n_3 = 0. \tag{35}$$

The boundary surface is free of external loads, the stress vector is a zero one

$$\vec{\mathrm{t}} = \sigma_{bk} n^k \vec{i}^{\,b} = \vec{0} \Rightarrow \sigma_{bk} n^k = 0 \ \text{ where } b, k = 1, 2, 3. \tag{36}$$

The conditions (36) correspond to three equations of equilibrium, which are explicit in form

$$\begin{cases} b = 1 & \rightarrow & \sigma_{11}n^1 + \sigma_{12}n^2 + \sigma_{13}n^3 = 0 \\ b = 2 & \rightarrow & \sigma_{21}n^1 + \sigma_{22}n^2 + \sigma_{23}n^3 = 0 \\ b = 3 & \rightarrow & \sigma_{31}n^1 + \sigma_{32}n^2 + \sigma_{33}n^3 = 0 \end{cases} \tag{37}$$

On the basis of (21) we state that in the variants $b = 1$ and $b = 2$, the equations are met identically, in the case of $b = 3$ we get

$$\sigma_{31}n^1 + \sigma_{32}n^2 = G\omega\left(-x_2 n^1 + x_1 n^2\right) = -G\omega\left(x_2\frac{dx_2}{ds} + x_1\frac{dx_1}{ds}\right) = 0. \tag{38}$$

Equation (38) is equivalent to a condition

$$x_1\,dx_1 + x_2\,dx_2 = 0. \tag{39}$$

Assuming that (39) is a complete differential equation, we find a solution by integration

$$\phi(x_1,x_2)=\int_{x_{10}}^{x_1} x_1\,dx_1+\int_{x_{10}}^{x_1} x_1\,dx_1=(x_1)^2+(x_2)^2-\left[(x_{10})^2+(x_{20})^2\right]=(x_1)^2+(x_2)^2-\rho^2\,, \quad (40)$$

geometric expression of solutions is a family of circles.

Quod erat demonstrandum.

8. Assumptions of the Saint-Venant theory

The development of Navier's tests is introduced by Saint-Venant, the function of warping/deplanation of the cross-section of the rod in pure torsion. Therefore the assumption VI) is changed and now it reads as follows –

VI') projections of the points of the warped surface onto the cross-section, behave like infinitely rigid discs. We mark the function of warping by

$$\vartheta=\vartheta(x_1,x_2)\,. \quad (41)$$

Consequently, in the case of the validity of formulas (3.1), (15), the component of the displacement vector is now zero and, as proposed by Saint-Venant, is expressed in the form

$$u_3=u_3(x_1,x_2,x_3)=\omega\,\vartheta(x_1,x_2)\,. \quad (42)$$

With the assumptions made (41), (42), now it is necessary to repeat the course of proceedings conducted in the Coulomb theory variant. We receive accordingly relationships:

3'. Strains

$$\varepsilon_{ij}=\frac{\omega}{2}\begin{bmatrix} 0 & 0 & -x_2+\vartheta,_1 \\ 0 & 0 & x_1+\vartheta,_2 \\ -x_2+\vartheta,_1 & x_1+\vartheta,_2 & 0 \end{bmatrix}. \quad (43)$$

4'. Stresses

$$\sigma_{13}=G\omega\left(-x_2+\vartheta,_1\right) \text{ and } \sigma_{23}=G\omega\left(x_1+\vartheta,_2\right). \quad (44)$$

5'. Equilibrium equations

$$\mathfrak{M}=G\omega\int_A\left[(x_1)^2+(x_2)^2+\vartheta,_2 x_1-\vartheta,_1 x_2\right]\,dA=G\omega\,J_s\,, \quad (45)$$

where from we get

$$\omega=\frac{\mathfrak{M}}{G J_s}\,. \quad (46)$$

where

$$J_s = J_o + J_\Delta = \int_A \left[(x_1)^2 + (x_2)^2 + \vartheta,_2 x_1 - \vartheta,_1 x_2 \right] dA \quad . \tag{47}$$

The deplanation function introduced must meet the equilibrium conditions on the unloaded side of the rod and, as we will show, it is a harmonic function.

The equilibrium condition on the unloaded side of the bar (38) at $b = 3$ is now given as

$$\sigma_{31} n^1 + \sigma_{32} n^2 = G\omega\left[\left(-x_2 + \vartheta,_1\right) n^1 + \left(x_1 + \vartheta,_2\right) n^2 \right] = 0 . \tag{48}$$

Using (34), (35), the above (48) can be expressed as

$$\vartheta,_1 n^1 + \vartheta,_2 n^2 = x_2 \frac{dx_2}{ds} + x_1 \frac{dx_1}{ds} . \tag{49}$$

We find that (49) expresses a normal derivative –

$$\frac{\partial \vartheta}{\partial n} = x_2 \frac{dx_2}{ds} + x_1 \frac{dx_1}{ds} . \tag{50}$$

When using the gradient vector definition

$$\vec{\nabla} = \nabla_m \vec{g}^m \xrightarrow{Cart.} \frac{\partial}{\partial x_m} \vec{i}^m , \tag{51}$$

and assuming VIII), we write the Lamé equation, i.e. the conditions of intrinsic balance in displacements

$$\nabla^2 \vec{u} + \frac{\mu + \lambda}{\mu} \vec{\nabla}\left(\vec{\nabla} \cdot \vec{u}\right) = grad \left(grad + \frac{\mu + \lambda}{\mu} div \right) \vec{u} = 0 , \tag{52}$$

which in the index notation in the Cartesian coordinates have the form

$$u_i,^j{}_j + \frac{\mu + \lambda}{\mu} u_j,^j{}_i = 0 . \tag{52.1}$$

On the basis of the relation (15), (42), we obtain the components of the displacement vector

$$div\, \vec{u} = \vec{\nabla} \cdot \vec{u} = u_j,^j = u_1,^1 + u_2,^2 + u_3,^3 = \varepsilon_1{}^1 + \varepsilon_2{}^2 + \varepsilon_3{}^3 = J_1 = 0 \tag{53}$$

and hence the equations (52) take the form

$$\begin{cases} i = 1 & \rightarrow \quad u_1,_1{}^1 + u_1,_2{}^2 + u_1,_3{}^3 \equiv 0 \\ i = 2 & \rightarrow \quad u_2,_1{}^1 + u_2,_2{}^2 + u_2,_3{}^3 \equiv 0 \\ i = 3 & \rightarrow \quad u_3,_1{}^1 + u_3,_2{}^2 + u_3,_3{}^3 = u_3,_1{}^1 + u_3,_2{}^2 = \omega\left(\vartheta,_{11} + \vartheta,_{22}\right) = \nabla^2_{(1,2)} \vartheta = 0 \end{cases} \tag{54}$$

The function $\vartheta(x_1, x_2)$, fulfilling the condition $\nabla^2_{(1,2)} \vartheta = 0$, is called a harmonic.

The problem of determining the function of deplanation $\vartheta = \vartheta(x_1, x_2)$, which is harmonic in open area of the cross-section, and on the edge of this area the function ϑ fulfils the condition for a normal derivative (50) is a problem of the potential theory often referred to as the *second boundary problem of Neumann*.

The problem of beam bending with shearing in tensor notation was derived in the paper [10]. The review of the deformation measures – strain tensors – was carried out in [11].

Conclusions

Any, even a simple mechanical problem, can be consistently analysed by means of tensor notation. Nevertheless, the reader should be familiar with the fundamentals of the theory of elasticity in tensor calculus. This necessitates additional classes/lectures for students, or self-study in tensor calculus.

The problem of pure torsion was selected due to the clarity of the assumptions and a relatively small scope of the issue.

The problem discussed in the paper was presented during the Math-Bridge Camp, Muenster 2018. The discussion pointed out that for people not familiar with the tensor calculus, the problem is not clear. Majority of workshop attendees preferred the classic approach to the problem of torsion. Such conservatism is characteristic of academic teachers, while to students the *tabula rasa* principle applies, i.e. they learn tensor calculus without prejudice.

For over 15 years, in the 1970s and 1980s, the team of the Stereomechanics Department of Lublin University of Technology, headed by prof. Jerzy Grycz taught classes in the strength of materials as well as the theory of elasticity and plasticity in tensor calculus. The author was a member of this team, and the classic torsion problem posted was one of several issues developed by the author.

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